

Generalized Dynamics of the Mass Point with Internal Degrees of Freedom

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An equation of motion of the mass point with internal degrees of freedom in scalar potential U depending on relative coordinates and time, velocity and accelerations is obtained both for non-relativistic and relativistic case. In non-relativistic case a generalization of the energy conservation law follows, if $\partial U/\partial t = 0$ fulfilled. A concept of work is generalized to relativistic case, leading to corresponding integral of motion, if $\partial U/\partial \tau = 0$ fulfilled, where τ is proper time of the point. In neglecting an internal degrees of freedom and absence of interaction this integral of motion gives standard Special Relativity.

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I. EQUATION OF MOTION AND THE ENERGY CONSERVATION

A long period of supremacy of quantum theories did not crush an interest in classical description of quantum systems. In this connect some conclusions, following from the basic equation of dynamics, the Second Newton's Law, should be noted. As it is well known from the Helmholtz epoch ([1]), the Second Newton's Law for conservative systems

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \quad (\text{I.1})$$

gives a force acting at the mass point in the form $\mathbf{F} = -\nabla U = -\partial U/\partial \mathbf{R}$, where $U = U(\mathbf{R})$ is potential function of coordinate of the mass point. As a result Eq.(I.1) and a definition of elementary work

$$dA = (\mathbf{F} \cdot d\mathbf{R}) = \left(\frac{d\mathbf{P}}{dt} \cdot d\mathbf{R}\right) = (\mathbf{V} \cdot d\mathbf{P}), \quad (\text{I.2})$$

where \mathbf{R} and $\mathbf{V} = d\mathbf{R}/dt$ are respectively radius vector and velocity of the mass point relative to origin of coordinate system, coupled with absolute rest reference frame (r.f.), give a conservation of total mechanical energy

$$E = \frac{m\mathbf{V}^2}{2} + U(\mathbf{R}). \quad (\text{I.3})$$

As soon as absolute r.f. be coupled with any physical object, one can say that the motion of mass point takes place in the field, created by this object and characterized by potential function $U(\mathbf{R})$.

It is clear from the common considerations that a motion of mass point in the field of some object should determined by potential function depending not only on relative coordinates \mathbf{R} , but also at least on relative velocity \mathbf{V} and accelerations, as well as on time, so that

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$U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, \dots, \mathbf{W}^{(N)})$, where $\mathbf{W}^{(k)} = d^k \mathbf{W} / dt^k$, a time dependence being specified by internal dynamics of the mentioned object. Remember W.Weber ([2]- [3]), who tried to explain electrical phenomena as a result of electric interaction of elementary particles, so called *electric atoms*, depending both on their relative disposition \mathbf{R} and on their relative velocity \mathbf{V} and acceleration $\mathbf{W} = d\mathbf{V}/dt$.

In this case corollaries $\mathbf{F} = -\nabla U$ and (I.3) from equation of motion (I.1) should be changed forasmuch as total differential of the function U is

$$dU = \frac{\partial U}{\partial t} dt + \left(\frac{\partial U}{\partial \mathbf{R}} \cdot d\mathbf{R} \right) + \left(\frac{\partial U}{\partial \mathbf{V}} \cdot d\mathbf{V} \right) + \sum_{k=0}^N \left(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot d\mathbf{W}^{(k)} \right). \quad (\text{I.4})$$

Indeed, definition of elementary work of the force (I.2) gives more general expression for force, namely

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{R}} + [\mathbf{C} \times \mathbf{V}], \quad (\text{I.5})$$

where \mathbf{C} is some pseudo-vector, inherent in mass point, and additional term $[\mathbf{C} \times \mathbf{V}]$ having a sense of gyroscopic force. As far back as Helmholtz in his work "On the conservation of force" ([1]; Addition 3) pointed out at formula (I.6).

Furthermore, when interaction takes place the momentum vector \mathbf{P} has a meaning of dynamical momentum. It can be written as a sum of kinematical momentum $m\mathbf{V}$ and some addition \mathbf{A} (a vector potential), connected both with internal structure of mass point, and with interaction

$$\mathbf{P} = m\mathbf{V} + \mathbf{A}. \quad (\text{I.6})$$

Then Eqs.(I.2) and (I.6) give

$$dA = (\mathbf{F} \cdot d\mathbf{R}) = (\mathbf{V} \cdot d(m\mathbf{V} + \mathbf{A})) = d \left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V}) \right) - (\mathbf{A} \cdot d\mathbf{V}), \quad (\text{I.7})$$

or

$$\begin{aligned} & d \left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V}) \right) + \left(\frac{\partial U}{\partial \mathbf{R}} \cdot d\mathbf{R} \right) - (\mathbf{A} \cdot d\mathbf{V}) = \\ & = d \left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V}) + U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, \dots, \mathbf{W}^{(N)}) \right) - \\ & - \frac{\partial U}{\partial t} dt - \left(\frac{\partial U}{\partial \mathbf{V}} \cdot d\mathbf{V} \right) - (\mathbf{A} \cdot d\mathbf{V}) - \sum_{k=0}^N \left(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot d\mathbf{W}^{(k)} \right) = 0. \end{aligned} \quad (\text{I.8})$$

Now, if one suppose

$$\mathbf{A} = -\frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}], \quad (\text{I.9})$$

where \mathbf{S} is some pseudo-vector, coupled with the mass point considered, \mathbf{W} is an acceleration of this point, then dynamical momentum (I.6) will get an expression

$$\mathbf{P} = m\mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}], \quad (\text{I.10})$$

whereas it follows from Eq.(I.8)

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} + \sum_{k=0}^N \left(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)} \right), \quad (\text{I.11})$$

where quantity

$$E = \frac{m\mathbf{V}^2}{2} + (\mathbf{V} \cdot [\mathbf{S} \times \mathbf{W}]) - (\mathbf{V} \cdot \frac{\partial U}{\partial \mathbf{V}}) + U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, \dots, \mathbf{W}^{(N)}) \quad (\text{I.12})$$

is a generalization of Eq.(I.3) for total mechanical energy. So, apart from standard kinetic and potential energies an additional energy arises due to both internal degrees of freedom and a dependence of potential energy on relative velocity.

Provided the condition

$$\frac{\partial U}{\partial t} + \sum_{k=0}^N \left(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)} \right) = 0 \quad (\text{I.13})$$

is fulfilled, the energy (I.12) will be an integral of motion. Condition $dE/dt > 0$ corresponds to absorption of energy by a mass point, and $dE/dt < 0$ corresponds to radiation of energy.

In view of stated above, the equation of motion (I.1) should be written down in the form

$$\frac{d}{dt} (m\mathbf{V} + [\mathbf{S} \times \mathbf{W}]) - [\mathbf{C} \times \mathbf{V}] = \frac{d}{dt} \frac{\partial U}{\partial \mathbf{V}} - \frac{\partial U}{\partial \mathbf{R}}. \quad (\text{I.14})$$

Let's note here that derivatives of potential function with respect to accelerations $\mathbf{W}^{(k)}$ do not enter into an equation of motion. Therefore one can be restricted to dependence of potential function only on acceleration \mathbf{W} : $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W})$.

Equation (I.14) gives a number of special cases.

1. First Newton's law ($U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}) = 0$, $\mathbf{V} = \mathbf{const}$) takes place, if the relation

$$\frac{d}{dt} [\mathbf{S} \times \mathbf{W}] = [\mathbf{C} \times \mathbf{V}], \quad (\text{I.15})$$

including also trivial absence of internal structure, $\mathbf{C} = \mathbf{0}$, $\mathbf{S} = \mathbf{0}$, is fulfilled.

2. If the force (I.5), acting at a mass point, becomes zero, i.e. the relation

$$\frac{\partial U}{\partial \mathbf{R}} = [\mathbf{C} \times \mathbf{V}] \quad (\text{I.16})$$

is fulfilled, then dynamical momentum (I.10) will be a conserved vector.

If besides the relation

$$\frac{\partial U}{\partial \mathbf{V}} = [\mathbf{S} \times \mathbf{W}], \quad (\text{I.17})$$

is fulfilled, then dynamical momentum coincides with kinematical momentum and there take place uniform and rectilinear motion.

For free mass point, when interaction is neglected, $U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}) = 0$, conservation law of dynamical momentum takes place only if $[\mathbf{C} \times \mathbf{V}] = \mathbf{0}$, and uniform and rectilinear motion is a result of additional condition $[\mathbf{S} \times \mathbf{W}] = \mathbf{0}$.

3. If $\partial U / \partial \mathbf{R} = \mathbf{0}$ and the relation

$$[\mathbf{C} \times \left([\mathbf{S} \times \mathbf{W}] - \frac{\partial U}{\partial \mathbf{V}} \right)] = 0 \quad (\text{I.18})$$

takes place, equation of motion (I.14) takes the form

$$\frac{d}{dt} \left(m\mathbf{V} + [\mathbf{S} \times \mathbf{W}] - \frac{\partial U}{\partial \mathbf{V}} \right) - \frac{1}{m} [\mathbf{C} \times \left(m\mathbf{V} + [\mathbf{S} \times \mathbf{W}] - \frac{\partial U}{\partial \mathbf{V}} \right)] = 0, \quad (\text{I.19})$$

from which it follows, that dynamical momentum (I.10) is precessing round a direction of pseudovector \mathbf{C} with angular velocity $\omega = \mathbf{C}/m$.

In general case the relation (I.15) or relations (I.16) and (I.17) necessary for performance of the first Newton's law, may not be satisfied. Therefore for mass points with internal degrees of freedom the inertia law in the form in which it has been formulated by Galilei and Newton, has no place and cannot be accepted as the first principle underlying mechanics. One can generalize it in the following way: *Mass point (body) with internal degrees of freedom, given to itself, moves according to equation (I.15) in which $U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}) = 0$.*

II. THE EQUATION OF MOMENTS FOR A MASS POINT WITH INTERNAL DEGREES OF FREEDOM

The equation (I.14) is insufficient for description of dynamics of physical system. There is necessary in addition an equation of moments, which for structureless mass point looks like $d\mathbf{L}/dt = \mathbf{M}$, where $\mathbf{L} = [\mathbf{R} \times \mathbf{P}] = m[\mathbf{R} \times \mathbf{V}]$ is angular momentum, $\mathbf{M} = [\mathbf{R} \times \mathbf{F}]$ is total moment of external forces, acting at the system. For individual mass point equation of moments follows from the Eq.(I.1).

For a mass point with internal degrees of freedom, describing by Eq.(I.1), in which force and momentum are specified by equations (I.5) and (I.10), respectively, we have the relation

$$[\mathbf{R} \times \frac{d\mathbf{P}}{dt}] = \frac{d}{dt} [\mathbf{R} \times \mathbf{P}] - [\mathbf{V} \times \left(-\frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}] \right)] = -[\mathbf{R} \times \frac{\partial U}{\partial \mathbf{R}}] + [\mathbf{R} \times [\mathbf{C} \times \mathbf{V}]], \quad (\text{II.1})$$

implying the following equation of moments

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} + \mathbf{T}, \quad (\text{II.2})$$

where

$$\mathbf{L} \doteq [\mathbf{R} \times \mathbf{P}] = m[\mathbf{R} \times \mathbf{V}] - [\mathbf{R} \times \frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{R} \times [\mathbf{S} \times \mathbf{W}]] \quad (\text{II.3})$$

is a dynamical angular momentum,

$$\mathbf{M} \doteq [\mathbf{R} \times \mathbf{F}] = -[\mathbf{R} \times \frac{\partial U}{\partial \mathbf{R}}] + [\mathbf{R} \times [\mathbf{C} \times \mathbf{V}]] \quad (\text{II.4})$$

is a moment of force, acting at the mass point,

$$\mathbf{T} \doteq [\mathbf{V} \times \mathbf{P}] = -[\mathbf{V} \times \frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{V} \times [\mathbf{S} \times \mathbf{W}]] \quad (\text{II.5})$$

is an additional twisting moment, or torque. In standard mechanics the concept "torque" is applied to the moment of force (II.4). Here we distinguish the moment of force (II.4) and torque (II.5).

It should be noticed, that in the same way both equation $d\mathbf{L}/dt = \mathbf{M}$ follows from Eq.(I.1) for usual mass point and equation (II.2) follows from Eq.(I.14) (i.e. Eq.(I.1), in which \mathbf{F} and \mathbf{P} are specified by equations (I.5) and (I.10)) for a mass point with internal degrees of freedom.

Solution of equation (I.14) may be obtained in principle, if potential function $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W})$ and time dependence of pseudo-vectors \mathbf{S} и \mathbf{C} , coupled with internal structure

of mass point, are known. As it is known, one of internal property of particles is spin, associated classically with proper angular momentum of particle. Therefore a temptation arises to connect pseudo-vectors \mathbf{S} and \mathbf{C} with spin. However, having only definition (II.3) for angular momentum it is impossible to define a concept of proper angular momentum. Therefore pseudo-vectors \mathbf{S} , \mathbf{C} and their equations of motion should either postulated here artificially or determined starting from additional arguments. In particular, one may go by the same way as a solid body in mechanics considered as a system of mass point. Then it is possible to define a concept of *particle* with internal degrees of freedom as a system of the same mass points, whose proper angular momentum is determined relative to center of inertia of particle. Such procedure will be made elsewhere. Here it is reasonably to generalize equations and concepts above to relativistic case.

III. RELATIVISTIC EQUATION OF MOTION

Relativistic generalization of the second Newton's law for mass point is

$$\frac{d\mathbf{P}}{d\lambda} = \frac{1}{c}\mathbf{F} \quad (\text{III.1})$$

where $\mathbf{P} = \{P^\mu\} = (P^0, \mathbf{P})$, $\mathbf{F} = \{F^\mu\} = (F^0, \mathbf{F})$, $\mu = 0, 1, 2, 3$, are relativistic generalizations of momentum and force, λ is invariant parameter determined by the interval

$$dS^2 = \eta_{\mu\nu}dR^\mu dR^\nu = (dR^0)^2 - d\mathbf{R}^2 = \sigma d\lambda^2, \quad \sigma = \pm 1, \quad (\text{III.2})$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Thus, for $\sigma = +1$ parameter $\lambda/c = \tau$ is a proper time of "concomitant observer" K' moving together with event defined by four-dimensional radius-vector $\mathbf{R} = \{R^\mu\} = (R^0, \mathbf{R})$. For $\sigma = -1$ parameter $\lambda = S$ coincides with length of arc of world line of the event \mathbf{R} .

Let's note an important fact, that standard Special Relativity with interval (III.2) is valid exceptionally for inertial r.f. Usually interval (III.2) is considered as a definition of distance between two points in the Minkowski space $\mathbf{E}_{1,3}^{\mathbf{R}}$. Then coordinates of a point in $\mathbf{E}_{1,3}^{\mathbf{R}}$, defined by radius-vector \mathbf{R} , are quantities relative to origin, coinciding with origin of the rest inertial r.f., and have absolute character in the meaning of absolute time and absolute space of Newton's mechanics. Relative character in the meaning of Special Relativity they acquire when interval (III.2) is coupled with r.f. K' , moving relative to the rest r.f. K with velocity $\mathbf{V} = cd\mathbf{R}/dR^0$. In this case radius-vector \mathbf{R} is said to be an event \mathbf{R} , whose world line is a trajectory of the origin of inertial r.f. K' , moving with velocity \mathbf{V} in $\mathbf{E}_{1,3}^{\mathbf{R}}$, i.e. in the space of the rest r.f. K .

For inertially moving r.f. K' r.h.s. of Eq.(III.1) vanishes, and we obtain conservation of 4-momentum, whence it follows conservation of

$$\mathbf{P}^2 = \eta_{\mu\nu}P^\mu P^\nu = (P^0)^2 - \mathbf{P}^2 = \sigma m_0^2 c^2, \quad (\text{III.3})$$

if 4-momentum is defined as

$$\mathbf{P}^\mu = m_0 c U^\mu = m_0 c dR^\mu / d\lambda = m_0 dR^\mu / d\tau. \quad (\text{III.4})$$

Relations (III.3)-(III.4) are standard relations of Special Relativity for kinematical momentum, which are extended on any asymptotically free physical systems without any reason. Between other things one may consider an expression (III.2) for relativistic interval as a corollary from relations (III.3), postulating connection between energy and momentum.

If some force be acting on moving r.f. K' , the latter is no longer inertial one. Then 4-momentum in Eq.(III.1) becomes dynamical momentum, whose definition ought to be analogous to Eq.(I.6)

$$P^\mu = m_0 c \frac{dR^\mu}{d\lambda} + K^\mu, \quad (\text{III.5})$$

where K^μ is some addition to kinematical 4-momentum (III.4) due to interaction between moving r.f. K' and rest r.f. K .

In Newton's mechanics an interaction force (I.5) between K' and K is determined by means of elementary work (I.2) which may be written as $dA = -\eta_{ij} F^i dR^j$. This work is a scalar under Galilei transformations, i.e. it is the same in all non-relativistic inertial r.f., but it is not covariant under Lorentz transformations.

Indeed, let L^μ_{ν} be matrix elements of the Lorentz transformation $dR'^\mu = L^\mu_{\nu} dR^\nu$, satisfying to condition $\eta_{\lambda\kappa} L^\lambda_{\mu} L^\kappa_{\nu} = \eta_{\mu\nu}$, so that ([4])

$$L^0_{.0} = \gamma_\sigma = (1 - \mathbf{B}_0^2) ^{-1/2}, \quad L^0_{.i} = \Gamma_\sigma V_{0i} / V_0, \quad L^i_{.0} = -\Gamma_\sigma V_0^i / V_0, \quad L^i_{.j} = \delta^i_j - \frac{\gamma_\sigma - 1}{\mathbf{V}_0^2} V_0^i V_{0j}, \quad (\text{III.6})$$

where

$$\mathbf{B}_0 = \mathbf{V}_0 / c, \quad V_0 = |\mathbf{V}_0| = \sqrt{\mathbf{V}_0^2}, \quad B_0 = |\mathbf{B}_0| = \sqrt{\mathbf{B}_0^2} = cV_0; \quad (\text{III.7})$$

$$\Gamma_\sigma = B_0^\sigma \gamma_\sigma, \quad \Gamma_+ = B_0 \gamma_+ = B_0 (1 - \mathbf{B}_0^2)^{-1/2}, \quad \Gamma_- = \gamma_- / B_0 = (\mathbf{B}_0^2 - 1)^{-1/2}, \quad (\text{III.8})$$

\mathbf{V}_0 is velocity of r.f. K' relative to r.f. K .

Then the Lorentz transformation takes form

$$dR'^0 = \gamma_\sigma \left[dR^0 - \frac{B_0^\sigma (\mathbf{V}_0 \cdot d\mathbf{R})}{V_0} \right], \quad (\text{III.9})$$

$$d\mathbf{R}' = d\mathbf{R} + \left[(\gamma_\sigma - 1) \frac{(\mathbf{V}_0 \cdot d\mathbf{R})}{c B_0^\sigma B_0} - \gamma_\sigma dR^0 \right] \frac{B_0^\sigma}{c B_0} \mathbf{V}_0. \quad (\text{III.10})$$

Transformation law of relativistic force looks as

$$F'^\mu = L^\mu_{\nu} F^\nu = L^\mu_{.0} F^0 + L^\mu_{.i} F^i, \quad (\text{III.11})$$

$$F'^0 = \gamma_\sigma \left[F^0 - \frac{B_0^\sigma (\mathbf{F} \cdot \mathbf{V}_0)}{V_0} \right], \quad (\text{III.12})$$

$$\mathbf{F}' = \mathbf{F} + \left[\frac{(\gamma_\sigma - 1) (\mathbf{F} \cdot \mathbf{V}_0)}{\mathbf{V}_0^2} - \frac{\gamma_\sigma B_0^\sigma F^0}{V_0} \right] \mathbf{V}_0. \quad (\text{III.13})$$

Hence Eqs.(III.10) and (III.13) give transformation law for elementary work

$$\begin{aligned} dA' &= (\mathbf{F}' \cdot d\mathbf{R}') = -\eta_{ij} F'^i dR'^j = -\eta_{ij} L^i_{. \mu} L^j_{. \nu} F^\mu dR^\nu = \\ &= dA + \Gamma_\sigma^2 \left[1 - \frac{(\mathbf{V} \cdot \mathbf{V}_0)}{c^2 B_0 B_0^\sigma} \right] c F^0 dt - \frac{\Gamma_\sigma^2}{B_0 B_0^\sigma} (\mathbf{F} \cdot \mathbf{V}_0) dt + \frac{\gamma_\sigma^2 - 1}{\mathbf{V}_0^2} (\mathbf{F} \cdot \mathbf{V}_0) (\mathbf{V} \cdot \mathbf{V}_0) dt, \end{aligned} \quad (\text{III.14})$$

whence it follows relativistic transformation of power $N = cdA/dR^0 = (\mathbf{F} \cdot \mathbf{V})$

$$\begin{aligned} N' &= c \frac{dA'}{dR'^0} = (\mathbf{F}' \cdot \mathbf{V}') = \\ &= \frac{N + L^0_{.0} L^0_{.i} F^0 V^i + c L^0_{.i} L^0_{.0} F^i + L^0_{.i} L^0_{.j} F^i V^j + c[(L^0_{.0})^2 - 1] F^0}{L^0_{.0} + L^0_{.i} V^i / c} = \end{aligned}$$

$$= \frac{N + c\Gamma_\sigma^2 \left[1 - \frac{(\mathbf{V} \cdot \mathbf{V}_0)}{c^2 B_0 B_0^\sigma} \right] F^0 - \frac{\Gamma_\sigma^2}{B_0 B_0^\sigma} (\mathbf{F} \cdot \mathbf{V}_0) + \frac{\gamma_\sigma^2 - 1}{V_0^2} (\mathbf{F} \cdot \mathbf{V}_0) (\mathbf{V} \cdot \mathbf{V}_0)}{\gamma_\sigma \left[1 - \frac{B_0^\sigma (\mathbf{V} \cdot \mathbf{V}_0)}{V_0} \right]}. \quad (\text{III.15})$$

Noncovariance of expression (I.2) and force transformation law (III.13) are inconsistent with principle of relativity, whose successive application means that equations and quantities, such as scalars, 4-vectors, tensors etc., should be covariant under Lorentz transformations in any theory. An expression, being a scalar in some inertial r.f., ought to be scalar in another inertial r.f. Therefore definition (I.2) should be generalized in the form

$$dW = -\eta_{\mu\nu} F^\mu dR^\nu = -F^0 dR^0 + dA = (-cF^0 + N)dt. \quad (\text{III.16})$$

In standard Special Relativity, dealing with interval (III.2), F^0 is defined from Eq.(III.1), where $P^0 = m_0 c dR^0/d\tau = m_0 c^2 \gamma = m_0 c^2 (1 - \mathbf{V}^2/c^2)^{-1/2}$, \mathbf{V} is absolute velocity of the mass point acquiring acceleration $\mathbf{W} = d\mathbf{V}/dt$ under action of the force \mathbf{F} . Then taking into account the relation

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} (\mathbf{V} \cdot \mathbf{W}), \quad (\text{III.17})$$

we obtain

$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = \gamma \frac{d(m_0 \gamma \mathbf{V})}{dt} = m_0 \gamma^2 \left[\mathbf{W} + \frac{\gamma^2}{c^2} (\mathbf{V} \cdot \mathbf{W}) \mathbf{V} \right], \quad (\text{III.18})$$

$$(\mathbf{F} \cdot \mathbf{V}) = m_0 \gamma^4 (\mathbf{V} \cdot \mathbf{W}), \quad (\text{III.19})$$

$$F^0 = \frac{dP^0}{d\tau} = m_0 c \frac{d\gamma}{d\tau} = m_0 c \gamma \frac{d\gamma}{dt} = \frac{m_0 \gamma^4}{c} (\mathbf{V} \cdot \mathbf{W}) = \frac{1}{c} (\mathbf{F} \cdot \mathbf{V}) = \frac{N}{c}. \quad (\text{III.20})$$

Comparison of Eq.(III.16) with Eq.(III.20) shows that scalar dW is identically zero in all inertial r.f. However, should moving r.f. K' be coupled with considered mass point, the latter ceases to be inertial one. Then in such r.f. an expression (III.2) for length of arc of the world line of the mass point and relations (III.18)-(III.20) become invalid. It means that scalar $dW = -\eta_{\mu\nu} F^\mu dR^\nu = -\eta_{\mu\nu} F'^\mu dR'^\nu$ does not equal to zero, conserving its covariant expression in all inertial r.f. Relativistic force, acting at rest mass point in r.f. K' , may be expressed in terms of potential function $U = U(\lambda, R^\mu, U^\mu, W^\mu, \dot{W}^\mu, \dots, (W^{(N)})^\mu)$ by analogy with (I.5) in non-relativistic mechanics, where $R^\mu, U^\mu, W^\mu = dU^\mu/d\lambda$, $(W^{(k)})^\mu = d^k W^\mu/d\lambda^k$ are relative radius-vector, 4-velocity and 4-accelerations of r.f. K' relative to r.f. K . Forasmuch as relation $\eta_{\mu\nu} U^\mu U^\nu = \sigma$ following from (III.2) becomes invalid, arguments of potential function should be considered as independent variables, so that total differential of U equals

$$dU = \frac{\partial U}{\partial \lambda} d\lambda + \frac{\partial U}{\partial R^\mu} dR^\mu + \frac{\partial U}{\partial U^\mu} dU^\mu + \sum_{k=0}^N \frac{\partial U}{\partial (W^{(k)})^\mu} d(W^{(k)})^\mu. \quad (\text{III.21})$$

Formulae (I.10) and (I.5) should be considered as non-relativistic limits of 4-momentum and 4-force

$$P^\mu = m_0 c U^\mu - \eta^{\mu\nu} \frac{\partial U}{c \partial U^\nu} + \frac{1}{2} \eta^{\mu\nu} \varepsilon_{\nu\lambda\kappa\rho} S^{\lambda\kappa} W^\rho, \quad (\text{III.22})$$

$$F^\mu = -\eta^{\mu\nu} \frac{\partial U}{\partial R^\nu} + \frac{1}{2} \eta^{\mu\nu} \varepsilon_{\nu\lambda\kappa\rho} C^{\lambda\kappa} U^\rho, \quad (\text{III.23})$$

respectively, where $S_{\lambda\kappa}$ and $C_{\lambda\kappa}$ are some antisymmetric tensors, characterizing internal structure of the mass point.

Substitution of Eqs.(III.22)-(III.23) into equation (III.1) gives next equation of motion

$$\frac{d}{d\lambda} \left[m_0 c U^\mu + \frac{1}{2} \eta^{\mu\nu} \varepsilon_{\nu\lambda\kappa\rho} S^{\lambda\kappa} W^\rho \right] - \frac{1}{2c} \eta^{\mu\nu} \varepsilon_{\nu\lambda\kappa\rho} C^{\lambda\kappa} U^\rho = \frac{1}{c} \eta^{\mu\nu} \left[\frac{d}{d\lambda} \frac{\partial U}{\partial U^\nu} - \frac{\partial U}{\partial R^\nu} \right]. \quad (\text{III.24})$$

Substitution of Eqs.(III.22)-(III.23) into equation (III.1) gives

$$c \eta_{\mu\nu} \frac{dP^\mu}{d\lambda} dR^\nu = c \eta_{\mu\nu} U^\mu dP^\nu = \eta_{\mu\nu} F^\mu dR^\nu = \eta_{\mu\nu} U^\mu F^\nu d\lambda, \quad (\text{III.25})$$

or

$$\eta_{\mu\nu} U^\mu \frac{d}{d\lambda} \left[m_0 c^2 U^\nu + \frac{c}{2} \eta^{\mu\nu} \varepsilon_{\nu\lambda\kappa\rho} S^{\lambda\kappa} W^\rho \right] = U^\mu \frac{d}{d\lambda} \frac{\partial U}{\partial U^\mu} - U^\mu \frac{\partial U}{\partial R^\mu}. \quad (\text{III.26})$$

Hence we obtain equation

$$\frac{dE}{d\lambda} = \frac{\partial U}{\partial \lambda} + \sum_{k=0}^N \frac{\partial U}{\partial (W^{(k)})^\mu} (W^{(k+1)})^\mu, \quad (\text{III.27})$$

where quantity

$$E = \frac{m_0 c^2}{2} \eta_{\mu\nu} U^\mu U^\nu + \frac{c}{2} \varepsilon_{\mu\nu\lambda\kappa} U^\mu W^\nu S^{\lambda\kappa} + U - U^\mu \frac{\partial U}{\partial U^\mu} = \frac{m_0 c^2}{2} \sigma = \text{const}, \quad (\text{III.28})$$

is an integral of motion provided a condition

$$\frac{\partial U}{\partial \lambda} + \sum_{k=0}^N \frac{\partial U}{\partial (W^{(k)})^\mu} (W^{(k+1)})^\mu = 0 \quad (\text{III.29})$$

is satisfied.

Neglecting internal structure of mass point and its interaction, $U = 0$, from (III.28) we obtain $\eta_{\mu\nu} U^\mu U^\nu = \sigma$, and expression (III.2) for interval of standard Special Relativity. In general case integral of motion σ does not equal to $+1$ or -1 . Specifically, an account of internal structure of free mass point gives

$$\eta_{\mu\nu} dR^\mu dR^\nu + \frac{1}{m_0} \varepsilon_{\mu\nu\lambda\kappa} S^{\lambda\kappa} dR^\mu dU^\nu = \left[1 + \frac{1}{m_0 c} \frac{\varepsilon_{\lambda\kappa\tau\omega} S^{\lambda\kappa} U^\tau W^\omega}{\eta_{\rho\sigma} U^\rho U^\sigma} \right] \eta_{\mu\nu} dR^\mu dR^\nu = \sigma d\lambda^2, \quad (\text{III.30})$$

i.e. the Minkowski space-time $\mathbf{E}_{1,3}^R$ effectively extends to 8-dimensional phase space with interval (III.30) and degenerate metric, which is equivalent to 4-dimensional conformally flat space with metric

$$g_{\mu\nu} = \left[1 + \frac{1}{m_0 c} \frac{\varepsilon_{\lambda\kappa\tau\omega} S^{\lambda\kappa} U^\tau W^\omega}{\eta_{\rho\sigma} U^\rho U^\sigma} \right] \eta_{\mu\nu}, \quad (\text{III.31})$$

coordinate dependence of which may be determined, as soon as solution of equation of motion (III.24) for $U = 0$ is found.

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