On the Possible Trajectories of Spinning Particles in an External Electromagnetic Fields

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By means of the method of moving Frenet-Serret frame the set of equations of motion is derived for spinning particle in an arbitrary external field, which is determined by potential depending from both position and the state of movement, as well as by two pseudo-vectors one of which is easily associated with external magnetic field, and another still remains undetermined. It is shown that description of the motion of both massive and massless particles with spin is possible. All solutions of the equations of motion of spinning particle in the absence of external fields were found, and besides, we give more precise definition of a free object. It turns out that the massive particles always possess a longitudinal polarization. There are possible transversal motions of the following types: 1) oscillatory motion with proper frequency, 2) circular motion, and 3) complicated motion along rosette trajectories round the center of inertia with the velocity, varying in the limits $v_{\min} < v < v_{\max}$. Free massless particles can either fluctuate or move along ellipses around fixed centers of balance, with the spin of particles can have any direction. The behavior of spin particles in a constant homogeneous magnetic field is also considered and all types of trajectories are found.

1. Introduction

It became clear after theoretical discovery of Zitterbewegung by Schrödinger and numerous papers devoted to it, that spin makes essential impact on a trajectory even for the free particles. However a possibility of taking account of spin in terms of classical nonrelativistic theory has not been properly analyzed. In the papers [1]-[3] an idea is proposed for description of the objects with internal degrees of freedom by generalization of the Second Newton's Law, and classic non-relativistic theory of such objects interacting with external fields by means of potential depending on both position and the state of motion of the object has been developed.

In [2]-[3] solutions are found for non-relativistic equation of motion for free mass point in the center-of-inertia reference frame (r. f.), which generally does not coincide with the center-of-mass r. f. moving through the complicated trajectory around the direction of movement of the center of inertia that can be interpreted as Zitterbewegung. In this connection a particle should be considered as non-inertial r. f. with some structure, whose spin is classical proper angular momentum. The main conclusion lies in the fact that the electric charge should not be seen as a physical quantity characterizing the electromagnetic interaction, but as a consequence of the presence of the particles spin. Then electromagnetic interaction can be interpreted as interaction of the spin with external field. So the question arises, how the motion of spinning particles in an electromagnetic field and their interaction with each other can be associated with the behavior of charged particles considered in the framework of the standard (classical or quantum) electrodynamics. In this paper, under the proposed equations of motion the effect of spin on the behavior of both free particles and particles in an external magnetic field is studied.

2. Equations of Motion in Arbitrary External Field

The equation of motion of a particle with spin in an external field can be written as the Second Newton's Law $d\mathbf{P}/dt = \mathbf{F} = \mathbf{F}_{\mathrm{g}} + \mathbf{F}^{\mathrm{ext}}$ ([1]-[3]), or

$$\frac{d}{dt}\, m_0\mathbf{V} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}] + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}] = -\frac{\partial U}{\partial \mathbf{R}} + \frac{d}{dt}\left(\frac{\partial U}{\partial \mathbf{V}}\right) + [\mathbf{C}^{\mathrm{ext}}\times\mathbf{V}] - \frac{d}{dt}[\mathbf{S}^{\mathrm{ext}}\times\dot{\mathbf{V}}], \quad (2.1)$$

where $\mathbf{F}_{\mathrm{g}} = -\varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}]$ is a gyroscopic force, arising due to non-inertial object, $\mathbf{F}^{\mathrm{ext}} = -\partial U/\partial\mathbf{R} + [\mathbf{C}^{\mathrm{ext}}\times\mathbf{V}]$ is external force, $U = U_0 + \varsigma\Omega_0^2([\mathbf{R}\times\mathbf{V}]\cdot\mathbf{s}) - ([\mathbf{R}\times\mathbf{V}]\cdot\mathbf{C}^{\mathrm{ext}})$ is potential energy of interaction, $\mathbf{\Omega}_0$ is Zitterbewegung frequency of free particle, $\mathbf{C}^{\mathrm{ext}}$ and $\mathbf{S}^{\mathrm{ext}}$ are some pseudo-vectors due to external field.

Dynamic momentum of the object $\mathbf{P} = \mathbf{P}_{\mathrm{kin}} + \mathbf{A}$ contains proper kinetic momentum $\mathbf{P}_{\mathrm{kin}} = m_0\mathbf{V} + \varsigma[\mathbf{s}\times\dot{\mathbf{V}}]$, and addition $\mathbf{A} = -\partial U/\partial\mathbf{V} + [\mathbf{S}^{\mathrm{ext}}\times\dot{\mathbf{V}}]$, arising due to interaction with external fields. This addition leads to a renormalization of the force **F**, so that the change of the proper

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kinetic momentum $\mathbf{P}_{\text{kin}}$ is determined by the right-hand side of the equation of motion (2.1), which can be compared with the Lorentz force, if we put

$$\mathbf{E} = -\frac{\partial U}{\partial \mathbf{R}} + \frac{d}{dt}\left(\frac{\partial U}{\partial \mathbf{V}} - [\mathbf{S}^{\text{ext}} \times \dot{\mathbf{V}}]\right) = -\frac{\partial U}{\partial \mathbf{R}} - \frac{d\mathbf{A}}{dt},\ \mathbf{B} = -\mathbf{C}^{\text{ext}}, \qquad (2.2)$$

which shows that addition $\mathbf{A}$ has a role of vector potential, introduced by F. E. Neumann. If we assume the definition (2.2), the equation of motion (2.1) takes the form

$$\frac{d}{dt}\ \ m_0\mathbf{V} + \varsigma[\mathbf{s} \times \dot{\mathbf{V}}]\ \ + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{V}] = \mathbf{E} + [\mathbf{V} \times \mathbf{B}], \qquad (2.3)$$

and the equation of motion of spin is given by

$$\dot{\mathbf{s}} = [\boldsymbol{\Omega} \times \mathbf{s}], \qquad (2.4)$$

where $\boldsymbol{\Omega}$ is angular velocity of spin precession.

Equation (2.1) leads to the balance of total energy

$$\mathcal{E} = \frac{m_0\mathbf{V}^2}{2} + \varsigma(\mathbf{V} \cdot [\mathbf{s} \times \dot{\mathbf{V}}]) + (\mathbf{V} \cdot [\mathbf{S}^{\text{ext}} \times \dot{\mathbf{V}}]) - (\mathbf{V} \cdot \frac{\partial U}{\partial \mathbf{V}}) + U. \qquad (2.5)$$

Let r. f. K', whose origin in K is specified by radius vector $\mathbf{R}_{(\mathrm{K}')}$, be the r. f. moving relative to absolute one with the velocity $\mathbf{V}_{(\mathrm{K}')}$. We write down the equation (2.1), written in absolute r. f. K, in moving r. f. K'. Then $\mathbf{R} = \mathbf{R}_{(\mathrm{K}')} + \mathbf{r}$, $\mathbf{V} = \mathbf{V}_{(\mathrm{K}')} + \mathbf{v}$, …, where $\mathbf{r}$ and $\mathbf{v}$ are radius vector and velocity of the object relative to the origin of K'. It can be shown that

$$\frac{\partial U}{\partial \mathbf{R}} = \frac{\partial U}{\partial \mathbf{r}},\ \frac{\partial U}{\partial \mathbf{V}} = \frac{\partial U}{\partial \mathbf{v}}, \ldots. \qquad (2.6)$$

Substitution of (2.6) into (2.1) leads it to the form

$$\begin{aligned} &\frac{d}{dt}\left(m_0\mathbf{v} + \varsigma[\mathbf{s} \times \dot{\mathbf{v}}]\right) + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{v}] + \frac{d}{dt}(m_0\mathbf{V}_{(\mathrm{K}')} + \varsigma[\mathbf{s} \times \dot{\mathbf{V}}_{(\mathrm{K}')}]) + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{V}_{(\mathrm{K}')}] = \\ &= -\frac{\partial U}{\partial \mathbf{r}} + \frac{d}{dt}\frac{\partial U}{\partial \mathbf{v}} + [\mathbf{v} \times \mathbf{B}] - \frac{d}{dt}[\mathbf{S}^{\text{ext}} \times \dot{\mathbf{v}}] + [\mathbf{V}_{(\mathrm{K}')} \times \mathbf{B}] - \frac{d}{dt}[\mathbf{S}^{\text{ext}} \times \dot{\mathbf{V}}_{(\mathrm{K}')}]. \end{aligned} \qquad (2.7)$$

There is no necessity to use vector potential to describe the motion in a constant electromagnetic field. In this case electric field is defined as

$$\mathbf{E} = -\frac{\partial U}{\partial \mathbf{R}} = -\frac{\partial U}{\partial \mathbf{r}}, \qquad (2.8)$$

whence it follows

$$U = -\int(\mathbf{E} \cdot d\mathbf{R}) + u(\mathbf{V}, \dot{\mathbf{V}}, \ddot{\mathbf{V}}, \ldots, \dot{\mathbf{V}}^{(N)}) = -\int(\mathbf{E} \cdot d\mathbf{r}) + u(\mathbf{v}, \dot{\mathbf{v}}, \ddot{\mathbf{v}}, \ldots, \dot{\mathbf{v}}^{(N)}). \qquad (2.9)$$

On the other hand, in the case of varying field its dependence on time may be considered in potential function together with conservation of the definition (2.8) instead of generally accepted definition $\mathbf{E} = -\nabla U - \partial\mathbf{A} / c\partial t$ in Maxwell electrodynamics. Then the equation (2.1) will become

$$\frac{d}{dt}\left(m_0\mathbf{V} - \frac{\partial u}{\partial \mathbf{V}} + \varsigma[\mathbf{s} \times \dot{\mathbf{V}}] + [\mathbf{S}^{\text{ext}} \times \dot{\mathbf{V}}]\right) = \mathbf{E} + [\mathbf{V} \times (\mathbf{B} + \varsigma\Omega_0^2\mathbf{s})]. \qquad (2.10)$$

In the moving r. f. K' relations $\mathbf{V}_{(\mathrm{K}')} = \mathbf{0}$, $\dot{\mathbf{V}}_{(\mathrm{K}')} = \mathbf{0}$ should be fulfilled. Therefore in view of (2.6), (2.8) equation (2.7) splits into two equations

$$\frac{d}{dt}\left(m_0\mathbf{v} - \frac{\partial u}{\partial \mathbf{v}} + \varsigma[\mathbf{s} \times \dot{\mathbf{v}}] + [\mathbf{S}^{\text{ext}} \times \dot{\mathbf{v}}]\right) + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{v}] = \mathbf{E} + [\mathbf{v} \times \mathbf{B}], \qquad (2.11)$$

$$\frac{d}{dt}\left(m_0\mathbf{V}_{(\mathrm{K}')} + \varsigma[\mathbf{s} \times \dot{\mathbf{V}}_{(\mathrm{K}')}] + [\mathbf{S}^{\text{ext}} \times \dot{\mathbf{V}}_{(\mathrm{K}')}]\right) + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{V}_{(\mathrm{K}')}] = [\mathbf{V}_{(\mathrm{K}')} \times \mathbf{B}]. \qquad (2.12)$$

We shall introduce in K' orthonormal basis Frenet-Serret

$$\mathbf{e}_\tau = \frac{\mathbf{v}}{v} = [\mathbf{e}_\mathrm{n} \times \mathbf{e}_\mathrm{b}],\ \mathbf{e}_\mathrm{n} = \frac{[\mathbf{v}\times[\dot{\mathbf{v}}\times\mathbf{v}]]}{|[\mathbf{v}\times[\dot{\mathbf{v}}\times\mathbf{v}]]|} = [\mathbf{e}_\mathrm{b}\times\mathbf{e}_\tau],\ \mathbf{e}_\mathrm{b} = \frac{[\mathbf{v}\times\dot{\mathbf{v}}]}{|[\mathbf{v}\times\dot{\mathbf{v}}]|} = [\mathbf{e}_\tau\times\mathbf{e}_\mathrm{n}]. \tag{2.13}$$

with equations of motion

$$\dot{\mathbf{e}}_\tau = [\mathbf{\Omega}_\mathrm{D}\times\mathbf{e}_\tau] = vK\mathbf{e}_\mathrm{n},\ \dot{\mathbf{e}}_\mathrm{n} = [\mathbf{\Omega}_\mathrm{D}\times\mathbf{e}_\mathrm{n}] = -vK\mathbf{e}_\tau + vT\mathbf{e}_\mathrm{b},\ \dot{\mathbf{e}}_\mathrm{b} = [\mathbf{\Omega}_\mathrm{D}\times\mathbf{e}_\mathrm{b}] = -vT\mathbf{e}_\mathrm{n}, \tag{2.14}$$

where $\mathbf{\Omega}_\mathrm{D} = v(T\mathbf{e}_\tau + K\mathbf{e}_\mathrm{b})$ is the Darboux vector defining an angular velocity of moving frame, $K = |[\mathbf{v}\times\dot{\mathbf{v}}]|/v^3$ is curvature, $T = (\mathbf{v}\cdot[\dot{\mathbf{v}}\times\ddot{\mathbf{v}}])/[\mathbf{v}\times\dot{\mathbf{v}}]^2$ is torsion of trajectory. If we decompose spin pseudo-vector in the basis (2.13) we find from (2.4) that $\mathbf{\Omega} = \mathbf{\Omega}_\mathrm{D}$, and spin components s_τ, s_n and s_b are constant.

We choose r. f. K' so that its velocity has to be orthogonal to the plane of vectors $\mathbf{v}$ and $\dot{\mathbf{v}}$, $\mathbf{V}_{(\mathrm{K}')} = V_{(\mathrm{K}')}\mathbf{e}_\mathrm{b}$. When it is inertial frame, then $\dot{\mathbf{V}}_{(\mathrm{K}')} = \mathbf{0}$, the torsion *T* vanishes, and the binormal direction is kept constant, which greatly simplifies the equation of motion.

For the sake of simplicity we assume that the source of the field *U* is at rest and function *u* does not depend on accelerations $\dot{\mathbf{v}}, \ddot{\mathbf{v}}, \ldots, \dot{\mathbf{v}}^{(N)}$ of the particle. Then

$$\frac{d}{dt}\frac{\partial u}{\partial \mathbf{v}} = \frac{d}{dt}\left(\frac{du}{vdv}\mathbf{v}\right) = \frac{du}{vdv}\dot{\mathbf{v}} + \frac{d}{dt}\left(\frac{du}{vdv}\right)\mathbf{v} = \frac{du}{vdv}\dot{\mathbf{v}} + (\mathbf{v}\cdot\dot{\mathbf{v}})\frac{d}{vdv}\left(\frac{du}{vdv}\right)\mathbf{v}. \tag{2.15}$$

Further, we seek a solution of equation (2.11) in the form

$$\mathbf{v}(t) = v(t)(\cos\Phi(t)\mathbf{e}_X + \sin\Phi(t)\mathbf{e}_Y), \tag{2.16}$$

where $\mathbf{e}_X$, $\mathbf{e}_Y$ are unit vectors of coordinate system in the plane which is orthogonal to the Z axis in absolute r. f., $\mathbf{e}_Z = \mathbf{e}_\mathrm{b}$. Then the spin of the particle is equal to

$$\mathbf{s} = (s_\tau\cos\Phi - s_\mathrm{n}\sin\Phi)\mathbf{e}_X + (s_\tau\sin\Phi + s_\mathrm{n}\cos\Phi)\mathbf{e}_Y + s_\mathrm{b}\mathbf{e}_Z. \tag{2.17}$$

In view of (2.15)-(2.17) equations of motion (2.11), (2.12) and spin precession (2.4) are

$$\begin{aligned}
&\frac{d}{dt}\left[\left(m_0 v - \frac{du}{dv} - (\varsigma s_\mathrm{b} + S_Z^\mathrm{ext})v\dot{\Phi}\right)\cos\Phi - (\varsigma s_\mathrm{b} + S_Z^\mathrm{ext})\dot{v}\sin\Phi\right]\mathbf{e}_X - \varsigma s_\mathrm{b} v\Omega_0^2\sin\Phi\mathbf{e}_X + \\
&+\frac{d}{dt}\left[\left(m_0 v - \frac{du}{dv} - (\varsigma s_\mathrm{b} + S_Z^\mathrm{ext})v\dot{\Phi}\right)\sin\Phi + (\varsigma s_\mathrm{b} + S_Z^\mathrm{ext})\dot{v}\cos\Phi\right]\mathbf{e}_Y + \varsigma s_\mathrm{b} v\Omega_0^2\cos\Phi\mathbf{e}_Y + \\
&+\frac{d}{dt}\left[(S_X^\mathrm{ext}\dot{v} + S_Y^\mathrm{ext}v\dot{\Phi})\sin\Phi + (S_X^\mathrm{ext}v\dot{\Phi} - S_Y^\mathrm{ext}\dot{v})\cos\Phi\right]\mathbf{e}_Z + \left[\varsigma s_\tau(v\ddot{\Phi} + \dot{v}\dot{\Phi}) - \varsigma s_\mathrm{n}(\ddot{v} + \Omega_0^2 v)\right]\mathbf{e}_Z = \\
&= (E_X + vB_Z\sin\Phi)\mathbf{e}_X + (E_Y - vB_Z\cos\Phi)\mathbf{e}_Y + (E_Z + vB_Y\cos\Phi - vB_X\sin\Phi)\mathbf{e}_Z,
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\dot{V}_{(\mathrm{K}')}(S_Y^\mathrm{ext} + \varsigma s_\tau\sin\Phi + \varsigma s_\mathrm{n}\cos\Phi)\right]\mathbf{e}_X + \left[\varsigma s_\tau\sin\Phi + \varsigma s_\mathrm{n}\cos\Phi\right]\Omega_0^2 V_{(\mathrm{K}')}\mathbf{e}_X - \\
&-\frac{d}{dt}\left[\dot{V}_{(\mathrm{K}')}(S_X^\mathrm{ext} + \varsigma s_\tau\cos\Phi - \varsigma s_\mathrm{n}\sin\Phi)\right]\mathbf{e}_Y + \left[\varsigma s_\mathrm{n}\sin\Phi - \varsigma s_\tau\cos\Phi\right]\Omega_0^2 V_{(\mathrm{K}')}\mathbf{e}_Y + \\
&+m_0\dot{V}_{(\mathrm{K}')}\mathbf{e}_\mathrm{Z} = -V_{(\mathrm{K}')}B_Y\mathbf{e}_X + V_{(\mathrm{K}')}B_X\mathbf{e}_Y,
\end{aligned} \tag{2.19}$$

$$\dot{\mathbf{s}} = [\mathbf{\Omega}_\mathrm{D}\times\mathbf{s}] = \dot{\Phi}\left[-(s_\tau\sin\Phi + s_\mathrm{n}\cos\Phi)\mathbf{e}_X + (s_\tau\cos\Phi - s_\mathrm{n}\sin\Phi)\mathbf{e}_Y\right], \tag{2.20}$$

where $\mathbf{\Omega}_\mathrm{D} = \Omega_\mathrm{b}\mathbf{e}_\mathrm{b} = \dot{\Phi}\mathbf{e}_\mathrm{Z}$.

It follows from (2.19), that $\dot{V}_{(\mathrm{K}')} = 0$ at $m_0 \neq 0$, so that (2.19) is equivalent to equations

$$\varsigma s_\mathrm{n}\Omega_0^2 = B_X\sin\Phi - B_Y\cos\Phi,\ \varsigma s_\tau\Omega_0^2 = -B_X\cos\Phi - B_Y\sin\Phi, \tag{2.21}$$

whence it follows $\varsigma^2(s_\tau^2 + s_\mathrm{n}^2)\Omega_0^4 = B_X^2 + B_Y^2$, as well as

$$\sin\Phi = \frac{\varsigma\Omega_0^2(s_\mathrm{n}B_X - s_\tau B_Y)}{B_X^2 + B_Y^2},\ \cos\Phi = -\frac{\varsigma\Omega_0^2(s_\tau B_X + s_\mathrm{n}B_Y)}{B_X^2 + B_Y^2}. \tag{2.22}$$

In the case $m_0 = 0$ the moving r. f. K' may be non-inertial one. Then (2.19) reduces to the set of two equations

$$\begin{aligned}&\frac{d}{dt}\left[\dot V_{(\mathrm{K}')} S_Y^{\mathrm{ext}}\right] + \left[\varsigma s_\tau(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) - \varsigma s_\mathrm{n} \dot V_{(\mathrm{K}')}\dot\Phi\right]\sin\Phi + \\ &+\left[\varsigma s_\mathrm{n}(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) + \varsigma s_\tau \dot V_{(\mathrm{K}')}\dot\Phi\right]\cos\Phi + B_Y V_{(\mathrm{K}')} = 0\,,\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\frac{d}{dt}\left[\dot V_{(\mathrm{K}')} S_X^{\mathrm{ext}}\right] + \left[\varsigma s_\tau(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) - \varsigma s_\mathrm{n} \dot V_{(\mathrm{K}')}\dot\Phi\right]\cos\Phi - \\ &-\left[\varsigma s_\mathrm{n}(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) + \varsigma s_\tau \dot V_{(\mathrm{K}')}\dot\Phi\right]\sin\Phi + B_X V_{(\mathrm{K}')} = 0\,.\end{aligned} \tag{2.24}$$

In the r. f. K' the energy (2.5) is conserved and is the self-energy of particle

$$\begin{aligned}\mathcal{E} = \mathcal{E}_0 &= \frac{m_0\mathbf{v}^2}{2} - (\mathbf{v}\cdot\frac{\partial U}{\partial \mathbf{v}}) + U + (\mathbf{v}\cdot[(\varsigma\mathbf{s} + \mathbf{S}^{\mathrm{ext}})\times\dot{\mathbf{v}}]) = \\ &= \frac{m_0 v^2}{2} - v\frac{du}{dv} + u - v^2\dot\Phi(\varsigma s_\mathrm{b} + S_Z^{\mathrm{ext}})\;.\end{aligned} \tag{2.25}$$

which gives the relation

$$(\varsigma s_\mathrm{b} + S_Z^{\mathrm{ext}})(v\ddot\Phi + 2\dot v\dot\Phi) + v\dot S_Z^{\mathrm{ext}}\dot\Phi = \frac{d}{dt}\left(m_0 v - \frac{du}{dv}\right). \tag{2.26}$$

In view of (2.21), (2.26) and $\dot V_{(\mathrm{K}')} = 0$ for massive particles (2.18) reduces to the set of equations

$$E_X\cos\Phi + E_Y\sin\Phi = 0\,, \tag{2.27}$$

$$\left(m_0 v - \frac{du}{dv}\right)\dot\Phi + (\varsigma s_\mathrm{b} + S_Z^{\mathrm{ext}})(\ddot v - v\dot\Phi^2) + \dot S_Z^{\mathrm{ext}}\dot v + (\varsigma s_\mathrm{b}\Omega_0^2 + B_Z)v = -E_X\sin\Phi + E_Y\cos\Phi\,, \tag{2.28}$$

$$\frac{d}{dt}\left[\varsigma s_\tau v\dot\Phi - \varsigma s_\mathrm{n}\dot v + (S_X^{\mathrm{ext}}\dot v + S_Y^{\mathrm{ext}} v\dot\Phi)\sin\Phi + (S_X^{\mathrm{ext}} v\dot\Phi - S_Y^{\mathrm{ext}}\dot v)\cos\Phi\right] = E_Z\,. \tag{2.29}$$

For massless particles one should set $m_0 = 0$ in (2.28) and instead of (2.21) one should use (2.23), (2.24). We emphasize that obtained system of equations holds for an arbitrary external field.

3. The motion of free particle

In [2]-[3] there considered equations of motion (2.1) and (2.4) for the case $U_0 = 0$, $\mathbf{S}^{\mathrm{ext}} = \mathbf{0}$, $\mathbf{C}^{\mathrm{ext}} = \mathbf{0}$, and all solutions of them are found. By definition an object should be free if the latter two conditions are fulfilled and the force **F** vanishes at all time. Then $U_0 = 0$ is special case of the condition $\partial U_0 / \partial\mathbf{R} = \mathbf{0}$, whence it follows that U_0 may be function of the velocity and accelerations. On the other hand, if free object will be defined by the condition $\partial U / \partial\mathbf{R} = \mathbf{0}$, whence $U = u(t, \mathbf{V}, \dot{\mathbf{V}}, \ddot{\mathbf{V}}, ..., \dot{\mathbf{V}}^{(N)})$, then in the l. h. s. of (2.1) a term remains that make sense of gyroscopic force. If U does not depend explicitly on time, and velocity, and accelerations, then it is constant, and it may be set zero. It is easy to show in this case that joint solution of (2.1) and (2.4) leads to the fact that spin of the object is always collinear to the velocity **V** which remains constant, $\dot{\mathbf{V}} = \mathbf{0}$. Hence, the force **F** vanishes, $U = U_0$, so that the object in question moves inertially in total concordance with standard Newton's mechanics.

If $U = u \neq 0$, then one should be based on (2.27)-(2.29) and (2.21) for massive particles or (2.23), (2.24) for massless particles at $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ and $\mathbf{S}^{\mathrm{ext}} = \mathbf{0}$. It follows from (2.21) that $s_\tau = s_\mathrm{n} = 0$ and (2.23)-(2.24) reduce to the set

$$s_\tau(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) = s_\mathrm{n}\dot V_{(\mathrm{K}')}\dot\Phi\,,\;\; s_\mathrm{n}(\ddot V_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')}) = -s_\tau\dot V_{(\mathrm{K}')}\dot\Phi\,, \tag{3.1}$$

which are also valid when $s_\tau \neq 0$, $s_\mathrm{n} \neq 0$. Equations (2.27)-(2.29) reduce to

$$\left(m_0 v - \frac{du}{dv}\right)\dot\Phi = -\varsigma s_\mathrm{b}[\ddot v + (\Omega_0^2 - \dot\Phi^2)v]\,, \tag{3.2}$$

$$s_{\mathrm{n}}\ddot{v} = s_{\tau}(v\ddot{\Phi} + \dot{v}\dot{\Phi}),\; s_{\tau}v\dot{\Phi} - s_{\mathrm{n}}\dot{v} = \mathrm{const}, \tag{3.3}$$

where (3.3) takes place for massless particle and becomes identity for massive one.

Substitution of (2.26) at $S_Z^{\mathrm{ext}} = 0$ into (3.2) leads to

$$\frac{d}{dt}\left(\frac{\ddot{v} + \Omega_0^2 v}{\dot{\Phi}}\right) + \dot{v}\dot{\Phi} = 0, \tag{3.4}$$

whence it follows the first integral

$$\frac{(\ddot{v} + \Omega_0^2 v)^2}{\dot{\Phi}^2} + \dot{v}^2 + \Omega_0^2 v^2 = D^2 = \mathrm{const}, \tag{3.5}$$

and

$$\dot{\Phi} = \frac{\ddot{v} + \Omega_0^2 v}{\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2}}. \tag{3.6}$$

Now substituting (3.6) into (3.2), we obtain the equation

$$\frac{m_0 v - \dfrac{du}{dv}}{\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2}}(\ddot{v} + \Omega_0^2 v) = -\varsigma s_{\mathrm{b}}(\ddot{v} + \Omega_0^2 v)\left[1 - \frac{(\ddot{v} + \Omega_0^2 v)v}{D^2 - \dot{v}^2 - \Omega_0^2 v^2}\right], \tag{3.7}$$

which is valid in two cases: I. $\ddot{v} + \Omega_0^2 v = 0$, $\dot{\Phi} = 0$, and II. $\ddot{v} + \Omega_0^2 v \neq 0$, $\dot{\Phi} \neq 0$. Let us consider them in details.

I.1. $\ddot{v} + \Omega_0^2 v = 0$, $\dot{\Phi} = 0$, $s_\tau = 0$, $s_{\mathrm{n}} = 0$, $m_0 \geq 0$. In this case we have $v = v_0 \cos(\Omega_0 t + \varphi_0)$, whence the equation of the trajectory in the absolute r. f. is given by

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_0}[\sin(\Omega_0 t + \varphi_0) - \sin\varphi_0](\cos\Phi\mathbf{e}_X + \sin\Phi\mathbf{e}_Y) + V_{(\mathrm{K}')}t\mathbf{e}_Z, \tag{3.8}$$

i. e. the particle has a longitudinal polarization and oscillates in the plane (X,Y) around the center of inertia moving along the Z axis with velocity $\mathbf{V}_{\mathrm{C}} = \mathbf{V}_{(\mathrm{K}')}$. At $\Omega_0 = 0$ oscillations are absent, and particle moves along the Z axis in accordance with the law of inertia of Galileo-Newton.

I.2. $\ddot{v} + \Omega_0^2 v = 0$, $\dot{\Phi} = 0$, $m_0 = 0$. Equations (3.1) and (3.3) give $s_{\mathrm{n}} = 0$, $s_\tau \neq 0$, $\ddot{V}_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')} = 0$, $V_{(\mathrm{K}')} = V_{(\mathrm{K}')0}\cos(\Omega_0 t + \varphi_1)$. Thus, we obtain a trajectory

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_0}\sin(\Omega_0 t + \varphi_0)(\cos\Phi\mathbf{e}_X + \sin\Phi\mathbf{e}_Y) + \frac{V_{(\mathrm{K}')0}}{\Omega_0}\sin(\Omega_0 t + \varphi_1)\mathbf{e}_Z, \tag{3.9}$$

which in general is an ellipse, transforming into either circle at $\Delta\varphi = \varphi_1 - \varphi_0 = (2m+1)\pi/2$ or line segment at $\Delta\varphi = m\pi$, where *m* is integer.

II. The second case of equation (3.7) for function $u(v)$ is specified by conditions $\ddot{v} + \Omega_0^2 v \neq 0$, $\dot{\Phi} \neq 0$. We have

$$\frac{du}{dv} = \left(m_0 - \frac{\varsigma s_{\mathrm{b}}(\ddot{v} + \Omega_0^2 v)}{\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2}}\right)v + \varsigma s_{\mathrm{b}}\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2}\,. \tag{3.10}$$

whence it follows

$$u(v) = \frac{m_0 v^2}{2} + \varsigma s_{\mathrm{b}} v\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2} + \mathcal{E}_0. \tag{3.11}$$

Equation of motion has the infinite set of solutions, one of which corresponds to $\Phi = \Omega_{\mathrm{D}} t$, where $\Omega_{\mathrm{D}} = \pm\sqrt{\mathbf{\Omega}_{\mathrm{D}}^2} = \mathrm{const}$. In this case equation (3.6) admits first integral of motion $\sqrt{D^2 - \dot{v}^2 - \Omega_0^2 v^2} + \Omega_{\mathrm{D}} v = F = \mathrm{const}$, which reduces to equation for velocity

$$\dot{v} = \pm\sqrt{D^2 - F^2 + 2F\Omega_{\mathrm{D}}v - (\Omega_{\mathrm{D}}^2 + \Omega_0^2)v^2}\,. \tag{3.12}$$

Substitution of (3.12) into (3.6) leads to equation

$$\ddot{v} + (\Omega_0^2 + \Omega_{\mathrm{D}}^2)v = F\Omega_{\mathrm{D}}, \tag{3.13}$$

which has general solution

$$v(t) = \frac{F\Omega_{\mathrm{D}}}{\chi^2} + v_0\cos(\chi t + \varphi_0),\ \chi = \sqrt{\Omega_{\mathrm{D}}^2 + \Omega_0^2}\,. \tag{3.14}$$

It follows from (3.12) and (3.14), that velocity may vary in the limits $v_{\min} \le v \le v_{\max}$, where

$$v_{\min} = \frac{F\Omega_{\mathrm{D}}}{\chi^2} - v_0 \ge 0,\ v_{\max} = \frac{F\Omega_{\mathrm{D}}}{\chi^2} + v_0,\ v_0 = \frac{\sqrt{D^2\chi^2 - F^2\Omega_0^2}}{\chi^2}\,. \tag{3.15}$$

For $u(v)$ we obtain the expression

$$u(v) = \frac{m_0 v^2}{2} + \varsigma s_{\mathrm{b}} v\,|\,F - \Omega_{\mathrm{D}}v\,| + \mathcal{E}_0\,. \tag{3.16}$$

The law of motion for both massive and massless particles with longitudinal spin polarization is

$$\begin{aligned}\mathbf{R}(t) = {}& \mathbf{R}(0) + \frac{F}{\chi^2}(\sin\Omega_{\mathrm{D}}t\mathbf{e}_X - \cos\Omega_{\mathrm{D}}t\mathbf{e}_Y) + V_{(\mathrm{K}')}t\mathbf{e}_Z + \\ &+ \frac{v_0}{2\Omega_0^2}\big[(\chi - \Omega_{\mathrm{D}})\sin[(\chi + \Omega_{\mathrm{D}})t + \varphi_0] + (\chi + \Omega_{\mathrm{D}})\sin[(\chi - \Omega_{\mathrm{D}})t + \varphi_0]\big]\mathbf{e}_X - \\ &- \frac{v_0}{2\Omega_0^2}\big[(\chi - \Omega_{\mathrm{D}})\cos[(\chi + \Omega_{\mathrm{D}})t + \varphi_0] - (\chi + \Omega_{\mathrm{D}})\cos[(\chi - \Omega_{\mathrm{D}})t + \varphi_0]\big]\mathbf{e}_Y\,.\end{aligned} \tag{3.17}$$

Note that there is another possibility that the equation of motion (3.3)-(3.4) are satisfied.

III. $\dot{v} = 0$, $v = v_0 = \mathrm{const}$. Then it follows from (2.25), (2.26)

$$\frac{d}{dt}\left(\frac{du}{dv} + \varsigma s_{\mathrm{b}} v\dot{\Phi}\right) = 0,\ \frac{du}{dv} + \varsigma s_{\mathrm{b}} v\dot{\Phi} = C,\ u(v) = \mathcal{E}_0 - \frac{m_0 v^2}{2} + Cv\,. \tag{3.18}$$

Equation (3.2) reduces to

$$m_0 v + \varsigma s_{\mathrm{b}}\frac{\Omega_0^2 v}{\dot{\Phi}} = \frac{du}{dv} + \varsigma s_{\mathrm{b}} v\dot{\Phi} = C\,, \tag{3.19}$$

whence $\dot{\Phi} = \Omega_{\mathrm{D}} = \varsigma s_{\mathrm{b}}\Omega_0^2 v_0 (C - m_0 v_0)^{-1} = \mathrm{const}$. As a result we obtain following solutions.

III.1. $v = v_0$, $\Phi = \Omega_{\mathrm{D}}t$, $s_\tau = 0$, $s_{\mathrm{n}} = 0$, $m_0 \ge 0$. Trajectory looks like helix along the Z axis

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_{\mathrm{D}}}[\sin\Omega_{\mathrm{D}}t\mathbf{e}_X + (1 - \cos\Omega_{\mathrm{D}}t)\mathbf{e}_Y] + V_{(\mathrm{K}')}t\mathbf{e}_Z\,. \tag{3.20}$$

De facto this solution coincides with solutions I.1, I.2 from [2], [3].

III.2. $v = v_0$, $\Phi = \Omega_{\mathrm{D}}t$, $m_0 = 0$, $s_\tau = s_{\mathrm{n}} = 0$. From (3.1) we find $\ddot{V}_{(\mathrm{K}')} + \Omega_0^2 V_{(\mathrm{K}')} = 0$, $V_{(\mathrm{K}')} = V_{(\mathrm{K}')0}\cos(\Omega_0 t + \varphi_1)$. Trajectory is represented by radius vector

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{v_0}{\Omega_{\mathrm{D}}}[\sin\Omega_{\mathrm{D}}t\mathbf{e}_X + (1 - \cos\Omega_{\mathrm{D}}t)\mathbf{e}_Y] + \frac{V_{(\mathrm{K}')0}}{\Omega_0}[\sin(\Omega_0 t + \varphi_1) - \sin\varphi_1]\mathbf{e}_Z\,, \tag{3.21}$$

i. e. the particle performs complex movement around the stationary center of balance. At $\Omega_{\mathrm{D}} = \Omega_0$ the trajectory, which looks like three-dimensional Lissajous figure, becomes an ellipse ($V_{(\mathrm{K}')0} \ne v_0$) or circle (at $V_{(\mathrm{K}')0} = v_0$ or $V_{(\mathrm{K}')0} = 0$).

We summarize the results of this section. We suppose that in a homogeneous isotropic space behavior of a free particle with spin is described by equations (2.3) (with $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$) or (2.10) (with $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, $\mathbf{S}^{\mathrm{ext}} = \mathbf{0}$) and (2.4), which lead to the following types of motion.

1. Oscillatory motion of the particle with longitudinal polarization in the plane, which is orthogonal to direction of movement of the center of inertia, described by radius vector (3.8). At $\Omega_0 = 0$ oscillations are absent and particle moves uniformly in a straight line. The self-energy $\mathcal{E}_0$ of the particle may have any constant value, $u(v) = m_0 v^2 / 2 + Cv + \mathcal{E}_0$.

2. Oscillatory or cyclic motion of massless particles around fixed center of balance, described by the law (3.9). Spin $\mathbf{s} = s_\tau(\cos\Phi \mathbf{e}_X + \sin\Phi \mathbf{e}_Y) + s_\mathrm{b}\mathbf{e}_Z$ has a constant direction. The self-energy $\mathcal{E}_0$ of the particle may have any constant value, $u(v) = Cv + \mathcal{E}_0$.

3. A motion of particle with longitudinal spin polarization along complicated trajectory (3.17) clockwise ($\Omega_\mathrm{D} > 0$) or counter-clockwise ($\Omega_\mathrm{D} < 0$). The velocity of particle relative to the center of inertia varies in the limits $v_\mathrm{min} < v < v_\mathrm{max}$.

4. A motion of massive or massless particle with longitudinal spin polarization along helix (3.20) clockwise ($\Omega_\mathrm{D} > 0$, $s_\mathrm{b} < 0$) or counter-clockwise ($\Omega_\mathrm{D} < 0$, $s_\mathrm{b} > 0$) with constant velocity v_0 relative to the center of inertia, which moves in absolute r. f. with constant velocity $\mathbf{V}_\mathrm{C} = V_{(\mathrm{K}')}\mathbf{e}_Z$.

5. A motion of massless particle along complicated trajectory (3.21) around fixed center of balance. Spin $\mathbf{s} = s_\mathrm{b}\mathbf{e}_Z$ is precessing around Z axis with angular velocity Ω_D.

4. Motion in a constant homogeneous magnetic field

Now we consider the behavior of a spinning particle in a constant homogeneous magnetic field. The corresponding equations of motion are given by (2.27)-(2.29) and (2.21), $\dot{V}_{(\mathrm{K}')} = 0$ for massive particles or (2.23)-(2.24), $\dot{V}_{(\mathrm{K}')} \neq 0$ for massless particles at $\mathbf{E} = \mathbf{0}$, $\mathbf{S}^\mathrm{ext} = \mathbf{0}$. From the set of equations above we have several cases determined by the relation between the mass, spin and magnetic field.

I. $m_0 \neq 0$. Equation (2.28) is

$$\left(m_0 v - \frac{du}{dv}\right)\dot{\Phi} + \varsigma s_\mathrm{b}[\ddot{v} + (\Omega_0^2 + B_Z / \varsigma s_\mathrm{b} - \dot{\Phi}^2)v] = 0 . \tag{4.1}$$

From (2.21) for constant field $\mathbf{B}$ we have two possibilities: I. $\dot{\Phi} = 0$, and II. $\dot{\Phi} \neq 0$, $B_X = B_Y = 0$, which lead to the following solutions.

I.1. $m_0 \neq 0$, $\dot{\Phi} = 0$, $s_\mathrm{n} = 0$, $s_\tau \neq 0$, $\ddot{v} \neq 0$, $\Omega^2 = \Omega_0^2 + B_Z / \varsigma s_\mathrm{b} > 0$. Spin of a particle according to (2.17) and (2.22) is given by

$$\mathbf{s} = -\frac{1}{\varsigma\Omega_0^2}(B_X\mathbf{e}_X + B_Y\mathbf{e}_Y) + s_\mathrm{b}\mathbf{e}_Z . \tag{4.2}$$

Then (4.1) has a solution $v(t) = v_0\cos(\Omega t + \varphi_0)$, and trajectory is represented by

$$\mathbf{R}(t) = \mathbf{R}(0) - \frac{v_0}{\varsigma s_\tau \Omega_0^2 \Omega}[\sin(\Omega t + \varphi_0) - \sin\varphi_0](B_X\mathbf{e}_X + B_Y\mathbf{e}_Y) + V_{(\mathrm{K}')}t\mathbf{e}_Z , \tag{4.3}$$

i. e. particle with spin (4.2) oscillates in the plane (XY), and its center of inertia moves uniformly along Z axis.

I.2. $m_0 \neq 0$, $\dot{\Phi} = 0$, $s_\mathrm{n} = 0$, $s_\tau \neq 0$, $\ddot{v} \neq 0$, $\tilde{\Omega}^2 = -\Omega_0^2 - B_Z / \varsigma s_\mathrm{b} > 0$. Spin of the particle is defined by (4.2). Equation (4.1) has a solution $v(t) = v_0\cosh(\tilde{\Omega}t + \varphi_0)$, whence

$$\mathbf{R}(t) = \mathbf{R}(0) - \frac{v_0}{\varsigma s_\tau \Omega_0^2 \tilde{\Omega}}[\sinh(\tilde{\Omega}t + \varphi_0) - \sinh\varphi_0](B_X\mathbf{e}_X + B_Y\mathbf{e}_Y) + V_{(\mathrm{K}')}t\mathbf{e}_Z . \tag{4.4}$$

The path (4.4) is almost a straight line and deviates from it near $t = 0$.

I.3. $m_0 \neq 0$, $\dot{\Phi} = 0$, $s_\mathrm{n} \neq 0$, s_τ may have any value. (4.1) and (3.3) lead to $\ddot{v} = 0$,

$v(t) = v_0 + wt$, $w = \mathrm{const}$, $\Omega_0^2 = -B_Z / \varsigma s_{\mathrm{b}}$. Spin of the particle is defined by (4.2). The trajectory is parabolic,

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{B_Z(v_0 t + wt^2 / 2)}{s_{\mathrm{b}}(B_X^2 + B_Y^2)}[(s_\tau B_X + s_{\mathrm{n}} B_Y)\mathbf{e}_X - (s_{\mathrm{n}} B_X - s_\tau B_Y)\mathbf{e}_Y] + V_{(\mathrm{K}')} t \mathbf{e}_Z, \tag{4.5}$$

or straight line at $w = 0$.

In cases I.1-I.3 we find from (2.25)

$$u(v) = \frac{m_0 v^2}{2} + C_1 v + \mathcal{E}_0. \tag{4.6}$$

II. $m_0 \neq 0$, $\dot{\Phi} \neq 0$. In this case the solution of (4.1) is consistent with (2.21), only when $s_\tau = s_{\mathrm{n}} = 0$, $B_X = B_Y = 0$, i. e. magnetic field **B** and spin **s** are collinear with the Z axis. Then (2.25) gives

$$\varsigma s_{\mathrm{b}} \dot{\Phi} = \frac{m_0}{2} - \frac{d}{dv}\frac{u}{v} - \frac{\mathcal{E}_0}{v^2}. \tag{4.7}$$

Equations (4.1) and (4.7) are two equations for three unknown functions $\Phi(t)$, $v(t)$ and potential function $u(v)$. Here we consider a particular solution corresponding to constant cyclotron frequency $\dot{\Phi} = \Omega_B = \mathrm{const}$. Then from (4.7) we find the potential function

$$u(v) = \frac{(m_0 - 2\varsigma s_{\mathrm{b}} \Omega_B) v^2}{2} + C_1 v + \mathcal{E}_0, \tag{4.8}$$

and (4.1) reduces to

$$\ddot{v} + (\Omega_0^2 + \Omega_B^2 + B_Z / \varsigma s_{\mathrm{b}}) v = C_1 \Omega_B, \tag{4.9}$$

which gives three types of solutions.

II.1. $m_0 \neq 0$, $\dot{\Phi} = \Omega_B$, $s_\tau = s_{\mathrm{n}} = 0$, $B_X = B_Y = 0$, $\Omega^2 = \Omega_0^2 + \Omega_B^2 + B_Z / \varsigma s_{\mathrm{b}} > 0$. Equation (4.9) has the following solution $v(t) = v_1 + v_0 \sin(\Omega t + \varphi_0)$. The relevant trajectory is described by radius vector

$$\begin{aligned}\mathbf{R}(t) = \mathbf{R}(0) + \left[\rho(t)\sin\Omega_B t - \rho_B \cos(\Omega t + \varphi_0)\cos\Omega_B t + \rho_B \cos\varphi_0\right]\mathbf{e}_X - \\ -\left[\rho(t)\cos\Omega_B t + \rho_B \cos(\Omega t + \varphi_0)\sin\Omega_B t - \rho(0)\right]\mathbf{e}_Y + V_{(\mathrm{K}')} t\mathbf{e}_Z,\end{aligned} \tag{4.10}$$

where $\Omega = \Omega_B \sqrt{1 + \eta_B}$, $v_1 = C_1 \Omega_B / \Omega^2$, $\eta_B = (\Omega_0^2 + B_Z / \varsigma s_{\mathrm{b}}) / \Omega_B^2$,

$$\rho_B = \frac{\varsigma s_{\mathrm{b}} \Omega v_0}{\varsigma s_{\mathrm{b}} \Omega_0^2 + B_Z} = \frac{v_0 \sqrt{1 + \eta_B}}{\Omega_B \eta_B}, \ \rho(t) = \frac{\rho_B}{\sqrt{1 + \eta_B}}\left[\frac{v_1 \eta_B}{v_0} - \sin(\Omega t + \varphi_0)\right]. \tag{4.11}$$

II.2. If the magnetic field is such that the relation $\Omega_0^2 = -B_Z / \varsigma s_{\mathrm{b}}$ is valid, i. e. $\eta_B = 0$, then $\Omega = \Omega_B$, $v_1 = C_1 / \Omega_B$. Trajectory is presented by radius vector

$$\begin{aligned}\mathbf{R}(t) = \mathbf{R}(0) + \rho_0\left[k\sin\Omega_B t + 2\Omega_B t\sin\varphi_0 + \cos\varphi_0 - \cos(2\Omega_B t + \varphi_0)\right]\mathbf{e}_X + \\ +\rho_0\left[k(1 - \cos\Omega_B t) + 2\Omega_B t\cos\varphi_0 + \sin\varphi_0 - \sin(2\Omega_B t + \varphi_0)\right]\mathbf{e}_Y + V_{(\mathrm{K}')} t\mathbf{e}_Z,\end{aligned} \tag{4.12}$$

where $\rho_0 = v_0 / 4\Omega_B$, $k = 4v_1 / v_0$, and it is complicated curve in the center-of-inertia r. f., which moves with velocity $\mathbf{V}_{\mathrm{C}} = (v_0 / 2)(\sin\varphi_0 \mathbf{e}_X + \cos\varphi_0 \mathbf{e}_Y) + V_{(\mathrm{K}')}\mathbf{e}_Z$.

II.3. $m_0 \neq 0$, $\dot{\Phi} = \Omega_B$, $s_\tau = s_{\mathrm{n}} = 0$, $B_X = B_Y = 0$, $\tilde{\Omega}^2 = -\Omega_0^2 - \Omega_B^2 - B_Z / \varsigma s_{\mathrm{b}} > 0$, i. e. $\eta_B < -1$. It follows from (4.9) that $v(t) = v_1 + v_0 \sinh(\tilde{\Omega} t + \varphi_0)$,

$$\begin{gathered}\mathbf{R}(t)=\mathbf{R}(0)+\left[\tilde{\rho}(t)\sin\Omega_B t-\tilde{\rho}_B\cosh(\tilde{\Omega}t+\varphi_0)\cos\Omega_B t+\tilde{\rho}_B\cosh\varphi_0\right]\mathbf{e}_X-\\ -\left[\tilde{\rho}(t)\cos\Omega_B t+\tilde{\rho}_B\cosh(\tilde{\Omega}t+\varphi_0)\sinh\Omega_B t-\tilde{\rho}(0)\right]\mathbf{e}_Y+V_{(\mathrm{K}')}t\mathbf{e}_Z\ ,\end{gathered}\tag{4.13}$$

where $\tilde{\Omega}=\Omega_B\sqrt{-1-\eta_B}$,

$$\tilde{\rho}_B=\frac{\varsigma s_{\mathrm{b}}\tilde{\Omega}v_0}{\varsigma s_{\mathrm{b}}\Omega_0^2+B_Z}=\frac{v_0\sqrt{-1-\eta_B}}{\Omega_B\eta_B},\ \tilde{\rho}(t)=\frac{\tilde{\rho}_B}{\sqrt{-1-\eta_B}}\left[\frac{v_1\eta_B}{v_0}-\mathrm{sh}(\tilde{\Omega}t+\varphi_0)\right].\tag{4.14}$$

Trajectory (4.13) in the center-of-inertia r. f. is twisting ($t<0$), and then untwisting ($t>0$) helix.

II.4. $m_0\neq 0$, $\dot{\Phi}=\Omega_B$, $s_\tau=s_\mathrm{n}=0$, $B_X=B_Y=0$, $\Omega_0^2+\Omega_B^2+B_Z/\varsigma s_\mathrm{b}=0$, i. e. cyclotron frequency is equal to $\Omega_B=\sqrt{-\Omega_0^2-B_Z/\varsigma s_\mathrm{b}}$. Here (4.9) gives $v(t)=v_0+wt+C_1\Omega_B t^2/2$,

$$\begin{gathered}\mathbf{R}(t)=\mathbf{R}(0)+\frac{1}{\Omega_B^2}\left[(v(t)\Omega_B-C_1)\sin\Omega_B t+\frac{dv(t)}{dt}\cos\Omega_B t-\left.\frac{dv(t)}{dt}\right|_{t=0}\right]\mathbf{e}_X+\\ +\frac{1}{\Omega_B^2}\left[(v(t)\Omega_B-C_1)(1-\cos\Omega_B t)+\frac{dv(t)}{dt}\sin\Omega_B t\right]\mathbf{e}_Y+V_{(\mathrm{K}')}t\mathbf{e}_Z\ .\end{gathered}\tag{4.15}$$

The trajectories of this type at $w=0$, $C_1=0$ correspond to the trajectories of classical electrodynamics, where, as it is known, a charged particle, that is flying in uniform magnetic field, moves in a spiral or circle, when its velocity is perpendicular to field, and spin of the particle is not taken into account in no way. As it follows from the solutions obtained above, the spin of a particle that has fallen into magnetic field, has always arranged parallel or antiparallel to the field. This was conclusively proven by experiment of Stern and Gerlach. It is obvious that condition $B_Z<-\varsigma s_\mathrm{b}\Omega_0^2$ should be satisfied. Assuming for the electron $\varsigma=-c^{-2}$, $s_\mathrm{b}=s=\hbar/2$, $\Omega_0=m_e c^2/\hbar\approx 7{,}77\cdot 10^{20}$ Hz ([2], Eq. (4.50), or [3], Eq. (89)), we find limit value of magnetic field $B_{\max}=m_e^2c^2/2\hbar\approx 5{,}6\cdot 10^{-11}$ kg/s, that corresponds to $B_{\max}=m_e^2c^2/e\hbar\approx 3{,}5\cdot 10^{8}$ T in SI. Large values of magnetic field are occured in magnetars, that are neutron stars with strong magnetic field (up to 10^{11} T), wherein condition $B_Z<-\varsigma s_\mathrm{b}\Omega_0^2$ is not valid. For such fields, apparently can be realized cases II.1, II.2.

Finally, we get the usual solution, assuming $s_\mathrm{b}=0$. Then (4.1) looks like

$$\left(m_0 v-\frac{du}{dv}\right)\dot{\Phi}=-vB_Z\ ,\tag{4.16}$$

whence (at $\dot{\Phi}=\Omega_B$)

$$\frac{du}{dv}=(m_0+B_Z/\Omega_B)v\ ,\ u(v)=\frac{1}{2}(m_0+B_Z/\Omega_B)v^2\ .\tag{4.17}$$

$u(v)=0$ corresponds to standard dependence of potential function from relative distance. As it follows from (4.17) this is possible, when $\Omega_B=-B_Z/m_0$, which coincides with standard definition of cyclotron frequency. For constant field (4.16), (4.17) give $v=v_0=\mathrm{const}$. Then trajectory is described by (4.15), i. e.

$$\mathbf{R}(t)=\mathbf{R}(0)+\frac{v_0}{\Omega_B}\sin\Omega_B t\mathbf{e}_X+\frac{v_0}{\Omega_B}(1-\cos\Omega_B t)\mathbf{e}_Y+V_{(\mathrm{K}')}t\mathbf{e}_Z\,.\tag{4.18}$$

III. Detailed description of solutions for $m_0=0$ will be made in a separate article. Here we only note that the equation (4.1) gives the first integral

$$\frac{[\ddot{v}+(\Omega_0^2+B_Z/\varsigma s_\mathrm{b})v]^2}{\dot{\Phi}^2}+\dot{v}^2+(\Omega_0^2+B_Z/\varsigma s_\mathrm{b})v^2=D^2=\mathrm{const}\,.\tag{4.19}$$

Comparison of (4.19) with (3.5) shows that the set of solutions for massless particles in a constant uniform magnetic field at $\dot{\Phi} = \Omega_{\mathrm{D}}$ may be easily obtained from solutions I-III for free particles, if Ω_0^2 replaced by $\Omega_B^2 = \Omega_0^2 + B_Z / \varsigma s_{\mathrm{b}}$ together with taking into account of equations (2.23)-(2.24) instead of (2.21).

Analyzing the solutions (4.10) and (4.13), one can see that they are close to the classical solution (4.18). Therefore a question arises to what extent the classical solution corresponds to real situation. It should be noted also that the obtained solutions are found in the simplified assumption about dependence of potential function on the velocity. In addition, the motion takes place from the infinite past to the infinite future. Most of the real problems is that a free particle flies into a certain domain of space, where the field is present, and then flies out from it, becoming free again. Therefore, we can assume that solution of such problems can be obtained by combining the results obtained.

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