# Dynamical Relations in the System of Two Objects with Internal Degrees of Freedom

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**Abstract.** A system of $N$ interacting objects with internal degrees of freedom is considered. Derivation of system of equations for the description of two interacting objects with spin is given. Relations between the parameters describing subsystems and the parameters describing the system as a whole are obtained. In particular, relation between energies of subsystems and energy of system is found, on the basis of which the assumption is made that energy of subsystem should be oscillatory functions of time, so that interaction between objects is reduced to permanent energy exchange.

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# 1 Introduction

Solution of the two-body problem has exclusive meaning for decoding the structure of interactions between both macroscopic bodies and elementary particles. The unique way to determine it by experiment is a study the scattering of particles on each other at various energies, theoretical description of which is a problem of two particles. A motion of cosmic objects (such as satellites, planets, stars, etc.) or a motion of two charged particles also reduces to the two-body problem (classical Kepler-Coulomb problem), which a lot of works is dedicated to beginning from Newton's formulation of the World Gravity Law, Hooke's formulation of elastic deformations and Coulomb's law of electrostatic interaction of charges. Laws of Newton, Hooke and Coulomb made possible to solve a bulk of physical problems and give rise to a great many discoveries. The up-to-date enunciating of a mathematical problem of two bodies interacting via central potential, one can find in the book [1].

The two-object problem, i.e. determination of their trajectories, can be solved in principle, if interaction between them is known. On the other hand, Bertrand sets up an inverse problem of determining interaction with respect to known trajectories of motion of bodies ( [2]). As it is known, according to the Bertrand's theorem only two types of central potentials, of Coulomb and harmonic type, give closed circular or elliptic orbits (see, e.g., [3]). However Bertrand's problem is solved in the assumption that interaction between objects depends only on relative distance between them and is central, meanwhile as early as in 19 century it was considered that interaction should depend also from relative velocity and acceleration. On this way Weber has developed well-working electrodynamics which after creation by Maxwell of the theory of electromagnetic field undeservedly has been removed aside. Weber has obtained expression for the force of interaction between charged particles depending on the relative velocity and acceleration [4]- [7]. Special and general relativity have given a new push to a consideration of the two-body problem. In particular, Sommerfeld has considered the relativistic problem in approximation when one of masses is infinitely great [8]. Darwin has obtained expression for interaction force between charged particles using retarded potential in non-relativistic approximation of relativistic Lagrangian [9]. Later on Darwin's Lagrangian was used in quantum mechanical calculations.

Besides dependence on the relative velocities and accelerations, interaction between physical objects depends also on mutual orientation of their spins originally interpreted classically as angular velocity of rotation ( [10], p. 123), and now as proper moments of momenta. In the bound state the spins of interacting objects are arranged up in definite way. For example, it is established that in a deuteron the spins of proton and neutron are oriented in one direction. Spins of electrons in Cooper pairs have opposite orientation whereas electrons and positrons in positronium can have both the same orientations of spins (orthopositronium), and opposite orientations (parapositronium).

By far not simple question about interaction between objects can be clarified by means of studying of the equations of motion taking into account all parameters of objects. To clarify all aspects of a problem we will start from the classical equations of motion of a material point with internal degrees of freedom, obtained in [11]- [13]. The aim of this paper is to obtain a system of equations of motion, describing non-relativistic motion of two interacting spinning objects. Having non-relativistic classical theory it is easy later on to pass on to relativistic and quantum variants, using a correspondence principle. Equations, which describe any system of $N$ interacting objects with internal degrees of freedom, are considered in §2. These equations are applied to system of two objects in §3. Unlike Newton's mechanics here it is supposed that additivity of a momentum does not take place, i.e. momentum of the system differs from the sum of momenta of subsystems by "the momentum excess" which disappears if interaction does not depend on relative speed. Center-of-mass variables and relative variables are introduced in §4, as well as relations are established between quantities describing a motion of the system as a whole and relative motion of subsystems. The equations of motion of the center of mass of the system and of relative motion of subsystems are obtained, which contain undefined scalar and pseudo-vector functions describing interaction. For a concrete definition of these functions and establishing of the spin equations of motion in §5 the moments of momentum of the system and its subsystems are considered. Relations between total moments of momentum and angular momenta and spins of the system and its subsystems are obtained. Equations of motion of spins are considered in §6. Finally, §7 has to do with a question of correlation of the energy of system with energies of its subsystems.

## 2 System of $N$ objects with internal degrees of freedom

It is shown in Refs. [11]- [13] that the equation of motion of a mass-point with internal degrees of freedom whose position is defined by a radius-vector $\mathbf{R}_K$ and which interacts with external fields, can be presented in the form

$$\frac{d\mathbf{P}_K}{dt} = \mathbf{F}_K \ , \tag{1}$$

where $K = 1, 2, ..., N$ is subscript labeling a mass-point,

$$\mathbf{P}_K = m_K\mathbf{V}_K = m_{0K}\mathbf{V}_K - \frac{\partial U_K}{\partial \mathbf{V}_K} + [\mathbf{S}_K \times \mathbf{W}_K] \ , \tag{2}$$

is dynamical momentum of $K$-th point, $m_{0K}$ is a naked mass of the point without of taking into account an interaction and interior structure, $m_K$ is an effective mass of the point,

$$\mathbf{F}_K = -\frac{\partial U_K}{\partial \mathbf{R}_K} + [\mathbf{C}_K \times \mathbf{V}_K] \ , \tag{3}$$

is a force acting onto $K$-th point,

$$U_K = U_K(t, \mathbf{R}_K, \mathbf{V}_K, \mathbf{W}_K, \dot{\mathbf{W}}_{\mathbf{K}}, ..., \mathbf{W}_K^{(N)}) = U_{0K} - ([\mathbf{R}_K \times \mathbf{V}_K] \cdot \mathbf{C}_K) \ , \tag{4}$$

is a potential function of $K$-th point, which generally may be function of coordinates $\mathbf{R}_K$, velocities $\mathbf{V}_K = \dot{\mathbf{R}}_K$ and accelerations $\mathbf{W}^{(m)} = d^m\mathbf{W}_K/dt^m, \quad m = 0, 1, 2, ..., n$, of both points.

A mass-point with internal degrees of freedom should be considered as non-inertial extended object with internal structure, defined by pseudo-vectors $\mathbf{S}_K$ and $\mathbf{C}_K$, which also depend on the interaction of the object with external fields. For the system of $N$ objects, interacting both with external fields and with each other, functions $U_K$ and pseudo-vectors $\mathbf{S}_K$, $\mathbf{C}_K$ may be represented as sums

$$U_K = U_K^{ext} + \sum_{J=1}^{N} U_{JK}^{int} \ , \tag{5}$$

$$\mathbf{S}_K = \mathbf{S}_{0K} + \mathbf{S}_K^{ext} = \mathbf{S}_{0K} + \mathbf{S}_{0K}^{ext} + \sum_{J=1}^{N} \mathbf{S}_{JK}^{int} \ , \tag{6}$$

$$\mathbf{C}_K = \mathbf{C}_{0K} + \mathbf{C}_K^{ext} = \mathbf{C}_{0K} + \mathbf{C}_{0K}^{ext} + \sum_{J=1}^{N} \mathbf{C}_{JK}^{int} \ , \tag{7}$$

Here functions $U_K^{ext}$, $\mathbf{S}_{0K}^{ext}$, $\mathbf{C}_{0K}^{ext}$ are specified by interaction of objects with external fields, whereas $U_{JK}^{int}$, $\mathbf{S}_{JK}^{int}$, $\mathbf{C}_{JK}^{int}$ are specified by interaction of objects $J$ and $K$ with each other. Pseudo-vectors

$$\mathbf{S}_K^{ext} = \mathbf{S}_{0K}^{ext} + \sum_{J=1}^{N} \mathbf{S}_{JK}^{int} \ , \tag{8}$$

$$\mathbf{C}_K^{ext} = \mathbf{C}_{0K}^{ext} + \sum_{J=1}^{N} \mathbf{C}_{JK}^{int} \ , \tag{9}$$

may depend on the same variables as potential function (4). Internal parameters $\mathbf{S}_{0K}$ and $\mathbf{C}_{0K}$ are specified exclusively by internal structure of objects and do not depend on external variables. In Ref. [13] it is shown that for free objects (mass-points with internal degrees of freedom) they are expressed in terms of spin, being integral characteristic of this structure, as follows

$$\mathbf{S}_{0K} = \varsigma_K \mathbf{s}_K \ , \tag{10}$$

$$\mathbf{C}_{0K} = -\Omega_{0K}^2 \mathbf{S}_{0K} = -\varsigma_K \Omega_{0K}^2 \mathbf{s}_K \ , \tag{11}$$

where $\varsigma_K$ is a constant with dimensionality of inverse square of velocity, $\Omega_{0K}$ is a cyclic frequency of Zitterbewegung of $K$-th object. If $\varsigma_K = -c_K^{-2}$, where $c_K$ is some velocity the equation of motion for free $K$-th object is reduced to non-relativistic limit of Frenkel-Mathisson-Weyssenhoff equation ( [14]- [16]). If one adopt $\varsigma_K = +c_K^{-2}$, the corresponding equation will describe a particle with opposite direction of spin, i.e. antiparticle. One may assume further that all constants $c_K$ are equal among themselves and represent the velocity of light. However it should be noticed that constant with dimensionality of velocity is proportional to product $r_{0K}\Omega_{0K}$, where $r_{0K}$ is a radius of Zitterbewegung of

$K$-th object. If $c_K = r_{0K}\Omega_{0K}$, the center of mass of free $K$-th object in its center-of-inertia reference frame moves along circle round the direction of motion of its center of inertia with velocity $c_K$. It is reasonable to expect the relations (10)-(11), including the cases $\mathbf{S}_{0K} = \mathbf{0}$ ($c_K = \infty$) and $\mathbf{C}_{0K} = \mathbf{0}$ ($\Omega_{0K} = 0$), will be valid also for interacting objects.

Along with (1) it is necessary to write down the equations of motion for spins

$$\frac{d\mathbf{s}_K}{dt} = [\mathbf{\Omega}_{\mathbf{N}K}(t) \times \mathbf{s}_K] + \mathbf{m}_K(t) = \sigma_{\mathbf{N}K}(t)[\mathbf{N}_K \times \mathbf{s}_K] + \mathbf{m}_K(t) \ , \tag{12}$$

where $\mathbf{\Omega}_{\mathbf{N}K}(t)$ are angular velocities of precession of spins $\mathbf{s}_K$ round directions of vectors $\mathbf{N}_K(t)$, which can be the same vector for the system of objects, for example, the vector of velocity of the center of inertia of the system. $\mathbf{m}_K(t)$ has meaning of the moment of force acting onto extended object relative to its center of mass. One can try to define the structure of $\mathbf{m}_K(t)$ from analysis of interaction of internal substance of the extended object with external fields, which induces possible movement of this substance inside the object.

A geometrical structure of the object, on the one hand, is specified by distribution of its internal substance in some spatial volume $V$ which can change due to the interaction, and on the other hand, itself determines an interaction of the object with external fields. There are only two variants of consideration of extended object: either as a set (discrete or continuous) of structureless mass-points, in the same way as it is usually made in the mechanics of absolutely rigid or deformable body, or as a set of mass-points with internal degrees of freedom, i.e. similar objects. Then by definition expression for spin of $K$-th object looks like

$$\mathbf{s}_K = j_{0K}\boldsymbol{\omega}_{0K} = \sum_{i=1}^{N_K}[\mathbf{r}_i^{(K)} \times \boldsymbol{\pi}_i^{(K)}] \ , \tag{13}$$

for discrete set of points, or

$$\mathbf{s}_K = j_{0K}\boldsymbol{\omega}_{0K} = \int_{V_K(t)} [\boldsymbol{\rho}_K \times \boldsymbol{\pi}_K(\boldsymbol{\rho}_K)] dV_K \ , \tag{14}$$

for continuous set of points. Here $\mathbf{r}_i^{(K)}$ and $\boldsymbol{\rho}_K$ are radius-vectors of internal points relative to the center of mass of $K$-th object,

$$\boldsymbol{\pi}_i^{(K)} = m_i^{(K)}\mathbf{v}_i^{(K)} = m_i^{(K)}\frac{d\mathbf{r}_i^{(K)}}{dt} \ , \tag{15}$$

is momentum of $i$-th point with effective mass $m_i^{(K)}$ and velocity $\mathbf{v}_i^{(K)}$ in the case of discrete set of structureless points,

$$\boldsymbol{\pi}_K(\boldsymbol{\rho}_K) = \frac{d\mu_K}{dV}\boldsymbol{\upsilon}_K(\boldsymbol{\rho}_K) = \frac{d\mu_K}{dV}\frac{d\boldsymbol{\rho}_K}{dt} \ , \tag{16}$$

is momentum density of elementary mass $d\mu_K$, moving with velocity $\boldsymbol{\upsilon}_K(\boldsymbol{\rho}_K)$ inside of $K$-th object, representable as continuous distribution of structureless points. In the case when the object with internal degrees of freedom is a set of the points also endowed with internal degrees of freedom, expressions for $\boldsymbol{\pi}_i^{(K)}$ and $\boldsymbol{\pi}_K(\boldsymbol{\rho}_K)$ should have the same structure as in (2). This variant unjustifiably complicates the description of extended objects, but it may be relevant for the description of larger composite objects.

Differentiation of (13) and (14) with respect to time and comparison of result with (12) will give the possibility to determine the structure of $\mathbf{m}_K(t)$. However at first it is necessary to contemplate the two-object problem, and then the problem of many objects with internal degrees of freedom.

# 3 System of two objects with internal degrees of freedom

In the system of two interacting mass-points, each of which possesses internal degrees of freedom, pseudo-vectors $\mathbf{S}_K$ and $\mathbf{C}_K$, $K = 1, 2$, due to (6), (7), (10) and (11) are represented like

$$\mathbf{S}_1 = \varsigma_1 \mathbf{s}_1 + \mathbf{S}_{01}^{ext} + \mathbf{S}_{21}^{int} \ , \ \ \mathbf{S}_2 = \varsigma_2 \mathbf{s}_2 + \mathbf{S}_{02}^{ext} + \mathbf{S}_{12}^{int} \ , \tag{17}$$

$$\mathbf{C}_1 = -\varsigma_1 \Omega_{01}^2 \mathbf{s}_1 + \mathbf{C}_{01}^{ext} + \mathbf{C}_{21}^{int} \ , \ \ \mathbf{C}_2 = -\varsigma_2 \Omega_{02}^2 \mathbf{s}_2 + \mathbf{C}_{02}^{ext} + \mathbf{C}_{12}^{int} \ . \tag{18}$$

Momenta of these points with respect to (2), (5), (17) look like

$$\mathbf{P}_1 = m_{01}\mathbf{V}_1 - \frac{\partial(U_1^{ext} + U_{21}^{int})}{\partial \mathbf{V}_1} + [(\varsigma_1 \mathbf{s}_1 + \mathbf{S}_{01}^{ext} + \mathbf{S}_{21}^{int}) \times \mathbf{W}_1] = m_1 \mathbf{V}_1 \ , \tag{19}$$

$$\mathbf{P}_2 = m_{02}\mathbf{V}_2 - \frac{\partial(U_2^{ext} + U_{12}^{int})}{\partial \mathbf{V}_2} + [(\varsigma_2 \mathbf{s}_2 + \mathbf{S}_{02}^{ext} + \mathbf{S}_{12}^{int}) \times \mathbf{W}_2] = m_2 \mathbf{V}_2 \ , \tag{20}$$

where $m_{01}$, $m_1$ and $m_{02}$, $m_2$ are the naked and effective masses of constituents, respectively. Forces, acting upon the points, with respect to (3), (5), (18) are

$$\mathbf{F}_1 = -\frac{\partial(U_1^{ext} + U_{21}^{int})}{\partial \mathbf{R}_1} + [(-\varsigma_1 \Omega_{01}^2 \mathbf{s}_1 + \mathbf{C}_{01}^{ext} + \mathbf{C}_{21}^{int}) \times \mathbf{V}_1] \ , \tag{21}$$

$$\mathbf{F}_2 = -\frac{\partial(U_2^{ext} + U_{12}^{int})}{\partial \mathbf{R}_2} + [(-\varsigma_2 \Omega_{02}^2 \mathbf{s}_2 + \mathbf{C}_{02}^{ext} + \mathbf{C}_{12}^{int}) \times \mathbf{V}_2] \ . \tag{22}$$

System as a whole, as well as their constituents, is non-inertial object with internal degrees of freedom, whose dynamical momentum should have the structure similar to (2), i.e.

$$\mathbf{P} = m_0 \mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [(\varsigma \mathbf{s} + \mathbf{S}^{ext}) \times \mathbf{W}] = m\mathbf{V} \ , \tag{23}$$

where $m_0$ and $m$ are the naked and effective mass of the system, respectively, and potential function $U$ and pseudo-vector $\mathbf{S}^{ext}$ have to determined exclusively by interaction of the system as a whole with external fields.

In Newton's classical mechanics, where potential function does not depend on velocities, momentum is additive quantity (see, e.g., [17]). It means that the momentum of the system is determined as sum of momenta of its subsystems $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. However experimental data of nuclear and elementary particle physics testify to an absence of the momentum additivity. For example, momentum of nucleus considered as a system of interacting nucleons does not equal to sum of momenta of nucleons. Due to strong interaction arises a mass excess. Therefore the composition law for momenta should be written down in the form

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \Delta\mathbf{P} \ , \tag{24}$$

where $\Delta\mathbf{P}$ can be named similar to the mass excess as "*momentum excess*". In general case we shall consider momentum excess to be caused not only by the interaction of constituents of system with each other, but also their interaction with external fields. Therefore we suppose

$$\Delta\mathbf{P} = \Delta\mathbf{P}^{ext} + \Delta\mathbf{P}^{int} \ . \tag{25}$$

Momentum (24) should satisfy to the Second Newton's Law

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 \ , \tag{26}$$

the right hand side of which includes the resultant of all forces, acting onto points of the system. This resultant force should be of the same structure as expression (3), i.e.

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{R}} + [(-\varsigma\Omega_0^2\mathbf{s} + \mathbf{C}^{ext}) \times \mathbf{V}] \ . \tag{27}$$

All quantities entering into expressions (23), (27), characterize system as a whole and can be expressed through corresponding partial quantities. Spin of the system $\mathbf{s}$ should also satisfy to equation of type (12), or

$$\frac{d\mathbf{s}}{dt} = \frac{\sigma_{\mathbf{P}}(t)}{m}[\mathbf{P} \times \mathbf{s}] + \mathbf{m}(t) \ . \tag{28}$$

Thus, on the one hand, the system of two objects with internal degrees of freedom is described by system of six equations (1) and six equations (12), which, on the other hand, should be equivalent to three equations (26) and three equations (28). The remaining six equations should describe internal movements in the system. These equations we shall consider in the separate section.

# 4 Relations between quantities, describing the system and its subsystems

Quantities $m_0$, $m$, $U$, $\varsigma$, $\Omega_0$, $\mathbf{s}$, $\mathbf{S}^{ext}$, $\mathbf{C}^{ext}$, entering into (23), (27), may be determined, if partial quantities $m_{0K}$, $m_K$, $U_K$, $\varsigma_K$, $\Omega_{0K}$, $\mathbf{s}_K$, $\mathbf{S}_{0K}^{ext}$, $\mathbf{C}_{0K}^{ext}$, entering into (1), are known.

To find a connection between all these quantities we transform system of equations (1) and (12) in standard way by introducing the relative variables

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \ , \ \ \mathbf{v} = \mathbf{V}_2 - \mathbf{V}_1 \ , \ \ \mathbf{w}^{(m)} = \mathbf{W}_2^{(m)} - \mathbf{W}_1^{(m)} = d^m\mathbf{w}/dt^m \ , \tag{29}$$

and the center-of-mass variables

$$\mathbf{R} = \frac{m_{01}\mathbf{R}_1 + m_{02}\mathbf{R}_2}{m_0} \ , \ \ \mathbf{V} = \frac{m_{01}\mathbf{V}_1 + m_{02}\mathbf{V}_2}{m_0} \ , \mathbf{W}^{(m)} = \frac{m_{01}\mathbf{W}_1^{(m)} + m_{02}\mathbf{W}_2^{(m)}}{m_0} \ , \tag{30}$$

where

$$m_0 = m_{01} + m_{02} \ . \tag{31}$$

We define also the internal variables, namely, radius-vectors, velocities and accelerations of points relative to the center of mass

$$\mathbf{r}_K = \mathbf{R}_K - \mathbf{R} \ , \ \ \mathbf{v}_K = \mathbf{V}_K - \mathbf{V} \ , \ \ \mathbf{w}_K^{(m)} = \mathbf{W}_K^{(m)} - \mathbf{W}^{(m)} \ . \tag{32}$$

From (29)-(32) we obtain

$$\mathbf{r}_1 = \mathbf{R}_1 - \mathbf{R} = -\frac{m_{02}}{m_0}\mathbf{r} \ , \ \ \mathbf{r}_2 = \mathbf{R}_2 - \mathbf{R} = \frac{m_{01}}{m_0}\mathbf{r} \ , \tag{33}$$

$$m_{01}\mathbf{r}_1 + m_{02}\mathbf{r}_2 = m_{01}\mathbf{R}_1 + m_{02}\mathbf{R}_2 - m_0\mathbf{R} = \mathbf{0} \ , \tag{34}$$

as well as corresponding relations for velocities and accelerations. If radius-vector $\mathbf{R}(t)$ of the center of mass and relative radius-vector $\mathbf{r}(t)$ are known, then radius-vectors $\mathbf{R}_K(t)$ of points in absolute reference frame can be easily determined from (33).

For arbitrary functions $f(\mathbf{R}_1, ...; \mathbf{R}_2, ...)$, depending on $\mathbf{R}_1$, $\mathbf{R}_2$ and their derivatives with respect to time we have

$$\frac{\partial f}{\partial \mathbf{R}_1} = \frac{m_{01}}{m_0}\frac{\partial f}{\partial \mathbf{R}} - \frac{\partial f}{\partial \mathbf{r}} \ , \ \ \frac{\partial f}{\partial \mathbf{R}_2} = \frac{m_{02}}{m_0}\frac{\partial f}{\partial \mathbf{R}} + \frac{\partial f}{\partial \mathbf{r}} \ , \tag{35}$$

and corresponding derivatives with respect to $\mathbf{V}_1$, $\mathbf{V}_2$, ... .

Substitution of quantities $\mathbf{R}_K(t)$, $\mathbf{V}_K(t)$, $\mathbf{W}_K(t)$, obtained from (33) into (24)-(26) and comparison of the result with (23), (27), give rise to equations, connecting quantities relating to the whole system with partial ones

$$\frac{\partial U}{\partial \mathbf{R}} = \frac{1}{m_0}\frac{\partial[m_{01}(U_1^{ext} + U_{21}^{int}) + m_{02}(U_2^{ext} + U_{12}^{int})]}{\partial \mathbf{R}} +$$

$$+\frac{\partial(U_2^{ext} + U_{12}^{int} - U_1^{ext} - U_{21}^{int})}{\partial \mathbf{r}} +$$

$$+[(\varsigma_1\Omega_{01}^2\mathbf{s}_1 + \varsigma_2\Omega_{02}^2\mathbf{s}_2 - \varsigma\Omega_0^2\mathbf{s} + \mathbf{C}^{ext} - \mathbf{C}_{01}^{ext} - \mathbf{C}_{21}^{int} - \mathbf{C}_{02}^{ext} - \mathbf{C}_{12}^{int}) \times \mathbf{V}] -$$

$$-\frac{1}{m_0}[(m_{02}\varsigma_1\Omega_{01}^2\mathbf{s}_1 - m_{01}\varsigma_2\Omega_{02}^2\mathbf{s}_2 - m_{02}\mathbf{C}_{01}^{ext} - m_{02}\mathbf{C}_{21}^{int} + m_{01}\mathbf{C}_{02}^{ext} + m_{01}\mathbf{C}_{12}^{int}) \times \mathbf{v}] \ , \tag{36}$$

$$\frac{\partial U}{\partial \mathbf{V}} = \frac{1}{m_0}\frac{\partial[m_{01}(U_1^{ext}+U_{21}^{int})+m_{02}(U_2^{ext}+U_{12}^{int})]}{\partial \mathbf{V}}+$$
$$+\frac{\partial(U_2^{ext}+U_{12}^{int}-U_1^{ext}-U_{21}^{int})}{\partial \mathbf{v}}-\Delta\mathbf{P}^{ext}-\Delta\mathbf{P}^{int}+$$
$$+[(\varsigma\mathbf{s}-\varsigma_1\mathbf{s}_1-\varsigma_2\mathbf{s}_2+\mathbf{S}^{ext}-\mathbf{S}_{01}^{ext}-\mathbf{S}_{21}^{int}-\mathbf{S}_{02}^{ext}-\mathbf{S}_{12}^{int})\times\mathbf{W}]+$$
$$+\frac{1}{m_0}[(m_{02}\varsigma_1\mathbf{s}_1-m_{01}\varsigma_2\mathbf{s}_2+m_{02}\mathbf{S}_{01}^{ext}+m_{02}\mathbf{S}_{21}^{int}-m_{01}\mathbf{S}_{02}^{ext}-m_{01}\mathbf{S}_{12}^{int})\times\mathbf{w}]\,. \tag{37}$$

For free system which is not undergo to action of external fields it is necessary to put $U=U_1^{ext}=U_2^{ext}=0$, $\Delta\mathbf{P}^{ext}=\mathbf{0}$, $\mathbf{S}^{ext}=\mathbf{S}_{01}^{ext}=\mathbf{S}_{02}^{ext}=\mathbf{0}$, $\mathbf{C}^{ext}=\mathbf{C}_{01}^{ext}=\mathbf{C}_{02}^{ext}=\mathbf{0}$. It is reasonably to admit also that $U_{12}^{int}$ and $U_{21}^{int}$ depend only on relative variables whereas $U_1^{ext}$ and $U_2^{ext}$ do not depend on them. Since they are scalar functions depending on relative variables we have

$$U_{12}^{int}=U_{12}^{int}\equiv U^{int}\,. \tag{38}$$

Then (36), (37) reduce to equations

$$[(\varsigma_1\Omega_{01}^2\mathbf{s}_1+\varsigma_2\Omega_{02}^2\mathbf{s}_2-\varsigma\Omega_0^2\mathbf{s}-\mathbf{C}_{21}^{int}-\mathbf{C}_{12}^{int})\times\mathbf{V}]-$$
$$-\frac{1}{m_0}[(m_{02}\varsigma_1\Omega_{01}^2\mathbf{s}_1-m_{01}\varsigma_2\Omega_{02}^2\mathbf{s}_2-m_{02}\mathbf{C}_{21}^{int}+m_{01}\mathbf{C}_{12}^{int})\times\mathbf{v}]=\mathbf{0}\,, \tag{39}$$
$$\Delta\mathbf{P}^{int}=[(\varsigma\mathbf{s}-\varsigma_1\mathbf{s}_1-\varsigma_2\mathbf{s}_2-\mathbf{S}_{21}^{int}-\mathbf{S}_{12}^{int})\times\mathbf{W}]+$$
$$+\frac{1}{m_0}[(m_{02}\varsigma_1\mathbf{s}_1-m_{01}\varsigma_2\mathbf{s}_2+m_{02}\mathbf{S}_{21}^{int}-m_{01}\mathbf{S}_{12}^{int})\times\mathbf{w}]\,, \tag{40}$$

where one may get rid of dependence on the center-of-mass variables when assuming

$$\varsigma\mathbf{s}=\varsigma_1\mathbf{s}_1+\varsigma_2\mathbf{s}_2+\mathbf{S}_{21}^{int}+\mathbf{S}_{12}^{int}\,, \tag{41}$$

$$\mathbf{C}_{21}^{int}+\mathbf{C}_{12}^{int}=\varsigma_1\Omega_{01}^2\mathbf{s}_1+\varsigma_2\Omega_{02}^2\mathbf{s}_2-\varsigma\Omega_0^2\mathbf{s}=$$
$$=\varsigma_1(\Omega_{01}^2-\Omega_0^2)\mathbf{s}_1+\varsigma_2(\Omega_{02}^2-\Omega_0^2)\mathbf{s}_2-\Omega_0^2(\mathbf{S}_{21}^{int}+\mathbf{S}_{12}^{int})\,. \tag{42}$$

These relations follow from the Galileo's relativity principle, according to which equations (39), (40) should be covariant relative to Galileo's transformations, so that in the center-of-mass reference frame ($\mathbf{V}=\mathbf{0}$, $\mathbf{W}=\mathbf{0}$) they take the form

$$[(m_{02}\varsigma_1\Omega_{01}^2\mathbf{s}_1-m_{01}\varsigma_2\Omega_{02}^2\mathbf{s}_2-m_{02}\mathbf{C}_{21}^{int}+m_{01}\mathbf{C}_{12}^{int})\times\mathbf{v}]=\mathbf{0}\,, \tag{43}$$

$$\Delta\mathbf{P}^{int}=\frac{1}{m_0}[(m_{02}\varsigma_1\mathbf{s}_1-m_{01}\varsigma_2\mathbf{s}_2+m_{02}\mathbf{S}_{21}^{int}-m_{01}\mathbf{S}_{12}^{int})\times\mathbf{w}]\,. \tag{44}$$

It is naturally to accept equation (43) to be valid in the case of interaction of the system with external fields. Then substitution of (41)-(44) in (36), (37) leads to equations

$$\frac{\partial U}{\partial \mathbf{R}}=\frac{1}{m_0}\frac{\partial(m_{01}U_1^{ext}+m_{02}U_2^{ext})}{\partial \mathbf{R}}+[(\mathbf{C}^{ext}-\mathbf{C}_{01}^{ext}-\mathbf{C}_{02}^{ext})\times\mathbf{V}]+$$

$$+\frac{1}{m_0}[(m_{02}\mathbf{C}_{01}^{ext} - m_{01}\mathbf{C}_{02}^{ext}) \times \mathbf{v}] \ , \tag{45}$$

$$\frac{\partial U}{\partial \mathbf{V}} = \frac{1}{m_0}\frac{\partial(m_{01}U_1^{ext} + m_{02}U_2^{ext})}{\partial \mathbf{V}} + [(\mathbf{S}^{ext} - \mathbf{S}_{01}^{ext} - \mathbf{S}_{02}^{ext}) \times \mathbf{W}] +$$

$$+\frac{1}{m_0}[(m_{02}\mathbf{S}_{01}^{ext} - m_{01}\mathbf{S}_{02}^{ext}) \times \mathbf{w}] - \Delta\mathbf{P}^{ext} \ . \tag{46}$$

To get rid of relative variables it is sufficient to put

$$\mathbf{S}_{0K}^{ext} = \frac{m_{0K}}{m_0}(\mathbf{S}^{ext} + \Sigma\mathbf{W}_K) \ , \ \ \mathbf{C}_{0K}^{ext} = \frac{m_{0K}}{m_0}(\mathbf{C}^{ext} + \Gamma\mathbf{V}_K) \ , \tag{47}$$

where functions or constants $\mathbf{S}^{ext}$, $\mathbf{C}^{ext}$, $\Sigma$ and $\Gamma$ are specified by external fields. It follows from (47)

$$\mathbf{S}^{ext} = \mathbf{S}_{01}^{ext} + \mathbf{S}_{02}^{ext} \ , \ \ \mathbf{C}^{ext} = \mathbf{C}_{01}^{ext} + \mathbf{C}_{02}^{ext} \ . \tag{48}$$

Now equations (45), (46) take simple form

$$\frac{\partial U}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}}\frac{m_{01}U_1^{ext} + m_{02}U_2^{ext}}{m_0} \ , \tag{49}$$

$$\Delta\mathbf{P}^{ext} = \frac{\partial}{\partial \mathbf{V}}\left(\frac{m_{01}U_1^{ext} + m_{02}U_2^{ext}}{m_0} - U\right) \ . \tag{50}$$

It follows from (49)

$$U(t;\mathbf{R},\mathbf{V},...) = \frac{m_{01}}{m_0}U_1^{ext} + \frac{m_{02}}{m_0}U_2^{ext} + u(t;\mathbf{V},\mathbf{W},...) \ , \tag{51}$$

where functions $U_K^{ext}$ are equal to $U(t;\mathbf{R},\mathbf{V},...)$ up to arbitrary function $u(t;\mathbf{V},\mathbf{W},...)$, determined from initial and boundary conditions

$$U_K^{ext}(t;\mathbf{R},\mathbf{V},...) = U(t;\mathbf{R},\mathbf{V},...) + u(t;\mathbf{V},\mathbf{W},...) \ . \tag{52}$$

In accordance to (50) it is connected with external momentum excess

$$\Delta\mathbf{P}^{ext}(t;\mathbf{V},\mathbf{W},...) = \frac{\partial u(t;\mathbf{V},\mathbf{W},...)}{\partial \mathbf{V}} \ , \tag{53}$$

which hence is determined by dependence of initial and boundary conditions from the velocity of the center of mass of the system. If such dependence is absent, then we have $\Delta\mathbf{P}^{ext}(t;\mathbf{W},...) = \mathbf{0}$.

Thus, equation of motion (26) of the system as a whole takes the following form

$$\frac{d}{dt}\left[m_0\mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [(\varsigma_1\mathbf{s}_1 + \varsigma_2\mathbf{s}_2 + \mathbf{S}_{21}^{int} + \mathbf{S}_{12}^{int} + \mathbf{S}^{ext}) \times \mathbf{W}]\right] =$$

$$= -\frac{\partial U}{\partial \mathbf{R}} - \Omega_0^2[(\varsigma_1\mathbf{s}_1 + \varsigma_2\mathbf{s}_2 + \mathbf{S}_{21}^{int} + \mathbf{S}_{12}^{int}) \times \mathbf{V}] + [\mathbf{C}^{ext} \times \mathbf{V}] \ , \tag{54}$$

where functions $U = U(t;\mathbf{R},\mathbf{V},...)$, $\mathbf{S}^{ext} = \mathbf{S}^{ext}(t;\mathbf{R},\mathbf{V},...)$, $\mathbf{C}^{ext} = \mathbf{C}^{ext}(t;\mathbf{R},\mathbf{V},...)$, as well as $\mathbf{S}_{12}^{int}(\mathbf{r},\mathbf{v},...)$ and $\mathbf{S}_{21}^{int}(\mathbf{r},\mathbf{v},...)$, specified by both structure of constituents and their spins $\mathbf{s}_1$ and $\mathbf{s}_2$, should be determined in advance.

Let us define now the relative momentum which in view of relations (47), (52) is

$$\mathbf{p} = \mathbf{P}_2 - \mathbf{P}_1 = \frac{m_{02}-m_{01}}{m_0}\left[m_0\mathbf{V} - \frac{\partial(U+u)}{\partial\mathbf{V}} + [\mathbf{S}^{ext}\times\mathbf{W}]\right] +$$
$$[(\varsigma_2\mathbf{s}_2 - \varsigma_1\mathbf{s}_1 + \mathbf{S}_{12}^{int} - \mathbf{S}_{21}^{int})\times\mathbf{W}] - 2\frac{\partial U^{int}}{\partial\mathbf{v}} + \frac{2m_{01}m_{02}}{m_0^2}\left(m_0\mathbf{v} + [\mathbf{S}^{ext}\times\mathbf{w}]\right) + \tag{55}$$
$$+\varsigma[\mathbf{s}\times\mathbf{w}] - \frac{1}{m_0}[(m_{01}\varsigma_1\mathbf{s}_1 + m_{02}\varsigma_2\mathbf{s}_2 + m_{01}\mathbf{S}_{21}^{int} + m_{01}\mathbf{S}_{12}^{int})\times\mathbf{w}]$$

and due to (1) and (19)-(22) satisfies to equation

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_2 - \mathbf{F}_1\ . \tag{56}$$

Thus, system of six equations (54) and (56) describes six translational degrees of freedom of the system.

For free system ($U_1^{ext} = U_2^{ext} = 0$, $\mathbf{C}^{ext} = \mathbf{0}$, $\mathbf{S}^{ext} = \mathbf{0}$) the relative momentum equals

$$\mathbf{p} = (m_{02}-m_{01})\mathbf{V} + [(\varsigma_2\mathbf{s}_2 - \varsigma_1\mathbf{s}_1 + \mathbf{S}_{12}^{int} - \mathbf{S}_{21}^{int})\times\mathbf{W}] + \frac{2m_{01}m_{02}}{m_0}\mathbf{v} -$$
$$-2\frac{\partial U^{int}}{\partial\mathbf{v}} + \frac{1}{m_0}[(m_{02}\varsigma_1\mathbf{s}_1 + m_{01}\varsigma_2\mathbf{s}_2 + m_{02}\mathbf{S}_{21}^{int} + m_{01}\mathbf{S}_{12}^{int})\times\mathbf{w}]\ , \tag{57}$$

whence it is obvious that it depends not only on the state of movement of system constituents, but also on the state of motion of the center of mass of the system. Equations of motion (54) and (56) in this case look like

$$\frac{d}{dt}(m_0\mathbf{V} + \varsigma[\mathbf{s}\times\mathbf{W}]) + \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{V}] = \mathbf{0}\ , \tag{58}$$

$$\frac{d}{dt}\left[(m_{02}-m_{01})\mathbf{V} + [(\varsigma_2\mathbf{s}_2 - \varsigma_1\mathbf{s}_1 - \mathbf{S}_{21}^{int} + \mathbf{S}_{12}^{int})\times\mathbf{W}] + \frac{2m_{01}m_{02}}{m_0}\mathbf{v}\right] -$$
$$-\frac{d}{dt}\left[2\frac{\partial U^{int}}{\partial\mathbf{v}} - \varsigma[\mathbf{s}\times\mathbf{w}] + \frac{1}{m_0}[(m_{01}(\varsigma_1\mathbf{s}_1 + \mathbf{S}_{21}^{int}) + m_{02}(\varsigma_2\mathbf{s}_2 + \mathbf{S}_{12}^{int}))\times\mathbf{w}]\right] =$$
$$= -2\frac{\partial U^{int}}{\partial\mathbf{r}} + [(\varsigma_1\Omega_{01}^2\mathbf{s}_1 - \varsigma_2\Omega_{02}^2\mathbf{s}_2 - \mathbf{C}_{21}^{int} + \mathbf{C}_{12}^{int})\times\mathbf{V}] - \varsigma\Omega_0^2[\mathbf{s}\times\mathbf{v}] +$$
$$+\frac{1}{m_0}[(m_{01}(\varsigma_1\Omega_{01}^2\mathbf{s}_1 - \mathbf{C}_{21}^{int}) + m_{02}(\varsigma_2\Omega_{02}^2\mathbf{s}_2 - \mathbf{C}_{12}^{int}))\times\mathbf{v}]\ . \tag{59}$$

In (55)-(59) scalar function $U^{int}(\mathbf{r},\mathbf{v},...)$ and pseudo-vector functions $\mathbf{S}_{12}^{int}(\mathbf{r},\mathbf{v},...)$, $\mathbf{S}_{21}^{int}(\mathbf{r},\mathbf{v},...)$, $\mathbf{C}_{12}^{int}(\mathbf{r},\mathbf{v},...)$ and $\mathbf{C}_{21}^{int}(\mathbf{r},\mathbf{v},...)$, satisfying to relations (41), (42), remain indeterminate. They may be obtained from additional equations, following from spin equations of motion, which will be considered below.

# 5 Moment of momentum of the system

As it was told above, each of two constituents of the system may be considered either as a system of structureless mass-points or as a system of mass-points with internal degrees of freedom. In this paragraph we deal with first variant, when for every $i$-th point in (1) it is necessary to put $\mathbf{S}_i = \mathbf{0}$, $\mathbf{C}_i = \mathbf{0}$ ($i = 1, 2, ..., N_1, N_1 + 1, N$, $N = N_1 + N_2$ is amount of points in the whole system, $N_K$ is amount of points in $K$-th subsystem). Then, if potential function $U_i$ depends on the velocity of the point, the momentum (2) and force (3) take standard form $\mathbf{P}_i = m_i \mathbf{V}_i$, $\mathbf{F}_i = -\partial U_i / \partial \mathbf{R}_i$, where $m_i \equiv m_i^{(K)}$, $\mathbf{R}_i \equiv \mathbf{R}_i^{(K)}$ and $\mathbf{V}_i \equiv \mathbf{V}_i^{(K)} = d\mathbf{R}_i^{(K)}/dt$ are effective mass, radius-vector and velocity of $i$-th mass-point of $K$-th subsystem relative to the origin of coordinates, respectively.

Both system and its subsystems are characterized by the moments of momentum relative to origin $\mathbf{J}$, $\mathbf{J}_1$, $\mathbf{J}_2$. Then, assuming the moment of momentum to be additive quantity, we have

$$\mathbf{J} = \sum_{i=1}^{N} [\mathbf{R}_i \times \mathbf{P}_i] = \sum_{i=1}^{N} m_i [\mathbf{R}_i \times \mathbf{V}_i] = \mathbf{J}_1 + \mathbf{J}_2 \ , \tag{60}$$

where

$$\mathbf{J}_K = \sum_{i=1}^{N_K} m_i^{(K)} [\mathbf{R}_i^{(K)} \times \mathbf{V}_i^{(K)}] \ . \tag{61}$$

In each of two subsystems one may determine the center of mass defined by radius-vector

$$\mathbf{R}_K = \frac{1}{m_K} \sum_{i=1}^{N_K} m_i^{(K)} \mathbf{R}_i^{(K)} \ , \quad m_K = \sum_{i=1}^{N_K} m_i^{(K)} \ , \tag{62}$$

whereas radius-vector of the center of inertia of the whole system is

$$\mathbf{R} = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{R}_i = \frac{1}{m} \left( \sum_{i=1}^{N_1} m_i^{(1)} \mathbf{R}_i^{(1)} + \sum_{i=1}^{N_2} m_i^{(2)} \mathbf{R}_i^{(2)} \right) = \frac{m_1 \mathbf{R}_1 + m_2 \mathbf{R}_2}{m} \ , \tag{63}$$

where $m = m_1 + m_2$. Differentiation of (62), (63) with respect to time gives corresponding relations for velocities and accelerations.

Let's note here that if potential function $U = U(t; \mathbf{R}, \mathbf{V}, ...)$ explicitly depends on time, then effective masses can also depend on time. Hence it follows from (62), (63)

$$\mathbf{V}_K = \frac{1}{m_K} \sum_{i=1}^{N_K} m_i^{(K)} \mathbf{V}_i^{(K)} + \frac{1}{m_K} \sum_{i=1}^{N_K} \dot{m}_i^{(K)} \mathbf{R}_i^{(K)} - \frac{\dot{m}_K}{m_K} \mathbf{R}_K \ , \tag{64}$$

$$\mathbf{V} = \frac{m_1 \mathbf{V}_1 + m_2 \mathbf{V}_2}{m} + \frac{\dot{m}_1 \mathbf{R}_1 + \dot{m}_2 \mathbf{R}_2}{m} - \frac{\dot{m}}{m} \mathbf{R} \ . \tag{65}$$

If we introduce the relative coordinates of the center of inertia of second subsystem relative to the center of inertia of first subsystem

$$\mathbf{r}_{21} = \mathbf{R}_2 - \mathbf{R}_1 \ , \tag{66}$$

then (63) gives

$$\mathbf{R}_1 = \mathbf{R} - \frac{m_2}{m}\mathbf{r}_{21}\ , \quad \mathbf{R}_2 = \mathbf{R} + \frac{m_1}{m}\mathbf{r}_{21}\ . \tag{67}$$

Substituting (67) in (65) we obtain

$$\mathbf{V} = \frac{m_1\mathbf{V}_1 + m_2\mathbf{V}_2}{m} + \frac{m_1\dot{m}_2 - \dot{m}_1 m_2}{m^2}\mathbf{r}_{21}\ . \tag{68}$$

Let $\mathbf{r}_i^{(K)}$ be a radius-vector of $i$-th point of $K$-th subsystem relative to its center of mass $\mathrm{M}_K$, $\mathbf{r}_i$ be a radius-vector of the same point relative to the center of mass M of the whole system, $\mathbf{r}_K$ be a radius-vector of the center of mass $\mathrm{M}_K$ of $K$-th subsystem relative to the center of mass M of the whole system, which is determined similar to (62)

$$\mathbf{r}_K = \frac{1}{m_K}\sum_{i=1}^{N_K} m_i^{(K)}\mathbf{r}_i\ , \tag{69}$$

Then we have following geometric relations

$$\mathbf{R}_i^{(K)} = \mathbf{R} + \mathbf{r}_i = \mathbf{R}_K + \mathbf{r}_i^{(K)}\ , \tag{70}$$

$$\mathbf{r}_i = \mathbf{r}_K + \mathbf{r}_i^{(K)}\ , \tag{71}$$

$$\mathbf{R}_K = \mathbf{R} + \mathbf{r}_K\ . \tag{72}$$

Differentiation of (69)-(72) with respect to time gives corresponding relations for velocities and accelerations.

Substituting (70) into (63) we obtain

$$\sum_{i=1}^{N} m_i\mathbf{r}_i = \sum_{i=1}^{N_1} m_i^{(1)}\mathbf{r}_i + \sum_{i=1}^{N_2} m_i^{(2)}\mathbf{r}_i = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}\ , \tag{73}$$

whence it follows

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \frac{m_1\dot{m}_2 - \dot{m}_1 m_2}{m^2}\mathbf{r}_{21} = \mathbf{0}\ . \tag{74}$$

Substitution of (71) into (69) gives respectively

$$\sum_{i=1}^{N_K} m_i^{(K)}\mathbf{r}_i^{(K)} = \mathbf{0}\ , \quad \sum_{i=1}^{N_K}(m_i^{(K)}\mathbf{v}_i^{(K)} + \dot{m}_i^{(K)}\mathbf{r}_i^{(K)}) = \mathbf{0}\ . \tag{75}$$

Now taking into account relations (70) and (75) expressions (61) for partial moments of momentum look like

$$\mathbf{J}_K = \mathbf{J}_{\mathrm{M}K} + \mathbf{s}_K = \mathbf{L}_K + \mathbf{J}_K^{int} + \mathbf{s}_K\ , \tag{76}$$

where

$$\mathbf{J}_{\mathrm{M}K} = \mathbf{L}_K + \mathbf{J}_K^{int} = m_K[\mathbf{R}_K \times \mathbf{V}_K] + [\mathbf{R}_K \times \sum_{i=1}^{N_K} m_i^{(K)}\mathbf{v}_i^{(K)}]\ , \tag{77}$$

is a moment of momentum of the center of mass of $K$-th subsystem relative to the origin,

$$\mathbf{s}_K = \sum_{i=1}^{N_K} m_i^{(K)} [\mathbf{r}_i^{(K)} \times \mathbf{v}_i^{(K)}] \ , \tag{78}$$

is proper moment of momentum (spin) of $K$-th subsystem (i.e. total moment of momentum of all points of $K$-th subsystem relative to its center of mass, see formula (13)),

$$\mathbf{J}_K^{int} = [\mathbf{R}_K \times \sum_{i=1}^{N_K} m_i^{(K)} \mathbf{v}_i^{(K)}] \ , \tag{79}$$

is a moment of internal momentum of $K$-th subsystem relative to the origin of coordinates, which according to (75) vanishes, if effective masses of points of subsystem do not depend on time.

Substitution of (72) into (77) gives

$$\mathbf{J}_{\mathrm{M}K} = \mathbf{L}_K + \mathbf{L}_{\mathrm{M}K} + m_K [\mathbf{r}_K \times \mathbf{V}] \ , \tag{80}$$

where

$$\mathbf{L}_K = m_K [\mathbf{R} \times \mathbf{V}_K] + [\mathbf{R} \times \sum_{i=1}^{N_K} m_i^{(K)} \mathbf{v}_i^{(K)}] \ , \tag{81}$$

is orbital angular momentum of $K$-th subsystem relative to the origin,

$$\mathbf{L}_{\mathrm{M}K}^{(0)} = m_K [\mathbf{r}_K \times \mathbf{v}_K] \ , \tag{82}$$

$$\mathbf{L}_{\mathrm{M}K} = \mathbf{L}_{\mathrm{M}K}^{(0)} - [\mathbf{r}_K \times \sum_{i=1}^{N_K} \dot{m}_i^{(K)} \mathbf{r}_i^{(K)}] \ , \tag{83}$$

are orbital angular momenta of $K$-th subsystem relative to the center of mass M of the whole system with and without taking into account of dependence of effective masses on time, respectively.

Substitution of (76) and (80) into (60) and using relations (73), (74), gives rise to

$$\mathbf{J} = \mathbf{J}_{\mathrm{M}1} + \mathbf{s}_1 + \mathbf{J}_{\mathrm{M}2} + \mathbf{s}_2 = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_{\mathrm{M}1} + \mathbf{L}_{\mathrm{M}2} + \mathbf{s}_1 + \mathbf{s}_2 \ . \tag{84}$$

Quantity

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = m[\mathbf{R} \times \mathbf{V}] + [\mathbf{R} \times \left( \sum_{i=1}^{N_1} m_i^{(1)} \mathbf{v}_i^{(1)} + \sum_{i=1}^{N_2} m_i^{(2)} \mathbf{v}_i^{(2)} \right) \tag{85}$$

represents orbital angular momentum of the system relative to origin of coordinates O, whereas

$$\mathbf{s} = \sum_{i=1}^{N} [\mathbf{r}_i \times \mathbf{v}_i] = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{L}_{\mathrm{M}1} + \mathbf{L}_{\mathrm{M}2} = \mathbf{s}_1 + \mathbf{s}_2 + m_1 [\mathbf{r}_1 \times \mathbf{v}_1] + m_2 [\mathbf{r}_2 \times \mathbf{v}_2] \tag{86}$$

is spin, or proper moment of momentum of the system, i.e. total moment of momentum of all points of the system relative to its center of mass M.

Relative coordinates and velocity of the center of mass of $J$-th subsystem relative to the center of mass of $K$-th subsystem are

$$\mathbf{r}_{JK} = \mathbf{r}_J - \mathbf{r}_K = \mathbf{R}_J - \mathbf{R}_K \ , \quad \mathbf{v}_{JK} = \mathbf{v}_J - \mathbf{v}_K = \mathbf{V}_J - \mathbf{V}_K \ . \tag{87}$$

Taking into account relations (73), (74), we obtain from (86) finally

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2 + \frac{m_1 m_2}{m}[\mathbf{r}_{21} \times \mathbf{v}_{21}] \ . \tag{88}$$

Comparison of this expression with relation (41) is reduced to relations

$$\mathbf{S}^{int}_{21} + \mathbf{S}^{int}_{12} = (\varsigma - \varsigma_1)\mathbf{s}_1 + (\varsigma - \varsigma_2)\mathbf{s}_2 + \varsigma\frac{m_1 m_2}{m}[\mathbf{r}_{21} \times \mathbf{v}_{21}] \ , \tag{89}$$

$$\mathbf{C}^{int}_{21} + \mathbf{C}^{int}_{12} = (\varsigma_1\Omega^2_{01} - \varsigma\Omega^2_0)\mathbf{s}_1 + (\varsigma_2\Omega^2_{02} - \varsigma\Omega^2_0)\mathbf{s}_2 - \varsigma\frac{m_1 m_2 \Omega^2_0}{m}[\mathbf{r}_{21} \times \mathbf{v}_{21}] \ . \tag{90}$$

Quantity

$$\mathbf{l} = \frac{m_1 m_2}{m}[\mathbf{r}_{21} \times \mathbf{v}_{21}] \tag{91}$$

is an orbital angular momentum characterizing the relative motion of subsystems.

Expressions (89)-(91) concern to both objects and are symmetric relative to their permutation. Therefore $\mathbf{S}^{int}_{JK}$ and $\mathbf{C}^{int}_{JK}$ may be represented as sums of symmetric and antisymmetric terms

$$\mathbf{S}^{int}_{JK} = \frac{1}{2}(\varsigma_{JK} - \varsigma_J)\mathbf{s}_J + \frac{1}{2}(\varsigma_{JK} - \varsigma_K)\mathbf{s}_K + \varsigma_{JK}\frac{m_J m_K}{2m_{JK}}[\mathbf{r}_{JK} \times \mathbf{v}_{JK}] + \mathbf{S}^{int}_{[JK]} \ , \tag{92}$$

$$\mathbf{C}^{int}_{JK} = \frac{1}{2}(\varsigma_J\Omega^2_{0J} - \varsigma_{JK}\Omega^2_{0JK})\mathbf{s}_J + \frac{1}{2}(\varsigma_K\Omega^2_{0K} - \varsigma_{JK}\Omega^2_{0JK})\mathbf{s}_K -$$
$$-\varsigma_{JK}\frac{m_J m_K \Omega^2_{0JK}}{m_{JK}}[\mathbf{r}_{JK} \times \mathbf{v}_{JK}] + \mathbf{C}^{int}_{[JK]} \ , \tag{93}$$

where $m_{JK} = m_J + m_K$, $\varsigma_K$ and $\varsigma_{JK}$ are constants associated both with $K$-th subsystem and system composed from $K$-th and $J$-th subsystem, respectively.

Let's try to determine antisymmetric terms $\mathbf{S}^{int}_{[JK]}$ and $\mathbf{C}^{int}_{[JK]}$ basing on following arguments. In the process of evolution the state of system changes from some initial state to finite one, in which spins of subsystems $\mathbf{s}_K$ are oriented relative to each other by some definite way. For example, $\mathbf{s}_2 = +\mathbf{s}_1$, i.e. $\Delta\mathbf{s} = \mathbf{s}_2 - \mathbf{s}_1 = \mathbf{0}$, in finite state of electron-positron system corresponds to orthopositronium, and at $\mathbf{s}_2 = -\mathbf{s}_1$, i.e. $\Delta\mathbf{s} = \mathbf{s}_2 - \mathbf{s}_1 = -2\mathbf{s}_1$, the finite state is parapositronium. Thus, variation of difference $\Delta\mathbf{s}_{JK} = -\Delta\mathbf{s}_{KJ} = \mathbf{s}_J - \mathbf{s}_K$ with time characterizes variation of relative orientation of spin of constituents. From (69), (78), (83) we obtain

$$\Delta\mathbf{s}_{JK} = \mathbf{s}_J - \mathbf{s}_K = \sum_{i=1}^{N_J} m^{(J)}_i[\mathbf{r}^{(J)}_i \times \mathbf{v}^{(J)}_i] - \sum_{i=1}^{N_K} m^{(K)}_i[\mathbf{r}^{(K)}_i \times \mathbf{v}^{(K)}_i] =$$

$$= \sum_{i=1}^{N_J} m_i^{(J)}[\mathbf{r}_i \times \mathbf{v}_i] - \sum_{i=1}^{N_K} m_i^{(K)}[\mathbf{r}_i \times \mathbf{v}_i] - m_J[\mathbf{r}_J \times \mathbf{v}_J] + m_K[\mathbf{r}_K \times \mathbf{v}_K] =$$

$$= \mathbf{j}_{\mathrm{M}J} - \mathbf{j}_{\mathrm{M}K} - \mathbf{L}_{\mathrm{M}J}^{(0)} + \mathbf{L}_{\mathrm{M}K}^{(0)} \ , \qquad (94)$$

where

$$\mathbf{j}_{\mathrm{M}K} = \sum_{i=1}^{N_K} m_i^{(K)}[\mathbf{r}_i \times \mathbf{v}_i] = \mathbf{L}_{\mathrm{M}K}^{(0)} + \mathbf{s}_K \ , \qquad (95)$$

is a moment of momentum of $K$-th subsystem relative to the center of mass M of the whole system.

We find from (88), (91), (94)

$$\mathbf{s}_1 = \frac{\mathbf{s} - \mathbf{l} - \Delta\mathbf{s}}{2} \ , \quad \mathbf{s}_2 = \frac{\mathbf{s} - \mathbf{l} + \Delta\mathbf{s}}{2} \ . \qquad (96)$$

Now, writing down the equation (56) in terms of the relative variables and the center-of-mass variables and assuming it to be covariant under Galileo transformations, one may come to conclusion, that following relations

$$\frac{d}{dt}\left[\frac{\partial u}{\partial \mathbf{V}} + \varsigma[\mathbf{s} \times \mathbf{W}]\right] + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{V}] = \mathbf{0} \ , \qquad (97)$$

$$\mathbf{S}_{[21]}^{int} = \frac{1}{2}(\varsigma_2\mathbf{s}_2 - \varsigma_1\mathbf{s}_1) = \frac{\varsigma_1 + \varsigma_2}{4}\Delta\mathbf{s} + \frac{\varsigma_2 - \varsigma_1}{4}(\mathbf{s} - \mathbf{l}) \ , \qquad (98)$$

$$\mathbf{C}_{[21]}^{int} = \frac{1}{2}(\varsigma_1\Omega_{01}^2\mathbf{s}_1 - \varsigma_2\Omega_{02}^2\mathbf{s}_2) =$$

$$= -\frac{\varsigma_1\Omega_{01}^2 + \varsigma_2\Omega_{02}^2}{4}\Delta\mathbf{s} + \frac{\varsigma_1\Omega_{01}^2 - \varsigma_2\Omega_{02}^2}{4}(\mathbf{s} - \mathbf{l}) \ . \qquad (99)$$

should be fulfilled.

Then the equations of motion (54), (56), describing six translational degrees of freedom, will take the following form

$$\frac{d\mathbf{P}}{dt} + \frac{\partial U}{\partial \mathbf{R}} + \varsigma\Omega_0^2[\mathbf{s} \times \mathbf{V}] - [\mathbf{C}^{ext} \times \mathbf{V}] =$$

$$= \frac{d}{dt}\left[m_0\mathbf{V} - \frac{\partial(U + u)}{\partial \mathbf{V}} + [\mathbf{S}^{ext} \times \mathbf{W}]\right] + \frac{\partial U}{\partial \mathbf{R}} - [\mathbf{C}^{ext} \times \mathbf{V}] = \mathbf{0} \ , \qquad (100)$$

$$\frac{d}{dt}\left[\frac{m_{01}m_{02}}{m_0}\mathbf{v} - \frac{\partial U^{int}}{\partial \mathbf{v}} + \frac{1}{4}\varsigma[\mathbf{s} \times \mathbf{w}] + \frac{m_{01}m_{02}}{m_0^2}[\mathbf{S}^{ext} \times \mathbf{w}]\right] +$$

$$+ \frac{\partial U^{int}}{\partial \mathbf{r}} - \frac{m_{01}m_{02}}{m_0}[\mathbf{C}^{ext} \times \mathbf{v}] + \frac{1}{4}\varsigma\Omega_0^2[\mathbf{s} \times \mathbf{v}] = \mathbf{0} \ , \qquad (101)$$

where the momentum of the system is given by expression (23). Equation (100) shows, that the system in question moves in such a way as if it has no internal degrees of freedom, but interaction with external fields is specified not by potential function $U$, but function $U + u$.

Substitution of relations (88), (92), (98) into (55) gives following expression for the relative momentum

$$\mathbf{p} = \frac{m_{02} - m_{01}}{m_0}\left[m_0\mathbf{V} - \frac{\partial(U+u)}{\partial\mathbf{V}} + [\mathbf{S}^{ext}\times\mathbf{W}]\right] - 2\frac{\partial U^{int}}{\partial\mathbf{v}} +$$
$$+ \frac{2m_{01}m_{02}}{m_0^2}(m_0\mathbf{v} + [\mathbf{S}^{ext}\times\mathbf{w}]) + \frac{1}{2}\varsigma[\mathbf{s}\times\mathbf{w}] = m_2\mathbf{V}_{C2} - m_1\mathbf{V}_{C1}\,. \tag{102}$$

It follows from here that the relative momentum depends on the state of motion not only of constituents but also of its center of mass.

For a complete solution of the two-body problem the spin equations of motion should be added to equations (100), (101) that will be considered in the next section.

# 6 Spin equations of motion

Equations of motion for spins may be written down in accordance with (12) as

$$\frac{d\mathbf{s}_K}{dt} = [\mathbf{\Omega}_{\mathbf{V}}(t)\times\mathbf{s}_K] + \mathbf{m}_K(t)\ , \tag{103}$$

$$\frac{d\mathbf{s}}{dt} = [\mathbf{\Omega}_{\mathbf{V}}(t)\times\mathbf{s}] + \mathbf{m}(t)\ , \tag{104}$$

$$\frac{d\Delta\mathbf{s}}{dt} = [\mathbf{\Omega}_{\mathbf{V}}(t)\times\Delta\mathbf{s}] + \Delta\mathbf{m}(t)\ , \tag{105}$$

where $\Delta\mathbf{m}(t) = \mathbf{m}_2 - \mathbf{m}_1 \neq \mathbf{0}$, as the relative direction of spins of interacting subsystems can change.

On the other hand, differentiation of spin (88) with respect to time gives

$$\frac{d\mathbf{s}}{dt} = \frac{d\mathbf{s}_1}{dt} + \frac{d\mathbf{s}_2}{dt} + \frac{d}{dt}\left(\frac{m_1m_2}{m}[\mathbf{r}\times\mathbf{v}]\right)\ , \tag{106}$$

or

$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 + \frac{d\mathbf{l}}{dt} - [\mathbf{\Omega}_{\mathbf{V}}\times\mathbf{l}]\ . \tag{107}$$

This relation suggests that pseudo-vectors $\mathbf{m}$, $\mathbf{m}_K$ should have identical structure. Thus we assume that

$$\mathbf{m} = \mathbf{m}_1 - \frac{1}{2}\Delta\mathbf{m} = \mathbf{m}_2 + \frac{1}{2}\Delta\mathbf{m} = -\frac{d\mathbf{l}}{dt} + [\mathbf{\Omega}_{\mathbf{V}}\times\mathbf{l}]\ . \tag{108}$$

Hence, equations of motion (104) may be finally written down in the form

$$\frac{d}{dt}(\mathbf{s}+\mathbf{l}) = [\mathbf{\Omega}_{\mathbf{V}}\times(\mathbf{s}+\mathbf{l})]\ , \tag{109}$$

whence it follows that pseudo-vector $\mathbf{s} + \mathbf{l}$ is constant over module. Thus, system of equations (100)-(101) and (105), (109) is sufficient for description of all movements of the system of two objects with internal degrees of freedom, where angular velocity of precession $\mathbf{\Omega_V}$ is constant, if dynamical momentum (23) is conserved.

It is necessary to notice that the equation of motion of spin of the system in the form (109) can take place only in the case when in the system one may distinguish two subsystems divided by a relative radius-vector $\mathbf{r}$. If the system is imagined as indivisible "atomic object, for which introduction of relative variables has no sense, then in the system of two vector equations, (100)-(101), and two pseudo-vector equations, (105), (109), two equations, (100) and (109), (with $\mathbf{m}(t) = \mathbf{0}$), are independent, whereas equations (101) and (105) lose meaning because of $\Delta\mathbf{s} = \mathbf{0}$, $U^{int} = 0$, $u(t; \mathbf{V}, ...) = 0$, $m_{02} = m_{01}$, $\mathbf{v} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$. In this case spin equation of motion (107) takes the standard form

$$\frac{d\mathbf{s}}{dt} = [\mathbf{\Omega_V}(t) \times \mathbf{s}] \ . \tag{110}$$

The similar situation takes place for the object represented as a set of noninteracting mass-points (with or without of internal degrees of freedom). The equations of motion of such object are (100) and (110).

# 7 Energy of system

In modern physics the conservation energy law has fundamental meaning. It is shown in [11]- [13] that possible explicit dependence of potential function on time and accelerations of the higher order leads to violation of this law for separately taken non-inertial object even if its internal degrees of freedom are not considered. Let us consider a problem of the energy of the system in question in detail.

If the energy conservation law takes place, but is broken for separately taken object, it means that the object in question is an open system interacting with its environment. Increment of energy of the object is compensated by decrease of energy of external medium so that total energy of object and medium remains constant. These general intuitive reasoning may be illustrated by example of two interacting objects with internal degrees of freedom.

For one of two objects, both environment and second object are external. Equations of motion for objects are given by (1), where expressions for momenta of subsystems and forces acting on them, are obtained by substitution of expressions (17)-(18), (47), (52)-(53), (92)-(93), (98)-(99) into (19)-(22). It gives

$$\mathbf{P}_K = m_{0K}\mathbf{V}_K - \frac{\partial(U + u + U^{int})}{\partial \mathbf{V}_K} + [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{W}_K] = m_K\mathbf{V}_K \ , \tag{111}$$

$$\mathbf{F}_K = -\frac{\partial(U + u + U^{int})}{\partial \mathbf{R}_K} + [(\frac{m_{0K}}{m_0}\mathbf{C}^{ext} - \frac{1}{2}\varsigma\Omega_0^2\mathbf{s}) \times \mathbf{V}_K] \ . \tag{112}$$

The equation of energy balance, which energy conservation as a special case follows from, is due to equations of motion (1) which for the system of two objects with interior degrees of freedom are reduced to equations (97), (100) and (101). All these equations have identical structure and lead to following equations of energy balance.

It follows from (1)

$$\frac{dE_K}{dt} = \frac{\partial(U+u+U^{int})}{\partial t} + \sum_{k=0}^{N}\left(\frac{\partial(U+u+U^{int})}{\partial \mathbf{W}_K^{(k)}} \cdot \mathbf{W}_K^{(k+1)}\right) , \tag{113}$$

where

$$E_K = \frac{m_{0K}\mathbf{V}_K^2}{2} + (\mathbf{V}_K \cdot [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{W}_K] -$$

$$-(\mathbf{V}_K \cdot \frac{\partial(U+u+U^{int})}{\partial \mathbf{V}_K}) + U + u + U^{int} , \tag{114}$$

is total energy of $K$-th subsystem in absolute reference frame. On account of relations (33), (35) the energy (114) may be represented as

$$E_K = E_{\mathrm{M}K} + E_{0K} , \tag{115}$$

where

$$E_{\mathrm{M}K} = \frac{m_{0K}\mathbf{V}^2}{2} + (\mathbf{V} \cdot [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{W}] - (\mathbf{V} \cdot \frac{\partial(U+u+U^{int})}{\partial \mathbf{V}_K}) +$$

$$+U + u + m_{0K}(\mathbf{V} \cdot \mathbf{v}_K) + (\mathbf{v}_K \cdot [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{W}] +$$

$$+(\mathbf{V} \cdot [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{w}_K] - \frac{m_{0K}}{m_0}(\mathbf{v}_K \cdot \frac{\partial(U+u)}{\partial \mathbf{V}}) , \tag{116}$$

$$E_{0K} = \frac{m_{0K}\mathbf{v}_K^2}{2} + (\mathbf{v}_K \cdot [(\frac{1}{2}\varsigma\mathbf{s} + \frac{m_{0K}}{m_0}\mathbf{S}^{ext}) \times \mathbf{w}_K] - (-1)^K(\mathbf{v}_K \cdot \frac{\partial U^{int}}{\partial \mathbf{v}}) + U^{int} , \tag{117}$$

is energy of $K$-th subsystem in the reference frame of the center of mass of the whole system. It is obvious that in the center-of-mass reference frame ($\mathbf{V} = \mathbf{0}$, $\mathbf{W} = \mathbf{0}$, ...) the energy of K-th subsystem is $E_K = E_{0K}$.

On the other hand, equations (97), (100) and (101) lead to

$$\frac{dE_\nu}{dt} = -\frac{\partial u}{\partial t} - \sum_{k=0}^{N}\left(\frac{\partial u}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)}\right) , \tag{118}$$

$$\frac{dE}{dt} = -\frac{dE_\nu}{dt} + \frac{\partial U}{\partial t} + \sum_{k=0}^{N}\left(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)}\right) , \tag{119}$$

$$\frac{dE_{\mathrm{r}}}{dt} = \frac{\partial U^{int}}{\partial t} + \sum_{k=0}^{N}\left(\frac{\partial U^{int}}{\partial \mathbf{w}^{(k)}} \cdot \mathbf{w}^{(k+1)}\right) , \tag{120}$$

where on account of (47)-(48)

$$E = \frac{m_0 \mathbf{V}^2}{2} + (\mathbf{V} \cdot [\mathbf{S}^{ext} \times \mathbf{W}] - (\mathbf{V} \cdot \frac{\partial (U+u)}{\partial \mathbf{V}}) + U + u =$$

$$= \frac{m_0 \mathbf{V}^2}{2} + (\mathbf{V} \cdot [(\varsigma \mathbf{s} + \mathbf{S}^{ext}) \times \mathbf{W}] - (\mathbf{V} \cdot \frac{\partial U}{\partial \mathbf{V}}) + U - E_\nu \qquad (121)$$

is total energy of the system,

$$E_\mathrm{r} = \frac{m_{01} m_{02}}{2 m_0} \mathbf{v}^2 + \frac{\varsigma}{4} (\mathbf{v} \cdot [\mathbf{s} \times \mathbf{w}]) + \frac{m_{01} m_{02}}{m_0^2} (\mathbf{v} \cdot [\mathbf{S}^{ext} \times \mathbf{w}]) - (\mathbf{v} \cdot \frac{\partial U^{int}}{\partial \mathbf{v}}) + U^{int} \qquad (122)$$

is total energy of relative movement in the system,

$$E_\nu = \varsigma (\mathbf{V} \cdot [\mathbf{s} \times \mathbf{W}] + (\mathbf{V} \cdot \frac{\partial u}{\partial \mathbf{V}}) - u \qquad (123)$$

is additional energy arising because of internal degrees of freedom.

It is not difficult to show that total energy (121) is expressed in terms of $E_1$, $E_2$, $E_\nu$, and $E_\mathrm{r}$ in the following way

$$E = \mu_1 E_1 + \mu_2 E_2 + u - \mathcal{E} \ , \qquad (124)$$

where

$$\mu_K = \frac{m_0 m_{0K}}{m_{01}^2 + m_{02}^2} \ , \qquad (125)$$

$$\mathcal{E} = \frac{2 m_{01} m_{02} (U + E_\mathrm{r}) + m_0^2 u}{m_{01}^2 + m_{02}^2} + U^{int} - \frac{m_0 (m_{02} - m_{01})}{m_{01}^2 + m_{02}^2} (\mathbf{V} \cdot \frac{\partial U^{int}}{\partial \mathbf{v}}) +$$

$$+ \frac{\varsigma m_0^2}{2 (m_{01}^2 + m_{02}^2)} (\mathbf{V} \cdot [\mathbf{s} \times \mathbf{W}]) - \frac{m_{01} m_{02} (m_{02} - m_{01})}{m_0 (m_{01}^2 + m_{02}^2)} (\mathbf{v} \cdot \frac{\partial (U+u)}{\partial \mathbf{V}}) + \qquad (126)$$

$$+ \frac{m_{01} m_{02} (m_{02} - m_{01})}{m_0 (m_{01}^2 + m_{02}^2)} \left[ m_0 (\mathbf{V} \cdot \mathbf{v}) + (\mathbf{v} \cdot [\mathbf{S}^{ext} \times \mathbf{W}]) + (\mathbf{V} \cdot [\mathbf{S}^{ext} \times \mathbf{w}]) \right] \ .$$

In the center-of-mass reference frame of the whole system relation (124) reduces to

$$E_0 - u(t; \mathbf{0}, \mathbf{0}, ...) = \mu_1 E_{01} + \mu_2 E_{02} - \mathcal{E}_0 \ , \qquad (127)$$

where

$$\mathcal{E}_0 = \frac{m_{01} m_{02}}{m_{01}^2 + m_{02}^2} \left[ 2 (U + E_\mathrm{r}) - \frac{(m_{02} - m_{01})}{m_0} (\mathbf{v} \cdot \frac{\partial (U+u)}{\partial \mathbf{V}}) \right]_{\mathbf{V}=\mathbf{0}, \mathbf{W}=\mathbf{0}, ...} +$$

$$+ \frac{m_0^2 u}{m_{01}^2 + m_{02}^2} + U^{int} \ . \qquad (128)$$

For free system ($U = 0$, $\mathbf{S}^{ext} = \mathbf{0}$) equation (119) in the center-of-mass reference frame reduces to $d(E_0 - u)/dt = 0$, that gives $E_0 - u = \mu_1 E_{01} + \mu_2 E_{02} - \mathcal{E}_0 = \mathrm{const}$. It does not follow from equations of motion (1) that energies (117) will be conserved, as the interaction potential $U^{int}$, generally speaking, can depend on time. Then, for example, if $dE_{01}/dt > 0$, i.e. energy $E_{01}$ increases, function $\mu_2 E_{02} - \mathcal{E}_0$ should decrease, because $E_{01}$ cannot increase infinitely. Since both subsystems are in equivalent positions, energy $E_{02}$ behaves similarly. Thus, energies of interacting objects should oscillate with time. It means that potential energy $U^{int}$ of interaction also should be oscillatory function of time. Thereby interaction between objects is reduced to permanent energy exchange. Moreover, presence of function $u(t; \mathbf{V}, \mathbf{W}, ...)$, apparently, allows to describe the high-energy interaction considered usually from the point of view of relativistic quantum theories. Determination of explicit dependence of interaction energy and an ascertainment of a role of function $u(t; \mathbf{V}, \mathbf{W}, ...)$ requires separate careful consideration.

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