# Zitterbewegung as purely classical phenomenon

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**Abstract**

Nonrelativistic formalism is developed, which allows describing systems with internal degrees of freedom in the scalar potential field $U$, which is a function both on relative coordinates and time, and on relative speed and accelerations. The equation for energy, which is an integral of motion when $U$ satisfies to certain differential condition, is derived for the general case. For a free mass point all solutions of the equations of motion in the center-of-inertia reference frame, moving with constant velocity, are found. As a result, the center of mass follows along helical and more complicated trajectories round a direction of motion of the center of inertia. This motion can be interpreted as trembling movement (Zitterbewegung). On this basis a conclusion is done that Zitterbewegung has purely classical origin, arising even in a nonrelativistic case if internal degrees of freedom are taken into account. The general equation of motion for a spin which can be interpreted from positions of a classical mechanics is written down. Application of the obtained results to the electron leads to new conception of electric charge, sign of which corresponds to a sign of spin polarization.

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## 1 Introduction

Experimental data on high energy physics say that elementary particles are complex systems with internal degrees of freedom. Fundamental particle which in certain relation can pretend to simplicity, on the one hand, and underlie all known interactions, on the

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other hand, is the electron. Nonexistence of the electron, being a source of electromagnetic field, in the Maxwell theory and its generalizations is paradoxical. It may be explained by that the Maxwell theory describes macroscopical averaged fields created by systems consisting of large quantity of moving charges. Therefore the electron theory, whose foundations were laid by Lorentz, is necessary for the description of the majority of the electromagnetic phenomena in material systems.

The electron theory has been developing in classical trend after discovery of the electron by J.J.Thomson in cathode rays. The theory by Abraham [1], based on the Maxwell-Lorentz electrodynamics, became the first theory considering internal structure of the electron. During two decades preceding experience by Stern-Gerlach and the Kronig-Uhlenbeck-Goudsmit hypothesis about electron spin, there were appeared a few alternative theories and guesses about electron structure. It should be mentioned here works by Compton [2]- [5], influenced by both the paper by Parson [6] and stereoscopic photos of tracks of the $\beta$-rays, made by Wilson, some of which had almost ideal form of a helix. Compton has come to a conclusion that the model of the electron, "spinning like a tiny gyroscope", can eliminate difficulties in an explanation of curvature of these tracks, as well as of Richardson-Barnett effect and diffraction of X-rays by magnetic crystals. Nevertheless, the Parson-Compton theory continuing early idea of vortex atoms, going back to Greek atomists, Kepler, Descartes, Leibnitz, Svedenborg, Boskovich, Ampere, Kelvin, and theories of some other researchers had rather natural philosophic and empirical nature, than they gave any mathematical instrument for future theory. The first mathematical realization of the Parson-Compton theory was the work by Frenkel [7], where the electron was considered as a point with six-vector of magnetic moment, what has allowed to explain the anomalous Zeeman effect.

After creation of quantum mechanics, especially after works by Pauli and Dirac, the electron theory began to develop chiefly in quantum direction. In 1930 Schrödinger has shown that the trembling motion (Zitterbewegung) of the electron takes place in the Dirac theory where eigenvalues of any component of the velocity operator are equal to $\pm c$. Microscopic trembling motion with the velocity of light of imaginary center of the cloud of charge, whose amplitude is about half Compton wave length, is imposed on translational macroscopic motion of the center of mass of the electron. As Shrödinger talks, "exclusively entangled relations, which are present according to the Dirac equation already at free movement of a mass point, seem to me worthy of enunciating though I cannot present some completed result of this research" [8]. In 1952 Huang has shown that Zitterbewegung of a free Dirac electron may be looked upon as a circular motion about the direction of the electron spin that in turn may be interpreted as the "orbital angular momentum" of this motion. As a result the electric current produced by Zitterbewegung is seen to give rise to the intrinsic magnetic moment of the electron [9].

However, as early as in 1937 Mathisson has written down general relativistic equations of motion for systems possessed multipole momenta [10]. He has shewed also, that their application to a particle with spin treated as a dipole gives rise to equations of motion, describing Zitterbewegung [11]. Mathisson has considered the motion of a free uncharged particle with spin and free electron taking account of the reaction of radiation. Here he has assumed spin to be a constant (pseudo-)vector, what was rather stringent assumption, for spin direction in general can change. Nevertheless, having associated trembling motion of the electron with the de Broglie's wave, he has obtained the well-known value of a spin $s = \hbar/2$.

Following basic articles by Schrödinger and Mathisson a lot of works were appeared, developing both quantum and classical electron theory and establishing connection between them. Sufficiently full list of references one may find in the books [12], [13]. Despite a considerable quantity of researches devoted to the theory of the electron, there are many unclear matters associated both with radiation of the electron and with dependence of its trajectory from its spin (see, e.g., [14]).

An interest in classical theory of the electron has especially increased at the recent years, and this work contains some arguments in favour of a classical origin of Zitterbewegung. Relating of the electron spin with its proper rotation allows considering the electron as non-inertial object which can be described as a mass point with internal degrees of freedom [15], [16]. The formalism of such description based on generalization of the second Newton's law is considered in §2. Here the equations of motion of the point interacting with an external field described by the potential function depending on relative variables are obtained. Consequence of the equation of motion is the equation of balance of energy which is integral of motion only if a certain condition is fulfilled. Non-conservation of the energy, in general, seems to be caused by the fact that mass point in question is non-inertial system. It is supposed that internal degrees of freedom are described by the pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$ connected both with internal structure of the point, and with its interaction with external fields. It is shown in §3 that equation of motion of free mass point reduces to a conservation of the velocity of the center of inertia. To obtain solutions natural equations of motion are introduced for internal degrees of freedom describing precession of pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$ round the direction of the velocity of the center of inertia. All solutions of the equation of motion for a free mass point (when $\mathbf{S} = \mathbf{S}_0$ and $\mathbf{C} = \mathbf{C}_0$) in the center-of-inertia reference frame are found in §4. The center of inertia proves to be does not coincide with the center of mass. As a result, the center of mass moves by a complicated trajectory round the direction of motion of the center of inertia. Some solutions in the center-of-inertia reference frame are infinite. It is shown that they become finite at zero energy of the mass point. It is of interest that equation of motion admits also solutions for zero mass and transversal polarization. In §5 equation of moments is considered and speculations are contained about the physical sense of pseudo-vectors $\mathbf{S}_0$ and $\mathbf{C}_0$ and their relation with spin whose equation of motion in general case we deal with in §6, which contains also conclusive remarks on possible interpretation of obtained solutions.

# 2 Description of the mass point with internal degrees of freedom

A mass point with internal degrees of freedom can be considered as a non-inertial system whose equation of motion taking into account its interaction with an external field has the form of the Newton's Second Law ( [15]-[16])

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \; , \tag{2.1}$$

where

$$\mathbf{P} = m_0 \mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}] \; , \tag{2.2}$$

is a dynamical momentum of the point, $m_0$ is its rest mass,

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{R}} + [\mathbf{C} \times \mathbf{V}] \; , \tag{2.3}$$

is a force, acting to the point. Expressions (2.2) and (2.3) follow from the definition of the elementary work of force, $dA = (\mathbf{F} \cdot d\mathbf{R})$, if potential function $U$ depends on velocity. Potential function characterizes both medium in which the point moves and interaction of the point with physical objects which are in this medium. Therefore it should be assumed that $U$ generally can depend on time $t$, relative coordinates $\mathbf{R}$, velocity $\mathbf{V}$ and accelerations $\mathbf{W}^{(k)} = d^k\mathbf{W}/dt^k$, $k = 0, 1, 2, ..., N$, so that $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, ..., \mathbf{W}^{(N)})$.

Internal degrees of freedom are characterized by pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$ connected with both internal structure of mass point and interaction. Hence, they can be represented as sums

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}^{ext} \; , \; \mathbf{C} = \mathbf{C}_0 + \mathbf{C}^{ext} \; , \tag{2.4}$$

where $\mathbf{S}^{ext}$ and $\mathbf{C}^{ext}$ are connected exclusively with interaction and depend on the same variables as potential function; $\mathbf{S}_0$ and $\mathbf{C}_0$ are connected exclusively with internal structure of mass point and during its motion they can change only in the direction but not in the module provided an interaction is neglected.

If the function $U$ is represented in the form

$$U = U_0 - (\mathbf{R} \cdot [\mathbf{V} \times \mathbf{C}]) = U_0 + (\mathbf{V} \cdot [\mathbf{R} \times \mathbf{C}]) = U_0 - ([\mathbf{R} \times \mathbf{V}] \cdot \mathbf{C}) \; , \tag{2.5}$$

then Eqs. (2.2) and (2.3) take the form

$$\mathbf{P} = m_0 \mathbf{V} - \frac{\partial U_0}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}] - [\mathbf{R} \times \mathbf{C}] + ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{V}}) \; , \tag{2.6}$$

$$\mathbf{F} = -\frac{\partial U_0}{\partial \mathbf{R}} + ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{R}}) \; , \tag{2.7}$$

where

$$([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{R}})_i = \varepsilon_{klm} R^k V^l \frac{\partial (\mathbf{C}^{ext})^m}{\partial R^i} \; , \tag{2.8}$$

$$([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}})_i=\varepsilon_{klm}R^kV^l\frac{\partial(\mathbf{C}^{ext})^m}{\partial V^i}\ . \tag{2.9}$$

It follows from Eq.(2.1), which is reduced to equation

$$\frac{d}{dt}\left(m_0\mathbf{V}+[\mathbf{S}_0\times\mathbf{W}]-[\mathbf{R}\times\mathbf{C}_0]\right)=-\frac{\partial U_0}{\partial\mathbf{R}}+([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{R}})+$$
$$+\frac{d}{dt}\left(\frac{\partial U_0}{\partial\mathbf{V}}-([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}})-[\mathbf{S}^{ext}\times\mathbf{W}]+[\mathbf{R}\times\mathbf{C}^{ext}]\right)\ , \tag{2.10}$$

that there take place an equation for energy

$$\frac{dE}{dt}=\frac{\partial U}{\partial t}+\sum_{k=0}^{N}(\frac{\partial U}{\partial\mathbf{W}^{(k)}}\cdot\mathbf{W}^{(k+1)})=\frac{\partial U_0}{\partial t}-([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial t})+$$
$$+\sum_{k=0}^{N}(\frac{\partial U_0}{\partial\mathbf{W}^{(k)}}\cdot\mathbf{W}^{(k+1)})-\sum_{k=0}^{N}(([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{W}^{(k)}})\cdot\mathbf{W}^{(k+1)})\ , \tag{2.11}$$

where

$$E=\frac{m_0\mathbf{V}^2}{2}-([\mathbf{V}\times\mathbf{W}]\cdot\mathbf{S}_0)-([\mathbf{V}\times\mathbf{W}]\cdot\mathbf{S}^{ext})-(\mathbf{V}\cdot\frac{\partial U_0}{\partial\mathbf{V}})+$$
$$+(\mathbf{V}\cdot([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}}))+U_0\ , \tag{2.12}$$

$$(\mathbf{V}\cdot([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}}))=\varepsilon_{klm}R^kV^lV^i\frac{\partial(\mathbf{C}^{ext})^m}{\partial V^i}\ , \tag{2.13}$$

$$(([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{W}^{(k)}})\cdot\mathbf{W}^{(k+1)})=\varepsilon_{klm}R^kV^l(\mathbf{W}^{(k+1)})^i\frac{\partial(\mathbf{C}^{ext})^m}{\partial(\mathbf{W}^{(k)})^i}\ . \tag{2.14}$$

Obviously, the energy (2.12) is integral of motion, if r.h.s. of Eq. (2.11) becomes zero.

# 3 The motion of free mass point

In this section we will consider a motion of free mass point M with internal degrees of freedom, and will not concern its interaction and physical sense of quantities $\mathbf{S}$ and $\mathbf{C}$. Mass point in question will be free, if $U_0=0$, $\mathbf{S}^{ext}=0$, $\mathbf{C}^{ext}=0$. Then equation (2.10) gives rise to conservation of the vector

$$\mathbf{P}_{\mathrm{C}}=m_0\mathbf{V}+[\mathbf{S}_0\times\mathbf{W}]-[\mathbf{R}\times\mathbf{C}_0]=m\mathbf{V}_{\mathrm{C}}\ , \tag{3.1}$$

where $m_0\mathbf{V}$ is kinetic momentum of the point M being *a center of mass*, $m$ is an effective mass. It is reasonably to term the vector (3.1) as the kinetic momentum associated with the point M. Here $\mathbf{V}_{\mathrm{C}}=d\mathbf{R}_{\mathrm{C}}/dt$ is a velocity of some point C specified by radius-vector

$$\mathbf{R}_{\mathrm{C}}(t)=\mathbf{R}_0+\mathbf{V}_{\mathrm{C}}t\ , \tag{3.2}$$

where $\mathbf{R}_0$ is a radius-vector of initial position of the point C.

It follows from Eq.(2.7) that the vector $\mathbf{P}_{\rm C}$ is a constant vector, if $U_0$ does not depend on the relative radius-vector. According to Eq.(3.2) the point C moves inertially with velocity $\mathbf{V}_{\rm C}$. Hence, it is *a center of inertia*, which in general does not coincide with the center of mass M, moving along some trajectory round the direction of $\mathbf{V}_{\rm C}$. Figure 1 shows parameters of this trajectory.

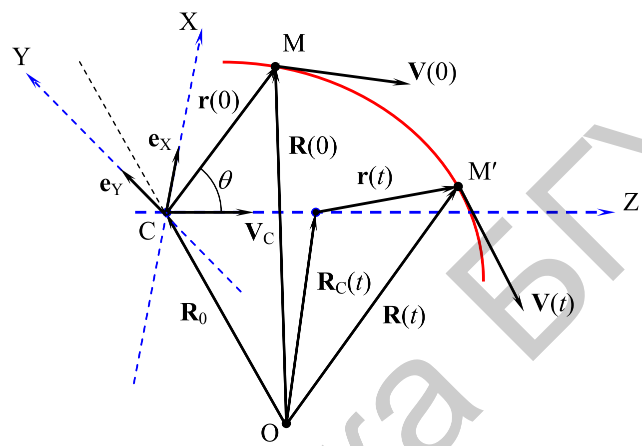



Figure 1. Parameters of the trajectory of the mass point M

Eq.(3.1) may be rewritten in the form

$$m_0\mathbf{v} + [\mathbf{S}_0 \times \mathbf{w}] - [\mathbf{R} \times \mathbf{C}_0] = (m - m_0)\mathbf{V}_{\rm C} \,, \tag{3.3}$$

where

$$\mathbf{r}(t) = \mathbf{R}(t) - \mathbf{R}_{\rm C}(t) \,,\; \mathbf{v}(t) = \mathbf{V}(t) - \mathbf{V}_{\rm C} \,,\; \mathbf{w}(t) = \mathbf{W}(t) \tag{3.4}$$

are radius-vector, velocity and acceleration of the center of mass M relative to the center of inertia C, respectively.

The energy (2.12) can be expressed as

$$E = K_{\rm C} + K_{0\rm C} + E_0 \,, \tag{3.5}$$

where

$$K_{\rm C} = \frac{m_0\mathbf{V}_{\rm C}^2}{2} \tag{3.6}$$

is a kinetic energy of the center of inertia as though the mass of the point M was in the point C,

$$K_{0\rm C} = m_0(\mathbf{v} \cdot \mathbf{V}_{\rm C}) + (\mathbf{V}_{\rm C} \cdot [\mathbf{S}_0 \times \mathbf{w}]) \tag{3.7}$$

is additional kinetic energy stipulated by both the motion of the center of inertia and the motion of the point M relative to the center of inertia C,

$$E_0 = \frac{m_0\mathbf{v}^2}{2} + (\mathbf{v} \cdot [\mathbf{S}_0 \times \mathbf{w}]) \tag{3.8}$$

is a kinetic energy of the point M stipulated by its motion relative to the center of inertia.

According to equation of motion (3.1) radius-vector $\mathbf{R}(t)$ should be determined by pseudo-vectors $\mathbf{S}_0$ and $\mathbf{C}_0$, which are precessing with the same velocity about the direction of the vector $\mathbf{V}_\mathrm{C}$. Therefore $\mathbf{S}_0$ and $\mathbf{C}_0$ ought to satisfy to following equations of motion

$$\frac{d\mathbf{S}_0}{dt} = [\mathbf{\Omega}_0 \times \mathbf{S}_0] \,, \frac{d\mathbf{C}_0}{dt} = [\mathbf{\Omega}_0 \times \mathbf{C}_0] \,, \tag{3.9}$$

where

$$\mathbf{\Omega}_0 = \sigma \mathbf{V}_\mathrm{C} = \Omega_0 \mathbf{e}_Z \,, \tag{3.10}$$

is an angular velocity of precession, $\sigma = \mathrm{const}$ has a dimension of inverse length. If we choose the axis Z to be coincided with the direction of $\mathbf{V}_\mathrm{C}$, i.e. $\mathbf{V}_\mathrm{C} = V_\mathrm{C}\mathbf{e}_Z$, then $\Omega_0 = \sigma V_\mathrm{C}$ can be both positive, and negative quantity. Unit vectors $\mathbf{e}_X$ and $\mathbf{e}_Y$ may be chosen in the following form

$$\mathbf{e}_X = K^{1/2}[\mathbf{r}_0 \times \mathbf{\Omega}_0] \,, \tag{3.11}$$

$$\mathbf{e}_Y = \Omega_0^{-1} K^{1/2} [\mathbf{\Omega}_0 \times [\mathbf{r}_0 \times \mathbf{\Omega}_0]] \,, \tag{3.12}$$

where

$$K = [\mathbf{\Omega}_0 \times \mathbf{r}_0]^{-2} = \frac{1}{\mathbf{r}_0^2 \mathbf{\Omega}_0^2 \sin^2\theta} \,, \tag{3.13}$$

$$[\mathbf{\Omega}_0 \times [\mathbf{r}_0 \times \mathbf{\Omega}_0]]^2 = \mathbf{\Omega}_0^2 K^{-1} \,, \tag{3.14}$$

$\mathbf{r}_0 = \mathbf{r}(0)$ is a radius-vector of the center of mass M relative to the center of inertia C at initial time $t = 0$; $\theta = \pi/2$ corresponds to that the vector $\mathbf{r}_0$ lies in a plane, perpendicular to a direction of motion.

Equations (3.9) have solutions

$$\mathbf{S}_0 = S_0(\sin\alpha_S \sin\Omega_0 t \mathbf{e}_X + \sin\alpha_S \cos\Omega_0 t \mathbf{e}_Y + \cos\alpha_S \mathbf{e}_Z) \,, \tag{3.15}$$

$$\mathbf{C}_0 = C_0(\sin\alpha_C \sin\Omega_0 t \mathbf{e}_X + \sin\alpha_C \cos\Omega_0 t \mathbf{e}_Y + \cos\alpha_C \mathbf{e}_Z) \,, \tag{3.16}$$

where $S_0 = |\mathbf{S}_0| = \mathrm{const}$, $C_0 = |\mathbf{C}_0| = \mathrm{const}$, $\alpha_S$ and $\alpha_C$ are constant angles between $\mathbf{S}_0$, $\mathbf{C}_0$ and direction of $\mathbf{V}_\mathrm{C}$, respectively.

It is convenient to solve equations (3.3) in the center-of-inertia reference frame, where $\mathbf{R}_0 = \mathbf{0}$, $\mathbf{R}_\mathrm{C} = \mathbf{0}$, $\mathbf{V}_\mathrm{C} = \mathbf{0}$, $E = E_0$. Let us introduce dimensionless variable $\xi = \Omega_0 t$ and denotations

$$\mu_S = 1 + \frac{2\Omega_0 S_0}{m_0} \cos\alpha_S \,, \; \lambda_S = \frac{2\Omega_0 S_0}{m_0} \sin\alpha_S \,, \tag{3.17}$$

$$\mu_C = 1 - \frac{2C_0}{m_0 \Omega_0} \cos\alpha_C \,, \; \lambda_C = \frac{2C_0}{m_0 \Omega_0} \sin\alpha_C \,, \tag{3.18}$$

$$\varepsilon_0 = \frac{2E_0}{m_0 \mathbf{r}_0^2 \Omega_0^2} \,. \tag{3.19}$$

Then, representing $\mathbf{r}(t)$ in the form

$$\mathbf{r}(\xi) = r_0[\sin\xi \mathbf{e}_X + \cos\xi \mathbf{e}_Y + B(\xi)\mathbf{e}_Z] Z(\xi) \,, \tag{3.20}$$

where functions $B(\xi)$ and $Z(\xi)$ satisfy to initial condition

$$B(0) = \cot\theta \ , \ Z(0) = \frac{|[\mathbf{r}_0 \times \mathbf{\Omega}_0]|}{r_0\Omega_0} = \sin\theta \ , \tag{3.21}$$

we reduce equation of motion (3.3) and equation of energy (3.8) to following system

$$(\lambda_S B - \mu_S + 1)Z'' + 2\lambda_S B' Z' + (\lambda_S B'' + \lambda_C B + \mu_S + \mu_C)Z = 0 \ , \tag{3.22}$$

$$\mu_S Z' = 0 \ , \tag{3.23}$$

$$[(B - \lambda_S)Z]' = 0 \ . \tag{3.24}$$

$$\begin{gathered}(\lambda_S B - \mu_S + 1)ZZ'' + (B^2 - 2\lambda_S B + 2\mu_S - 1)Z'^2 + \\ +2BB'ZZ' + (\lambda_S B'' + B'^2 + \mu_S)Z^2 = \varepsilon_0 \ .\end{gathered} \tag{3.25}$$

# 4 Solutions of the equation of motion

The detailed analysis of equations (3.22)-(3.25) leads to following possible solutions ( [17]).

**I.** $m_0 \neq 0$, $\mathbf{S}_0 \neq \mathbf{0}$, $\mathbf{C}_0 \neq \mathbf{0}$.

**I.1.** $B = 0$, $Z = 1$, $\mu_S \neq 0$, $\lambda_S \neq 0$, $\lambda_C \neq 0$, $\mu_C = -\mu_S = -\varepsilon_0$.

$$\cos\alpha_S = \frac{2E_0 - m_0 r_0^2 \Omega_0^2}{2r_0^2 \Omega_0^3 S_0} \ , \tag{4.1}$$

$$\cos\alpha_C = \frac{2E_0 + m_0 r_0^2 \Omega_0^2}{2r_0^2 \Omega_0 C_0} \ . \tag{4.2}$$

The equation of a trajectory looks like

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y] \ , \tag{4.3}$$

i.e. the point M moves on a circle round Z-direction with angular velocity $\Omega_0$. The direction of pseudo-vector $\mathbf{S}_0$ is given by Eq.(4.1), and the direction of pseudo-vector $\mathbf{C}_0$ is determined by Eq.(4.2), following from the condition $\mu_C = -\mu_S$. An angle $\beta$ between $\mathbf{S}_0$ and $\mathbf{C}_0$ may be found from the relation

$$\begin{gathered}\cos\beta = \frac{(\mathbf{S}_0 \cdot \mathbf{C}_0)}{S_0 C_0} = \frac{1}{S_0 C_0}\left(\frac{E_0^2}{r_0^4\Omega_0^4} - \frac{m_0^2}{4}\right) + \left[1 - \frac{1}{\Omega_0^2 S_0^2}\left(\frac{E_0^2}{r_0^4\Omega_0^4} - \frac{m_0 E_0}{r_0^2\Omega_0^2} + \frac{m_0^2}{4}\right) - \right. \\ \left. -\frac{\Omega_0^2}{C_0^2}\left(\frac{E_0^2}{r_0^4\Omega_0^4} + \frac{m_0 E_0}{r_0^2\Omega_0^2} + \frac{m_0^2}{4}\right) + \frac{1}{S_0^2 C_0^2}\left(\frac{E_0^2}{r_0^4\Omega_0^4} - \frac{m_0^2}{4}\right)^2\right]^{\frac{1}{2}} \ .\end{gathered} \tag{4.4}$$

**I.2.** $B = 0$, $Z = 1$, $\mu_S = 0$, $\lambda_S \neq 0$, $\lambda_C \neq 0$, $\mu_C = 0$, $\varepsilon_0 = 0$. This case is obtained from the previous one at $E_0 = 0$.

$$\cos\alpha_S = -\frac{m_0}{2\Omega_0 S_0} \ , \tag{4.5}$$

$$\cos\alpha_C = \frac{m_0\Omega_0}{2C_0}\ . \tag{4.6}$$

The trajectory is described by Eq.(4.3) and represents a circle of radius $r_0$, lying in a plane, perpendicular to a direction of the motion of the center of inertia C. A direction of the motion of the point M in the cases I.1, I.2 is determined by sign of angular velocity of precession $\Omega_0$, consistent with equations (4.1) and (4.2). For example, for $E_0 = 0$ and $\cos\alpha_S > 0$ we have $\Omega_0 < 0$, whence it follows $\cos\alpha_C < 0$. Hence, such a point moves by left spiral directed along $\mathbf{V}_{\mathrm{C}}$. An angle $\beta$ between $\mathbf{S}_0$ and $\mathbf{C}_0$ may be found from the relation

$$\cos\beta = \left[1 - \frac{m_0^2(\Omega_0^4 S_0^2 + C_0^2)}{4\Omega_0^2 S_0^2 C_0^2} + \frac{m_0^4}{16 S_0^2 C_0^2}\right]^{\frac{1}{2}} - \frac{m_0^2}{4S_0C_0}\ . \tag{4.7}$$

**I.3.** $B = 0$, $Z = \pm\xi\sqrt{-\varepsilon_0} + 1$, $\mu_S = 0$, $\lambda_S = 0$, $\lambda_C \neq 0$, $\mu_C = 0$, $\varepsilon_0 \leq 0$.

$$\alpha_S = 0\ ,\ S_0 = -\frac{m_0}{2\Omega_0}\ ,\ \Omega_0 < 0\ , \quad \text{or} \quad \alpha_S = \pi\ ,\ S_0 = \frac{m_0}{2\Omega_0}\ ,\ \Omega_0 > 0\ ; \tag{4.8}$$

$$\cos\alpha_C = \frac{m_0\Omega_0}{2C_0}\ , \tag{4.9}$$

$$\cos\beta = -\frac{m_0^2}{4S_0C_0}\ . \tag{4.10}$$

The equation of a trajectory looks like

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y](\pm\Omega_\varepsilon t + 1)\ , \tag{4.11}$$

where $\Omega_\varepsilon = \Omega_0\sqrt{-\varepsilon_0}$. In the center-of-inertia reference frame the trajectory is a plane helix, perpendicular to the direction of $\mathbf{V}_{\mathrm{C}}$. In the laboratory reference frame the trajectory is convergent and then divergent conical spiral. At $E_0 = 0$ the trajectory (4.11) becomes finite and takes the form (4.3).

**I.4.** $B = 0$, $\mu_S = 0$, $\lambda_S = 0$, $\lambda_C \neq 0$, $\mu_C = -\varepsilon_0$.

$$\alpha_S = 0\ ,\ S_0 = -\frac{m_0}{2\Omega_0}\ ,\ \Omega_0 < 0\ , \quad \text{or} \quad \alpha_S = \pi\ ,\ S_0 = \frac{m_0}{2\Omega_0}\ ,\ \Omega_0 > 0\ ; \tag{4.12}$$

$$\cos\alpha_C = \frac{2E_0 + m_0 r_0^2\Omega_0^2}{2C_0 r_0^2\Omega_0}\ , \tag{4.13}$$

$$\cos\beta = -\frac{2m_0E_0 + m_0^2 r_0^2\Omega_0^2}{4S_0C_0r_0^2\Omega_0^2}\ ; \tag{4.14}$$

$$Z(\xi) = \cos\xi\sqrt{-\varepsilon_0}\ ,\ \ \varepsilon_0 < 0\ , \tag{4.15}$$

$$Z(\xi) = \cosh\xi\sqrt{\varepsilon_0}\ ,\ \ \varepsilon_0 \geq 0\ , \tag{4.16}$$

The equations of a trajectory are

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cos\Omega_\varepsilon t\ ,\ E_0 < 0\ , \tag{4.17}$$

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cosh\Omega_\varepsilon t\ ,\ E_0 \geq 0\ , \tag{4.18}$$

Here, as in the preceding case infinite trajectories (4.18) become finite ones at $E_0 = 0$.

**I.5.** $B = 0$, $\mu_S = 0$, $\lambda_S = 0$, $\lambda_C = 0$, $\mu_C \neq 0$.

$$\alpha_S = 0\ ,\ S_0 = -\frac{m_0}{2\Omega_0}\ ,\ \Omega_0 < 0\ , \quad \text{or} \quad \alpha_S = \pi\ ,\ S_0 = \frac{m_0}{2\Omega_0}\ ,\ \Omega_0 > 0\ ; \tag{4.19}$$

$$\alpha_C = 0\ ,\ \mu_C = 1 - \frac{2C_0}{m_0\Omega_0}\ , \quad \text{or} \quad \alpha_C = \pi\ ,\ \mu_C = 1 + \frac{2C_0}{m_0\Omega_0}\ ; \tag{4.20}$$

$$\cos\beta = \begin{cases} \cos\alpha_C\ ,\ \Omega_0 < 0\ , \\ -\cos\alpha_C\ ,\ \Omega_0 > 0\ ; \end{cases} \tag{4.21}$$

$$Z(\xi) = \cos(\xi\sqrt{\mu_C}) \pm \sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\,\sin(\xi\sqrt{\mu_C})\ ,\ 0 < \mu_C < -\varepsilon_0\ ,\ \varepsilon_0 < 0\ , \tag{4.22}$$

$$Z(\xi) = \cosh(\xi\sqrt{-\mu_C}) \pm \sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\,\sinh(\xi\sqrt{-\mu_C})\ ,\ \mu_C \leq 0\ ,\ \mu_C \leq -\varepsilon_0\ . \tag{4.23}$$

The equations of a trajectory are

$$\mathbf{r}(t) = r_0\sqrt{\frac{-\varepsilon_0}{\mu_C}}\,[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cos(\omega_C t \mp \varphi_C)\ ,\ 0 < \mu_C < -\varepsilon_0\ ,\ E_0 < 0\ , \tag{4.24}$$

$$\mathbf{r}(t) = r_0\sqrt{2 - \frac{-\varepsilon_0}{\mu_C}}\,[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cosh(\omega_C t \pm \varphi_C)\ ,\ \mu_C \leq 0\ ,\ \mu_C \leq -\varepsilon_0\ , \tag{4.25}$$

where

$$\omega_C = \sqrt{\mu_C}\Omega_0\ ,\ \varphi_C = \arctan\sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\ ,\ 0 < \mu_C < -\varepsilon_0\ , \tag{4.26}$$

$$\omega_C = \sqrt{-\mu_C}\Omega_0\ ,\ \varphi_C = \tanh^{-1}\sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\ ,\ \mu_C \leq 0\ ,\ \mu_C \leq -\varepsilon_0\ , \tag{4.27}$$

Infinite trajectories (4.25) can be excluded, having imposed a condition $\mu_C = \varepsilon_0 = 0$. Then (4.25) reduces to (4.3).

**I.6.** $BZ = \lambda_S(Z - 1)$ , $\mu_S = 0$ , $\lambda_S = -\tan\alpha_S = \sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}$ , $\lambda_C = 0$ , $\mu_C = 0$ , $\varepsilon_0 < 0$ .

$$\cos\alpha_S = -\frac{m_0}{2\Omega_0 S_0}\ , \tag{4.28}$$

$$\alpha_C = 0\ ,\ C_0 = \frac{m_0\Omega_0}{2}\ ,\ \Omega_0 > 0\ , \quad \text{or} \quad \alpha_C = \pi\ ,\ C_0 = -\frac{m_0\Omega_0}{2}\ ,\ \Omega_0 < 0\ ; \tag{4.29}$$

$$\cos\beta = -\frac{m_0}{2\Omega_0 S_0}\cos\alpha_C\ , \tag{4.30}$$

$$Z(\xi) = \pm\xi\sqrt{\frac{-\varepsilon_0}{\lambda_S^2 + 1}} + 1\ . \tag{4.31}$$

The equation of a trajectory is

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\left[\pm\frac{\sqrt{-2m_0E_0}}{2r_0\Omega_0^2 S_0}\Omega_0 t + 1\right]\pm$$

$$\pm\sqrt{-\frac{2E_0}{m_0\Omega_0^2}(1-\frac{m_0^2}{4\Omega_0^2 S_0^2})}\Omega_0 t\mathbf{e}_Z\ , \tag{4.32}$$

reducing to finite form (4.3) at $E_0 = 0$.

**I.7.** $BZ = \lambda_S(Z-1)$ , $\mu_S = 0$ , $\lambda_S = -\tan\alpha_S = \sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}$ , $\lambda_C = 0$ , $\mu_C \neq 0$ .

$$\cos\alpha_S = -\frac{m_0}{2\Omega_0 S_0}\ , \tag{4.33}$$

$$\alpha_C = 0\ ,\ \mu_C = 1 - \frac{2C_0}{m_0\Omega_0}\ ,\quad \text{or}\quad \alpha_C = \pi\ ,\ \mu_C = 1 + \frac{2C_0}{m_0\Omega_0}\ ; \tag{4.34}$$

$$\cos\beta = -\frac{m_0}{2\Omega_0 S_0}\cos\alpha_C\ ; \tag{4.35}$$

$$Z(\xi) = \cos\sqrt{\frac{\mu_C}{\lambda_S^2+1}}\xi \pm \sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\sin\sqrt{\frac{\mu_C}{\lambda_S^2+1}}\xi\ ,\ 0 < \mu_C < -\varepsilon_0\ ,\ \varepsilon_0 < 0\ , \tag{4.36}$$

$$Z(\xi) = \cosh\sqrt{\frac{-\mu_C}{\lambda_S^2+1}}\xi \pm \sqrt{\frac{-\varepsilon_0}{\mu_C} - 1}\sinh\sqrt{\frac{-\mu_C}{\lambda_S^2+1}}\xi\ ,\ \mu_C \leq 0\ ,\ \mu_C \leq -\varepsilon_0\ . \tag{4.37}$$

The equations of a trajectory are

$$\mathbf{r}(t) = r_0\sqrt{\frac{-\varepsilon_0}{\mu_C}}[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cos(\Omega_C t \pm \varphi_C) +$$

$$+r_0\sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}\left[\sqrt{\frac{-\varepsilon_0}{\mu_C}}\cos(\Omega_C t \pm \varphi_C) - 1\right]\mathbf{e}_Z\ ,\ 0 < \mu_C < -\varepsilon_0\ ; \tag{4.38}$$

$$\mathbf{r}(t) = r_0\sqrt{2 - \frac{-\varepsilon_0}{\mu_C}}\ [\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cosh(\Omega_C t \pm \varphi_C) +$$

$$+r_0\sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}\left[\sqrt{2 - \frac{-\varepsilon_0}{\mu_C}}\ \cosh(\Omega_C t \pm \varphi_C) - 1\right]\mathbf{e}_Z\ ,\ \mu_C \leq 0\ ,\ \mu_C \leq -\varepsilon_0\ , \tag{4.39}$$

where $\varphi_C$ is determined in Eqs.(4.26)-(4.27),

$$\Omega_C = \omega_C\cos\alpha_S\ . \tag{4.40}$$

Infinite trajectories (4.39) becomes finite ones, when $\mu_C = \varepsilon_0 = 0$. Then the equation (4.39) takes the form

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y] - r_0\sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}\mathbf{e}_Z\ , \tag{4.41}$$

and it follows from (4.34) that

$$\alpha_C = 0\ ,\ C_0 = \frac{m_0\Omega_0}{2}\ ,\ \Omega_0 > 0\ ,\quad \text{or} \quad \alpha_C = \pi\ ,\ C_0 = -\frac{m_0\Omega_0}{2}\ ,\ \Omega_0 < 0\ . \tag{4.42}$$

**II.** $m_0 \neq 0$, $\mathbf{S}_0 \neq \mathbf{0}$, $\mathbf{C}_0 = \mathbf{0}$.

In this case equation (2.10) is a generalization of non-relativistic Frenkel-Mathisson-Weyssenhoff equation ( [7], [11], [18]), describing the motion of point particle with constant spin $\mathbf{s} = -c^2\mathbf{S}_0$. It may be deduced from the cases I.5 and I.7 at $\mu_C = 1$, $\lambda_C = 0$. As a result we have following variants.

**II.1.** $B = 0\ \ ,\ \mu_S = 0\ \ ,\ \lambda_S = 0\ \ ,\ \lambda_C = 0\ \ ,\ \mu_C = 1\ \ .$

$$\alpha_S = 0\ ,\ S_0 = -\frac{m_0}{2\Omega_0}\ ,\ \Omega_0 < 0\ ,\quad \text{or} \quad \alpha_S = \pi\ ,\ S_0 = \frac{m_0}{2\Omega_0}\ ,\ \Omega_0 > 0\ ; \tag{4.43}$$

$$Z(\xi) = \cos\xi \pm \sqrt{-\varepsilon_0 - 1}\sin\xi\ ,\ \varepsilon_0 < 0\ . \tag{4.44}$$

The equation of a trajectory looks as

$$\mathbf{r}(t) = r_0\sqrt{-\varepsilon_0}\ [\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cos(\Omega_0 t \mp \varphi)\ ,\ E_0 < 0\ , \tag{4.45}$$

where

$$\varphi = \arctan\sqrt{-\varepsilon_0 - 1}\ . \tag{4.46}$$

**II.2.** $BZ = \lambda_S(Z-1)$, $\quad\mu_S = 0$, $\quad\lambda_S = -\tan\alpha_S = \sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}$, $\quad\lambda_C = 0$, $\quad\mu_C = 1$.

$$\cos\alpha_S = -\frac{m_0}{2\Omega_0 S_0}\ , \tag{4.47}$$

$$Z(\xi) = \cos\frac{\xi}{\sqrt{\lambda_S^2 + 1}} \pm \sqrt{-\varepsilon_0 - 1}\sin\frac{\xi}{\sqrt{\lambda_S^2 + 1}}\ ,\ \varepsilon_0 < 0\ . \tag{4.48}$$

The equation of a trajectory is

$$\mathbf{r}(t) = r_0\sqrt{-\varepsilon_0}\ [\sin\Omega_0 t\mathbf{e}_X + \cos\Omega_0 t\mathbf{e}_Y]\cos(\Omega_0 t\cos\alpha_S \mp \varphi) +$$

$$+ r_0\sqrt{\frac{4\Omega_0^2 S_0^2}{m_0^2} - 1}\,[\sqrt{-\varepsilon_0}\cos(\Omega_0 t\cos\alpha_S \mp \varphi) - 1]\mathbf{e}_Z\ ,\ E_0 < 0\ . \tag{4.49}$$

The case II.1 is deduced from the case II.2 at $\lambda_S = 0$, i.e. if conditions (4.43) fulfill. Assuming for the electron $S_0 = s/c^2 = \hbar/2c^2$, $m_0 = m_e$, we obtain $h\nu_Z = \hbar\Omega_Z = m_e c^2$, or

$$\nu_Z = \frac{\Omega_Z}{2\pi} = \frac{|\Omega_0|}{2\pi} \approx 1.236 \cdot 10^{20}\ \text{Hz} \tag{4.50}$$

is the frequency corresponding to generally accepted rest energy of the electron or its Compton wavelength $\lambda_e = h/m_e c = 2.42627 \cdot 10^{-12}$ m. The energy of the electron in the center-of-inertia reference frame is negative, $E_0 = -m_e r_0^2\Omega_0^2/2$.

Polarization, or spin projection to the direction of motion, is determined by the value

$$P = \frac{(\mathbf{S}_0 \cdot \mathbf{V}_\mathrm{C})}{S_0 V_\mathrm{C}} = \frac{(\mathbf{S}_0 \cdot \mathbf{\Omega})}{\sigma S_0 V_\mathrm{C}} = \cos\alpha_S \ . \tag{4.51}$$

Value $P = +1$ corresponds to $\alpha_S = 0$, $\Omega_0 < 0$, i.e. to counter-clockwise motion, whereas $P = -1$ corresponds to $\alpha_S = \pi$, $\Omega_0 > 0$, i.e. to clockwise motion. It suggests associating these motions with motions of the electron and positron, which hence should differ from each other by the type of motion rather than by charge. More strictly it it is possible to prove or refuse this hypothesis, having considered two-body problem taking into account their interaction with each other, as well as their motion in constant electric and magnetic fields.

**III.** $m_0 \neq 0$, $\mathbf{S}_0 = \mathbf{0}$, $\mathbf{C}_0 \neq \mathbf{0}$. In this case $\lambda_S = 0$, $\mu_S = 1$, and equations (3.22)-(3.25) give solutions $B = 0$, $Z = 1$, and relations $\varepsilon_0 = 1$, $\mu_C = -1$, or

$$E_0 = \frac{m_0 r_0^2 \Omega_0^2}{2} \ , \ \cos\alpha_C = \frac{m_0 \Omega_0}{C_0} \ . \tag{4.52}$$

It is easy to see that this case turns out from case I.1 at $\lambda_S = 0$. The equation of a trajectory looks like Eq.(4.3).

**IV.** The system of equations (3.22)-(3.25) admits the solution corresponding to zero mass, $m_0 = 0$. In this case $\mu_C = 1$, $\lambda_C = 0$, $Z = 1$, and equation of motion gives following variants.

**IV.1.** $m_0 = 0$, $\mathbf{S}_0 \neq \mathbf{0}$, $\mathbf{C}_0 \neq \mathbf{0}$.

$$B = 0 \ , \ \cos\alpha_S = \frac{E_0}{r_0^2 \Omega_0^3 S_0} \ , \ \cos\alpha_C = \frac{E_0}{r_0^2 \Omega_0 C_0} \ . \tag{4.53}$$

In the center-of-inertia the equation of a trajectory looks like Eq.(4.3).

**IV.2.** $m_0 = 0$, $\mathbf{S}_0 \neq \mathbf{0}$, $\mathbf{C}_0 = \mathbf{0}$.

$$B(\xi) = -\frac{\cot\alpha_S}{2}\xi^2 + B_1\xi \ , \ B_1 = \mathrm{const} \ , \ E_0 = 0 \ . \tag{4.54}$$

In the center-of-inertia the equation of a trajectory looks like

$$\mathbf{r}(t) = r_0[\sin\Omega_0 t \mathbf{e}_X + \cos\Omega_0 t \mathbf{e}_Y] + r_0 \left[-\frac{\cot\alpha_S}{2}\Omega_0^2 t^2 + B_1 \Omega_0 t\right] \mathbf{e}_Z \ . \tag{4.55}$$

To eliminate such divergent trajectories, it is sufficient to assume $B_1 = 0$, $\alpha_S = \pm\pi/2$, what corresponds to transversal polarization of pseudo-vector $\mathbf{S}_0$.

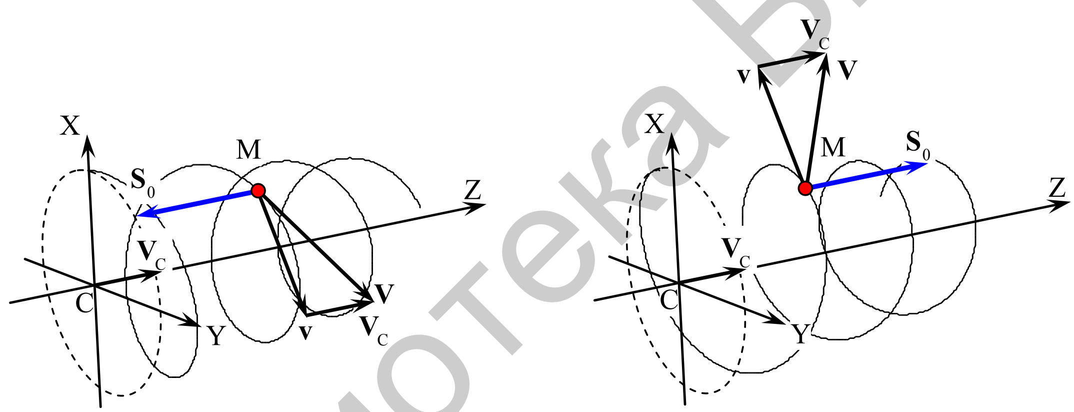
**IV.3.** $m_0 = 0$, $\mathbf{S}_0 = \mathbf{0}$, $\mathbf{C}_0 \neq \mathbf{0}$.

In this case we have $B = 0$, $\alpha_C = \pm\pi/2$, $E_0 = 0$, what corresponds to transversal polarization of pseudo-vector $\mathbf{C}_0$. The equation of a trajectory looks like Eq.(4.3).

Summarizing the results obtained, it is possible to assert that all finite trajectories of free mass points with internal degrees of freedom are subdivided into three types.

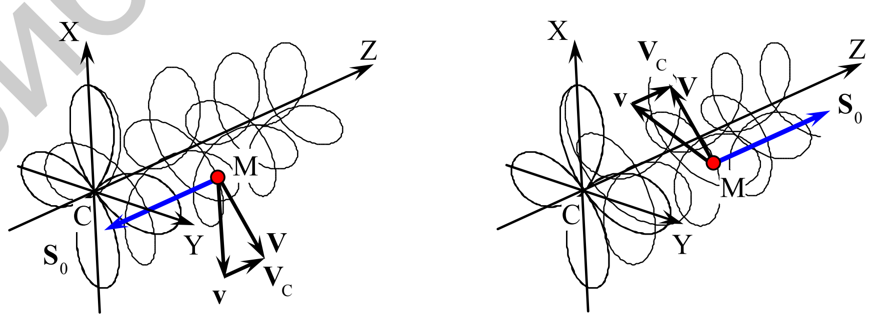
Trajectories of the first type are right or left helix along the direction of motion of the center of inertia. They are specific for the cases I.1-I.3, I.6, I.7 (at $E_0 = 0$), III and IV, and there is no restrictions in $E_0$ only for the cases I.1 and IV.1, whereas for the rest cases we have $E_0 = 0$. Figure 2 shows trajectories for the case I.3 for polarization $P = -1$ (clockwise motion along Z-axis, Figure 2a) and $P = +1$ (counter-clockwise motion along Z-axis, Figure 2b).

Trajectories of the second type in the center-of-inertia reference frame are plane multi-petal rosettes. They are specific for the cases I.4, I.5, I.7 (at $S_0^2 = m_0^2/4\Omega_0^2$) and II.1. They are closed $2N$-petal plane rosettes, when frequencies $\Omega_\varepsilon$, $\omega_C$ and $\Omega_C$ are multiple to the frequency $\Omega_0$ and represented in Figure 3 for the case I.4 ($E_0 \leq 0$, $\Omega_\varepsilon = 2\Omega_0$). As in preceding cases the motion is clockwise for $P = -1$ (Figure 3a) and counter-clockwise one for $P = +1$ (Figure3b). The direction of pseudo-vector $\mathbf{C}_0$ in Figures 2 and 3 does not pointed out.



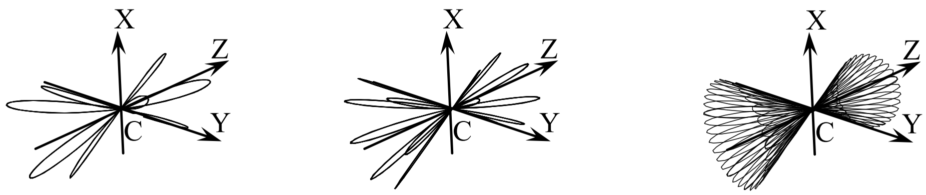

a) $P = -1$ b) $P = +1$

Figure 2.




a) $P = -1$ b) $P = +1$

Figure 3.




a) $N = 4$ b) $N = 7$ c) $N = 4,43$

Figure 4.

Trajectories of the third type are specific for the cases I.7 and II.2. They are multi-petal symmetric space curves in the center-of-inertia reference frame. The number of petals is defined by a condition of the maximum removal of the mass point from the center of inertia that takes place at $\cos(\Omega_C t \pm \varphi_C) = 1$, i.e. at

$$\frac{m_0\sqrt{\mu_C}}{2S_0}t \pm \arctan\sqrt{\frac{-\varepsilon_0}{\mu_C} - 1} = 2k\pi\ , \quad k = 0, \pm 1, \pm 2, \ldots\ . \tag{4.56}$$

These curves are closed $2N$-petal space rosettes, when frequency $\Omega_C$ is multiple to the frequency $\Omega_0$, $\Omega_C = N\Omega_0$, or

$$\frac{m_0\sqrt{\mu_C}}{2S_0\Omega_0} = \sqrt{\mu_C}\cos\alpha_S = N\ . \tag{4.57}$$

The motion is going clockwise for $P < 0$ and counter-clockwise for $P < 0$. Figure 4 shows samples of trajectories for $P < 0$, $N = 4$ (Figure 4a), $N = 7$ (Figure 4b) and for the nonintegral $N = 4.43$ (Figure 4c) in the interval $0 \leq \Omega_0 t \leq 50$.

# 5 Equation of moments and definition of spin

As it is known, one of internal property of particles is spin, associated classically with proper angular momentum of particle. Therefore a temptation arises to connect pseudo-vectors **S** and **C** with spin. For the sake of it we will consider the equation of moments

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} + \mathbf{T}, \tag{5.1}$$

where

$$\mathbf{L} \doteq [\mathbf{R}\times\mathbf{P}] = m_0[\mathbf{R}\times\mathbf{V}] - [\mathbf{R}\times\frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{R}\times[\mathbf{S}\times\mathbf{W}]] =$$

$$= m_0[\mathbf{R}\times\mathbf{V}] - [\mathbf{R}\times\frac{\partial U_0}{\partial \mathbf{V}}] + [\mathbf{R}\times[\mathbf{S}\times\mathbf{W}]] - [\mathbf{R}\times[\mathbf{R}\times\mathbf{C}]] + [\mathbf{R}\times([\mathbf{R}\times\mathbf{V}]\cdot\frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{V}})] \tag{5.2}$$

is a dynamical angular momentum,

$$\mathbf{M} \doteq [\mathbf{R}\times\mathbf{F}] = -[\mathbf{R}\times\frac{\partial U}{\partial \mathbf{R}}] + [\mathbf{R}\times[\mathbf{C}\times\mathbf{V}]] =$$

$$= -[\mathbf{R} \times \frac{\partial U_0}{\partial \mathbf{R}}] + [\mathbf{R} \times ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{R}})] \quad (5.3)$$

is a moment of force, acting at the mass point,

$$\mathbf{T} \doteq [\mathbf{V} \times \mathbf{P}] = -[\mathbf{V} \times \frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{V} \times [\mathbf{S} \times \mathbf{W}]] =$$

$$= -[\mathbf{V} \times \frac{\partial U_0}{\partial \mathbf{V}}] + [\mathbf{V} \times [\mathbf{S} \times \mathbf{W}]] - [\mathbf{V} \times [\mathbf{R} \times \mathbf{C}]] + [\mathbf{V} \times ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{V}})] \quad (5.4)$$

is an additional twisting moment, or torque. [1]

The equation of moments (5.1) follows from equation of motion (2.1). Therefore, having only definition (5.2) for angular momentum it is impossible to define a concept of proper angular momentum for the point with internal degrees of freedom. Indeed, let us consider free mass point, for which $U_0 = 0$, $\mathbf{S}^{ext} = \mathbf{0}$, $\mathbf{C}^{ext} = \mathbf{0}$. Then the moment of force (5.3) becomes zero, and the dynamical angular momentum (5.2) and torque (5.4) are equal to

$$\mathbf{L} = m_0[\mathbf{R} \times \mathbf{V}] + [\mathbf{R} \times [\mathbf{S}_0 \times \mathbf{W}]] - [\mathbf{R} \times [\mathbf{R} \times \mathbf{C}_0]] \ , \quad (5.5)$$

$$\mathbf{T} = [\mathbf{V} \times [\mathbf{S}_0 \times \mathbf{W}]] - [\mathbf{V} \times [\mathbf{R} \times \mathbf{C}_0]] \ . \quad (5.6)$$

respectively.

Introducing variables relative to the center of inertia (3.4), we have

$$\mathbf{L} = \mathbf{L}_{\mathrm{C}} + \mathbf{L}_{0\mathrm{C}} + \mathbf{L}_0 \ , \quad (5.7)$$

$$\mathbf{T} = \mathbf{T}_{\mathrm{C}} + \mathbf{T}_{0\mathrm{C}} + \mathbf{T}_0 \ , \quad (5.8)$$

where

$$\mathbf{L}_{\mathrm{C}} = m_0[\mathbf{R}_{\mathrm{C}} \times \mathbf{V}_{\mathrm{C}}] - [\mathbf{R}_{\mathrm{C}} \times [\mathbf{R}_{\mathrm{C}} \times \mathbf{C}_0]] \quad (5.9)$$

is an angular momentum of the center of inertia C relative to the origin O, as if total rest-mass $m_0$ was in the center of inertia C,

$$\mathbf{L}_{0\mathrm{C}} = m_0[\mathbf{r} \times \mathbf{V}_{\mathrm{C}}] + m_0[\mathbf{R}_{\mathrm{C}} \times \mathbf{v}] + [\mathbf{R}_{\mathrm{C}} \times [\mathbf{S}_0 \times \mathbf{w}]] - [\mathbf{R}_{\mathrm{C}} \times [\mathbf{r} \times \mathbf{C}_0]] - [\mathbf{r} \times [\mathbf{R}_{\mathrm{C}} \times \mathbf{C}_0]] \quad (5.10)$$

is an angular momentum of the center of mass M relative to the origin O, related with both its motion relative to the center of inertia C and a motion of the latter one in absolute reference frame,

$$\mathbf{L}_0 = m_0[\mathbf{r} \times \mathbf{v}] + [\mathbf{r} \times [\mathbf{S}_0 \times \mathbf{w}]] - [\mathbf{r} \times [\mathbf{r} \times \mathbf{C}_0]] \quad (5.11)$$

is an angular momentum of the center of mass M relative to the center of inertia C;

$$\mathbf{T}_{\mathrm{C}} = -[\mathbf{V}_{\mathrm{C}} \times [\mathbf{R}_{\mathrm{C}} \times \mathbf{C}_0]] \quad (5.12)$$

is a torque relative to the origin O, acting upon the center of mass M,

$$\mathbf{T}_{0\mathrm{C}} = [\mathbf{V}_{\mathrm{C}} \times [\mathbf{S}_0 \times \mathbf{w}]] - [\mathbf{V}_{\mathrm{C}} \times [\mathbf{r} \times \mathbf{C}_0]] - [\mathbf{v} \times [\mathbf{R}_{\mathrm{C}} \times \mathbf{C}_0]] \quad (5.13)$$

[1] In standard mechanics the concept "torque" is applied sometimes to the moment of force (5.3). Here we distinguish the moment of force (5.3) and torque (5.4).

is additional torque relative to the origin O, acting upon the center of mass M and related with both its motion relative to the center of inertia C and a motion of the latter one in absolute reference frame,

$$\mathbf{T}_0 = [\mathbf{v} \times [\mathbf{S}_0 \times \mathbf{w}]] - [\mathbf{v} \times [\mathbf{r} \times \mathbf{C}_0]] \tag{5.14}$$

is a torque relative to the center of inertia C, acting upon the center of mass M.

In the center-of-inertia reference frame we have $\mathbf{L}_{\mathrm{C}} = \mathbf{0}$, $\mathbf{L}_{0\mathrm{C}} = \mathbf{0}$, $\mathbf{T}_{\mathrm{C}} = \mathbf{0}$, $\mathbf{T}_{0\mathrm{C}} = \mathbf{0}$, so that the equation of moments (5.1) takes the form

$$\frac{d\mathbf{L}_0}{dt} = \mathbf{T}_0 \ , \tag{5.15}$$

or

$$[\mathbf{r} \times \frac{d}{dt}[m_0\mathbf{v} + [\mathbf{S}_0 \times \mathbf{w}] - [\mathbf{r} \times \mathbf{C}_0]]] = \mathbf{0} \ . \tag{5.16}$$

In the reference frame in question $m_0\mathbf{v} + [\mathbf{S}_0 \times \mathbf{w}] - [\mathbf{r} \times \mathbf{C}_0] = \mathbf{0}$, therefore equations (5.15) and hence (5.1) are identities.

On the other hand, since $\mathbf{r} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$, in the center-of-mass reference frame, then $\mathbf{L}_0 = \mathbf{0}$. Therefore $\mathbf{L}_0$ cannot play a role of proper angular momentum (spin) of the point M to which we want to relate pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$. For definition of their physical sense additional reasons are necessary. Point M with internal degrees of freedom should be considered as non-inertial extended object rotating with angular velocity $\boldsymbol{\omega}_0$ and possessing the proper angular momentum (spin). To take into account internal rotational degrees of freedom it is necessary to introduce the total moment of momentum instead of angular moment (5.7) (see, for example, [12])

$$\mathbf{J} = \mathbf{L} + \mathbf{s} = \mathbf{L}_{\mathrm{C}} + \mathbf{L}_{0\mathrm{C}} + \mathbf{L}_0 + \mathbf{s} \ , \tag{5.17}$$

which is defined as spin $\mathbf{s}$ in the center-of-mass reference frame, and equals to

$$\mathbf{J}_0 = \mathbf{L}_0 + \mathbf{s} \tag{5.18}$$

in the center-of-inertia reference frame.

When interaction is missing, equation of motion (2.1) reduces to Eq.(3.1). Consequently, pseudo-vectors $\mathbf{J}_0$, $\mathbf{L}_0$ and $\mathbf{s}$ in the center-of-inertia reference frame should be precessing round the direction of the center of inertia with the same angular velocity $\boldsymbol{\Omega}_0 = \sigma\mathbf{V}_{\mathrm{C}}$, as pseudo-vectors $\mathbf{S}_0$ and $\mathbf{C}_0$, i.e. they have to satisfy to equations of motion of the same form

$$\frac{d\mathbf{J}_0}{dt} = [\boldsymbol{\Omega}_0 \times \mathbf{J}_0] = \sigma[\mathbf{V}_{\mathrm{C}} \times \mathbf{J}_0] \ , \tag{5.19}$$

$$\frac{d\mathbf{L}_0}{dt} = [\boldsymbol{\Omega}_0 \times \mathbf{L}_0] = \sigma[\mathbf{V}_{\mathrm{C}} \times \mathbf{L}_0] \ , \tag{5.20}$$

$$\frac{d\mathbf{s}}{dt} = [\boldsymbol{\Omega}_0 \times \mathbf{s}] = \sigma[\mathbf{V}_{\mathrm{C}} \times \mathbf{s}] \ . \tag{5.21}$$

Comparison of Eq.(5.20) with Eq.(5.15) and taking into account Eq.(5.14) gives

$$\frac{d\mathbf{L}_0}{dt} = [\mathbf{\Omega}_0 \times \mathbf{L}_0] = \mathbf{T}_0 = [\mathbf{v} \times ([\mathbf{S}_0 \times \mathbf{w}] - [\mathbf{r} \times \mathbf{C}_0])] = -[\mathbf{v} \times m\mathbf{v}] = \mathbf{0} \ , \qquad (5.22)$$

whence it follows that $\mathbf{L}_0$ is parallel to the angular velocity of precession $\mathbf{\Omega}_0$,

$$\mathbf{L}_0 = m_0[\mathbf{r} \times \mathbf{v}] + [\mathbf{r} \times [\mathbf{S}_0 \times \mathbf{w}]] - [\mathbf{r} \times [\mathbf{r} \times \mathbf{C}_0]] = I_0\mathbf{\Omega}_0 = m_0 r_0^2 \mathbf{\Omega}_0 \ , \qquad (5.23)$$

whereas spin $\mathbf{s}$ is parallel to the angular velocity $\boldsymbol{\omega}_0$ of proper rotation of extended point M, if it is defined as its proper angular momentum. Then pseudo-vector $\mathbf{L}_0$ gets a sense of orbital angular momentum of the point M relative to the center of inertia. If $\hat{\mathbf{j}}_0$ is a proper tensor of inertia of the point M, the spin is defined as follows

$$\mathbf{s} = \hat{\mathbf{j}}_0 \boldsymbol{\omega}_0 = j_0 \boldsymbol{\omega}_0 \ , \qquad (5.24)$$

where $j_0$ is eigenvalue of $\hat{\mathbf{j}}_0$, i.e. proper moment of inertia of the point M relative to the axis of its rotation. Due to Eq.(5.23) and Eq.(5.24) the total moment of momentum (5.18) is equal to

$$\mathbf{J}_0 = \mathbf{L}_0 + \mathbf{s} = m_0 r_0^2 \mathbf{\Omega}_0 + j_0 \boldsymbol{\omega}_0 = \hat{\mathbf{I}} \mathbf{\Omega}_0 \ , \qquad (5.25)$$

where $\hat{\mathbf{I}}$ is the tensor of inertia of the point M relative to the center of inertia C, $r_0$ is the radius of Zitterbewegung.

Applying the expression (5.24) to electron roughly represented as a rigid sphere of radius $\rho_0$, consisting of structureless mass points, we have $j_0 = 2m_e\rho_0^2/5$, $s = \hbar/2$, whence it follows

$$\omega_0 = 2\pi\nu_0 = \frac{s}{j_0} = \frac{5\hbar}{4m_e\rho_0^2} \ . \qquad (5.26)$$

Substituting here a value of the electron radius $\rho_0 \approx 10^{-22}$ m (Dehmelt, [19]), we will obtain an estimate

$$\nu_0 = \frac{5\hbar}{8\pi m_e \rho_0^2} \approx 2.3 \cdot 10^{39} \text{ Hz} \ , \qquad (5.27)$$

i.e. frequency of proper rotation of the electron is at least 19 orders greater than the frequency of Zitterbewegung (4.50), $\nu_Z = m_e c^2/h \approx 1.24 \cdot 10^{20}$ Hz, whereas a velocity on the electron surface is $v = \nu_0\rho_0 \approx 2.3 \cdot 10^{17}$ m/s, what is 9 orders greater than the speed of light.

At high frequency of proper rotation in electron volume there should be arising huge centrifugal forces of inertia, which relocate interior substance of the electron to periphery. On the other hand, stability of the electron implies that centrifugal forces of inertia should be balanced by interior forces so that the equilibrium shape of the electron represented something like a toroid or a ring with a rigid surface. It is consistent with both earlier idea by Parson-Compton [5], [6], and modern toroidal or ring model (see, e.g., [20], [21]) or dumbbell model of the electron ( [22]). Ring (or dumbbell) is characterized, at least, by two sizes, by its radius (length) and thickness. Therefore it is difficult to say what value $\rho_0 \approx 10^{-22}$ m obtained by Dehmelt concerns. Even if $\rho_0$ is a classical electron radius, $r_e = \alpha\lambda_{\rm C}/2\pi \approx 2.9 \cdot 10^{-15}$ m, we will obtain instead of (5.27) an estimate for the frequency of rotation $\nu_0 \approx 2.7 \cdot 10^{24}$ Hz and for the velocity $v = \nu_0 r_e \approx 7.8 \cdot 10^9$ m/s, that also is greater than the speed of light. In due time Lorentz has refused an idea of extended electron as equatorial velocity of a surface of spinning electron has turned out to be more than the speed of light. However as long as we are in the frameworks of classical mechanics, we have not any restriction on speed.

Whatever the electron would be actually arranged, its internal structure should determine both field, created by it, and a type of its motion, depending on a spin being integral property of this structure. If the motion of free electron reduces to equations (3.3), (3.9), then in the center-of-inertia reference frame we obtain trajectories, described in §4. A motion of free extended electron relative to the center of inertia means that, on the one hand, its interior substance is acted upon by centrifugal forces of inertia, and, on the other hand, by centripetal forces which twist a trajectory. Resultant of these forces can be represented as $m'[\mathbf{v}\times\mathbf{\Omega}_0]$, where $m'$ is some coefficient with dimension of mass. For the point mass particle there takes place a kind of equivalence principle, $m' = m_0$, where the rest mass $m_0$ play a role of the measure of inertness of inertially moving mass point, and a mass $m'$ is a measure of non-inertiality of such a point moving with acceleration, whereas a mass $m$, entering to right hand side of Eq.(3.1), is a measure of inertness of extended object, which center of inertia moves inertially. Therefore the hypothesis $m' = m_0$ is not obvious for such extended particle as the electron, the more so mass of micro-object depends on its interaction with external fields and is not an additive quantity. Inasmuch as $m'[\mathbf{v} \times \mathbf{\Omega}_0]$ is resultant force acting upon free electron, it follows from equation (2.3), written in the center-of-inertia reference frame,

$$\mathbf{F} = \left(-\frac{\partial U}{\partial \mathbf{R}} + [\mathbf{C}\times\mathbf{V}]\right)_{U=0,\mathbf{V}_{\rm C}=\mathbf{0}} = [\mathbf{C}_0 \times \mathbf{v}] = m'[\mathbf{v}\times\mathbf{\Omega}_0]\ . \quad (5.28)$$

From here pseudo-vector $\mathbf{C}_0$ considering Eq.(3.10) may be determined as

$$\mathbf{C}_0 = -m'\mathbf{\Omega}_0 + \gamma\mathbf{v} = -m'\sigma\mathbf{V}_{\rm C} + \gamma\mathbf{v}\ , \quad (5.29)$$

where $\sigma$ and $\gamma$ are constant pseudo-scalars.

It should be noted that transformation (3.4) is a special case of the Galileo transformation

$$\mathbf{R}'(t) = \mathbf{R}(t) - \mathbf{V}_{\rm K'}t\ ,\ \mathbf{V}'(t) = \mathbf{V}(t) - \mathbf{V}_{\rm K'}\ ,\ \mathbf{W}'(t) = \mathbf{W}(t)\ . \quad (5.30)$$

where $\mathbf{V}_{\rm K'}$ is a velocity of inertial system K$'$ relative to absolute system K (here system K$'$ is the center-of-inertia reference frame, i.e. $\mathbf{V}_{\rm K'} = \mathbf{V}_{\rm C}$). If the Galileo's relativity principle is valid, equation (2.1) should be covariant relative to transformations (5.30), i.e. in the system K$'$ it should be $d\mathbf{P}'/dt = \mathbf{F}'$, where

$$\mathbf{P} = m_0\mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S}\times\mathbf{W}] = m_0\mathbf{V}' + m_0\mathbf{V}_{\rm K'} - \frac{\partial U}{\partial \mathbf{V}'} + [\mathbf{S}\times\mathbf{W}'] = \mathbf{P}' + m_0\mathbf{V}_{\rm K'}\ , \quad (5.31)$$

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{R}} + [\mathbf{C}\times\mathbf{V}] = -\frac{\partial U}{\partial \mathbf{R}'} + [\mathbf{C}\times\mathbf{V}'] + [\mathbf{C}\times\mathbf{V}_{\rm K'}] = \mathbf{F}' + [\mathbf{C}\times\mathbf{V}_{\rm K'}]\ , \quad (5.32)$$

whence it follows relation

$$[\mathbf{C} \times \mathbf{V}_{K'}] = \mathbf{0} . \qquad (5.33)$$

For free electron we have $\mathbf{C} = \mathbf{C}_0$ and it follows from Eq.(5.33) and Eq.(5.29) that $\gamma = 0$. Thus, finally

$$\mathbf{C}_0 = -m'\boldsymbol{\Omega}_0 = -m'\Omega_0\mathbf{e}_Z . \qquad (5.34)$$

Writing down expressions for $\mathbf{C}_0$, corresponding to cases I.1- I.7 ($\mathbf{S}_0 \neq \mathbf{0}$), and comparing them with Eq.(5.34), we find that the condition (4.6), corresponding to $\mu_C = 0$ and $E_0 = 0$, should satisfied, whence it follows $m' = -m_0/2$, and due to Eqs.(3.15), (4.1), (4.5), (4.8), (4.12), (4.19), (4.28) and (4.33) pseudo-vector $\mathbf{S}_0$ looks like

$$\mathbf{S}_0 = -\frac{m_0}{2\Omega_0}\mathbf{e}_Z , \qquad (5.35)$$

i.e. two kinds of motion, corresponding to polarizations $P = \pm 1$, are possible. Combining Eqs.(5.34) and (5.35), we obtain relation

$$\mathbf{C}_0 = -\Omega_0^2\mathbf{S}_0 . \qquad (5.36)$$

The case II ($\mathbf{C}_0 = \mathbf{0}$, $E_0 = -m_0\mathbf{r}_0^2\Omega_0^2/2$) corresponds to $m' = 0$.

Expression for $\mathbf{C}_0$, corresponding to the case III ($\mathbf{S}_0 = \mathbf{0}$, $E_0 = m_0\mathbf{r}_0^2\Omega_0^2/2$), which is similar to Eq.(5.34), gives $m' = -m_0$. It may be assumed that the state with $\mathbf{S}_0 = \mathbf{0}$ is a bound state of two particles with opposite polarizations, contribution of every of which in $\mathbf{C}_0$ is $m' = -m_0/2$. More strictly it can be confirmed after a detailed solution of the two-body problem for interacting mass points with internal degrees of freedom.

Trajectory of mass point in cases I and III is a circle (4.3) of radius $r_0$ in the center-of-inertia reference frame. The negative value of mass $m'$ means, that the force (5.28) is centripetal, rather than centrifugal one. The unique reason of such strange behavior is existence of internal rotational degrees of freedom, described by pseudo-vector $\mathbf{S}_0$. It is reasonably to express $\mathbf{S}_0$ in terms of spin (5.24), as follows

$$\mathbf{S}_0 = -\frac{1}{c^2}\mathbf{s} , \qquad (5.37)$$

where $c$ is some constant with dimension of velocity.

We obtain from Eqs.(5.24), (5.35) and (5.37)

$$\mathbf{s} = j_0\boldsymbol{\omega}_0 = \frac{m_0c^2}{2\Omega_0}\mathbf{e}_Z = \frac{m_0c^2}{2\Omega_0^2}\boldsymbol{\Omega}_0 . \qquad (5.38)$$

Equations of motion (3.9) give $\dot{\mathbf{s}} = \mathbf{0}$, implying conservation of spin direction when interaction is negligible. Substituting (5.38) in (5.25), we obtain for the total moment of momentum relative to the center of inertia

$$\mathbf{J}_0 = \left(1 + \frac{2r_0^2\Omega_0^2}{c^2}\right)\mathbf{s} . \qquad (5.39)$$

Introducing a denotation

$$\hbar = \frac{m_0 c^2}{|\Omega_0|} \ , \tag{5.40}$$

we have for spin $s = \hbar/2$. Here a question remains open whether $c$ and $\hbar$ be the velocity of light and Planck constant, respectively. Its solution will be determined by behavior of particles in external fields and their interaction with each other.

# 6 Equation of motion for spin

For obtaining complete solution of a problem about a motion of the system in question it is necessary to add the equations for internal degrees of freedom. In the previous paragraph we have found out that the equation of the moments (5.1) is a consequence of the equation (2.1), and it cannot be considered as the additional equation. For free system we have the equation for spin (5.21) which means that there exists a preferred direction, namely, a direction of motion of the center of inertia, round which the spin is precessing with constant angular velocity $\mathbf{\Omega}_0$.

Generally at every given instant spin is precessing round any instantaneous direction, simultaneously moving in space together with the center of mass. If $\mathbf{N}(t)$ is a vector pointing out in this direction the spin equation of motion can be written as

$$\frac{d\mathbf{s}}{dt} = [\mathbf{\Omega}_{\mathbf{N}} \times \mathbf{s}] + \mathbf{m}(t) = \sigma_{\mathbf{N}}(t)[\mathbf{N} \times \mathbf{s}] + \mathbf{m}(t) \ , \tag{6.1}$$

where $\mathbf{m}(t)$ is some pseudo-vector having a sense of the moment of force or torque acting to extended point. The structure of $\mathbf{m}(t)$, apparently, can be determined on specifying of an interaction of internal substance of the point with external fields. It follows from Eq.(6.1) the constancy of absolute value of spin if $(\mathbf{m} \cdot \mathbf{s}) = 0$. Otherwise spin changes not only over the direction, but also over absolute value. Assuming the interaction of internal substance with external fields to be much weaker than the interaction of the point as a whole object we will consider $\mathbf{m} = \mathbf{0}$ as first approximation. Besides, taking into account that equation (6.1) should be reduced to Eq.(5.21), when interaction is negligible, it is necessary to take as a vector $\mathbf{N}(t)$ the vector

$$\begin{aligned} \mathbf{P} = m\mathbf{V}_{\mathrm{C}} = m_0\mathbf{V} - \frac{\partial U_0}{\partial \mathbf{V}} - \frac{1}{c^2}[\mathbf{s} \times (\mathbf{W} - \Omega_0^2 \mathbf{R})] + \\ + [\mathbf{S}^{ext} \times \mathbf{W}] - [\mathbf{R} \times \mathbf{C}^{ext}] + ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{V}}) \ , \end{aligned} \tag{6.2}$$

which can be treated as a definition of the kinetic momentum related to a particle, specifying the motion of the center of inertia of the latter. In the presence of interaction the momentum (6.2) is not conserved, and spin is precessing round instantaneous direction of the momentum with instant angular velocity $\mathbf{\Omega}(t) = \sigma_{\mathbf{P}}(t)\mathbf{P}/m$. Thus, the equation of motion of the spin can be finally written down in the form

$$\frac{d\mathbf{s}}{dt} = \frac{\sigma_{\mathbf{P}}(t)}{m}[\mathbf{P} \times \mathbf{s}] + \mathbf{m}(t) =$$

$$= \frac{m_0 \sigma_{\mathbf{P}}(t)}{m}[\mathbf{V} \times \mathbf{s}] - \frac{\sigma_{\mathbf{P}}(t)}{m}[\frac{\partial U_0}{\partial \mathbf{V}} \times \mathbf{s}] + \frac{\sigma_{\mathbf{P}}(t)}{mc^2}[\mathbf{s} \times [\mathbf{s} \times (\mathbf{W} - \Omega_0^2 \mathbf{R})]] -$$

$$- \frac{\sigma_{\mathbf{P}}(t)}{m}[\mathbf{s} \times [\mathbf{S}^{ext} \times \mathbf{W}]] + \frac{\sigma_{\mathbf{P}}(t)}{m}[\mathbf{s} \times [\mathbf{R} \times \mathbf{C}^{ext}]] - \frac{\sigma_{\mathbf{P}}(t)}{m}[\mathbf{s} \times ([\mathbf{R} \times \mathbf{V}] \cdot \frac{\partial \mathbf{C}^{ext}}{\partial \mathbf{V}})] + \mathbf{m}(t) \ . \quad (6.3)$$

All known equations of motion of spin in non-relativistic approximation have structure of Eq.(6.3) with $\mathbf{m}(t) = 0$. System of equations (2.1) and (6.3) with momentum (6.2) and force (2.7) allow to solve a lot of problems about the motion of spinning particles in various fields and to compare these solutions with well-known results. Obtained here non-relativistic equations of motion suppose the relativistic generalization [16], however there are subtleties which should be considered carefully.

Being founded on the stated above it is possible to assert that the trembling motion of objects with internal degrees of freedom has origin in classical mechanics. This circumstance does not contradict numerous modern researches in which the classical theory of spin is developing (see, e.g., [13]). Experimental observation of the motion of individual micro-objects with internal structure is for now impossible at present status of experimental technique. Nevertheless, the quantum phenomena of Zitterbewegung type, which though are not experimentally observable now, but they can be simulated, and the first results are presented in work [23]. However, the results obtained say that this phenomenon should take place, in principle, for such classical objects as spinning top whose spin (proper moment of momentum) is not parallel to the velocity of translational motion of its center of inertia. Other examples are the projectile (bullet), which is shot through a rifle trunk, and a boomerang. It is possible to hope also that equations similar to the equations obtained in §4 will allow explaining such little-understandable phenomenon, as Dzhanibekov's effect. For micro-objects Zitterbewegung is manifested so as they have the wave nature, whereas this phenomenon is imperceptible for macro-objects because of both their large mass, and smallness of radius (amplitude) $r_0$ of trembling motion. Thus, there appears a possibility of classical interpretation of quantum phenomena.

Moreover, two types of motion, corresponding to opposite polarizations, give rise to new look at the origin of electric charge. The solution of non-relativistic problem of the motion of the point with internal degrees of freedom, given in §§4-5, and its application to free electron leads to new interpretation of the charge of elementary particle which sign is determined by its helicity. Right helicity $h = -P = +1$, $\alpha_S = \pi$, $\Omega_0 > 0$ (Figure 2a), corresponds to right polarization of spin for free antiparticles (positrons), charged positively, whereas left helicity $h = -P = -1$, $\alpha_S = 0$, $\Omega_0 < 0$ (Figure 2b), corresponds to left polarization of spin for free particles (electrons), charged negatively. For interacting particles helicity can be distinct from $h = \pm 1$, but its sign as before corresponds to the sign of charge. Hence, *the charge is the conventional concept characterizing type of the motion of spinning particle, corresponding to its helicity.* In this connection it would like to mention W. Ritz's opinion, according to which "these latter (electric charges) only playing, like the masses in Mechanics, the role of coefficients, conveniently chosen and invariable for a given ion or electron. In a certain sense it is a mechanical theory of electricity" ( [24], p. 149).

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