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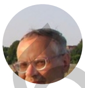
Chapter *in* Communications in Computer and Information Science · January 2015

DOI: 10.1007/978-3-319-20352-2



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Reconstruction of the Surface Heat Flux for a Quasi-linear System of the Hyperbolic Type Heat-Conduction Equations

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Abstract. The problem of the identification of the surface heat flux for a quasi-linear system of the hyperbolic type heat-conduction equations is studied. An approach is proposed based on the stage-by-stage suboptimal optimization of the cost functional and input data filtering using the HuberTikhonov functional. Results are presented for the numerical modeling of the identification problem in conditions of both standard noisy data and noise emissions.

Keywords: Inverse problem · Heat flux · The hyperbolic type system · HuberTikhonov functional · Suboptimal optimization

1 Introduction

To control processes of heat transfer, mass transfer, etc., different methods of mathematical modeling, the theory of optimal control and the theory of inverse problems (IP) of mathematical physics are widely applied [1–4].

In particular, in scientific literature the classical IP theory of heat conductivity consisting in reconstruction of the time-varying heat fluxes is frequently used (see [1–8] and the references therein).

As a rule, the mathematical model of transfer processes is based on the parabolic type heat conductivity equation. At the same time, a number of fast proceeding and intensive transfer processes can be described only within the theory of the hyperbolic type equations and systems of the equations [9–16].

In this paper, we consider IP of reconstruction of the time-varying surface heat flux for quasi-linear system of hyperbolic type differential equations [9,10]. The Dirichlet boundary conditions, initial conditions and time-varying temperature at a given interior point are used as the additional data for reconstruction.

The system under consideration describes transfer processes in the nonlinear mediums and takes into account both the heat flux relaxation time $\tau \geq 0$ and the convective component of the heat transfer [9,10]. When $\tau = 0$, this system of equations can be reduced to the parabolic type heat conductivity equation. Notice that generally, the system is not reducible to one equation [10].

A. Plakhov et al. (Eds.): EmC-ONS 2014, CCIS 499, pp. 49–67, 2015.
DOI: 10.1007/978-3-319-20352-2_4

As it is known, the heat fluxes reconstruction IP belongs to the class of ill-posed problems. Currently, there are several approaches to solving such problems [1–4,20,21]. However, there is no universal method among them that is caused by both difficulties of solving the ill-posed problems and the requirements of an effective realization and high speed of numerical procedures.

In this paper, we develop a method of stage-by-stage suboptimal optimization (SSO) (see [17–19]) combined with a method of the robust estimation on the base of the Huber loss function [22–25].

We consider the filtering procedure as an optimal control problem for the simplest differential equation of the first order [19]. The cost functional for the optimal control problem is the sum of the Huber functional for residuals and the Tikhonov functional for control. Under a suitable choice of settings of the filter, this procedure allows to smooth out and filter out both separate gross measurement errors ("wild" data points, outliers) and standard random errors. It should be mentioned here that the "wild" values of the measurements are quite common in the high-temperature experiences in industry, in the thermal protection methods, in the rocket engines testing, etc.

Notice that the SSO approach develops ideas of the sequential estimation [1,26] that allows to realize data processing in real time. This is important, for example, for the problems of the thermal processes' control.

2 Problem Statement

Let us consider the following initial boundary value problem for the quasi-linear system:

$$
\begin{gathered}
\rho(T)C(T)\frac{DT}{Dt} = -\frac{\partial q}{\partial x},\\
\tau\frac{Dq}{Dt} = -q - \lambda(T)\frac{\partial T}{\partial x}, \\
T = T(x,t),\ \ q = q(x,t),\ \ x_0 \le x \le x_*,\ \ t_0 \le t \le t_*,
\end{gathered}
\tag{1}
$$

with initial and boundary conditions

$$T(x,t_0) = T_{\text{in}}(x),\ \ q(x,t_0) = q_{\text{in}}(x),\ \ x_0 \le x \le x_*, \tag{2}$$

$$T(x_*,t) = T_*(t),\ \ t_0 \le t \le t_*, \tag{3}$$

$$q(x_0,t) = q_0(t),\ \ t_0 \le t \le t_*. \tag{4}$$

Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + \nu(x)\frac{\partial}{\partial x}$ is the material derivative, $\nu(x)\frac{\partial}{\partial x}$ is convection term, $\nu(x)$ is velocity, $T = T(x,t),\ x \in [x_0,x_*] \subset \mathbb{R},\ t \in [t_0,t_*] \subset \mathbb{R}$, is a temperature distribution, $q = q(x,t),\ x \in [x_0,x_*] \subset \mathbb{R},\ t \in [t_0,t_*] \subset \mathbb{R}$, is a heat flux, $\lambda(T)$ is the heat conduction coefficient, $\rho(T)$ is the material density, $C(T)$ is the specific heat of the material, τ is a parameter describing the heat flux relaxation time, $T_{\text{in}}(x), q_{\text{in}}(x), x \in [x_0,x_*]$ are given initial conditions, $T_*(t), t \in [t_0,t_*]$, is a prescribed temperature at the point x_*, $q_0(t), t \in [t_0, t_*]$, is a prescribed heat flux at the point x_0. All the functions $\lambda(T)$, $C(T)$, $\rho(T)$, $T \in \mathbb{R}$, $T_*(t), q_0(t)$, $t \in [t_0, t_*]$, and $T_{\rm in}(x), q_{\rm in}(x)$, $\nu(x)$, $x \in [x_0, x_*]$, are supposed to be sufficiently smooth.

In the problem under consideration, there is a convective component of the heat transfer. The presence of this component leads to instability of numerical methods for solving the direct and inverse problems. In particular, the standard **pdepe** -program of the computational MATLAB package doesn't allow to obtain the numerical solution of the direct problem with large values of parameter ν.

Notice that in the case $\nu(x) \equiv 0$, problem (1) can be reduced to a nonlinear heat conduction equation of the hyperbolic type

$$\tau \frac{\partial}{\partial t}\left(C(T)\rho(T)\frac{\partial T}{\partial t}\right) + \rho(T)C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)$$

with initial and boundary conditions that can be determined on the base of (2)–(4).

In the paper, for the case $\nu(x) \not\equiv 0$, we consider the problem of reconstruction of the heat flux $q_0(t) := q(x_0, t),\ \ t \in [t_0, t_*]$, at the point $x = x_0$ on the base of the known measurements

$$y(t) = T(x_1^*, t) + v(t),\ t \in [t_0, t_*], \tag{5}$$

of the temperature field at a given point x_1^*, $x_0 \le x_1^* \le x_*$. Here function $v(t)$ describes a measurement error.

Following [1,2], to model function $v(\cdot)$ we will apply the statistical description. Taking into account the discrete representation of the temperature measurements, this description takes the form

$$y(t_i) = T(x_1^*, t_i) + w(t_i)\sigma, i = 0, 1, ..., M; \tag{6}$$

where

$$t_i = t_0 + i\Delta t, \quad \Delta t = (t_* - t_0)/M, \tag{7}$$

σ is the standard deviation of the measurement errors, $w(t_i)$ is a realization of the random variable w with the normal distribution. Here we consider that the noisy measurements were carried out for the time stepsize $\Delta t > 0$ (see (6)).

In the nonstandard case, we assume that the probability density function of the measurement error has the form [22]

$$f(w) = \frac{1-\epsilon}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) + \epsilon g(w), \tag{8}$$

where ϵ is a weight parameter and $g(w)$ is an unknown function that perturbs the Gaussian density function. Note that the error probability density function of form (8) can be regarded as a model of large errors [22] in the measurement data. Having assumed in (8) that $\epsilon = 0$, one gets the standard situation.

Let the thermophysical parameters $\rho(T)$, $\lambda(T)$, $C(T)$, τ, $\nu(x)$, the temperatures $T_{\text{in}}(x), x \in [x_0, x_*]$, $T_*(t), t \in [t_0, t_*]$, the heat flux $q_{\text{in}}(x), x \in [x_0, x_*]$, the weight parameter ϵ, and the values of the error deviation σ be known. Then, the general problem of identifying the heat flux at the point $x = x_0$ consists in reconstruction of the function $q_0(t) := q(x_0, t),\ t \in [t_0, t_*]$, from the data (6), system (1) and conditions (2) and (3).

One of the distinguishing characteristics of the proposed approach is that, before solving the posed problem of reconstructing the heat flux from the given inaccurate measurements $y(t), t \in [t_0, t_*]$, a filtering procedure is employed. This procedure yields estimates $y^*(t)$ of the data (5), and the heat flux $q(x_0, t)$, $t \in [t_0, t_*]$, is reconstructed on the basis of these estimates. The filtering procedure is described in Sect. 3. The second specific feature of the proposed approach is that the problem of reconstructing the heat flux $q(x_0, t)$, $t \in [t_0, t_*]$, is solved by the SSO method [18,19]. The essence and advantages of the method are described in Sect. 4. It should be noted that the ideas of this method are also used in the prefiltering procedure.

3 Preliminary Filtering Procedure Using the Huber Function and Tikhonov Regularization

Let a given function $y(t),\ t \in [t_0, t_*]$, be representable in the form

$$y(t) = y^0(t) + v(t),\ t \in [t_0, t_*], \tag{9}$$

where $y^0(t),\ t \in T$, is some unknown smooth function, $v(t),\ t \in [t_0, t_*]$, is a function of unknown disturbances (a noise). It is required, to find a continuous smooth function $y^*(t),\ t \in [t_0, t_*]$, which approximates the function $y^0(t),\ t \in [t_0, t_*]$, on the basis of the given noisy function (9).

To solve this problem, taking into account the smoothness of the reconstructed function $y^0(t),\ t \in [t_0, t_*]$, we formulate the simplest optimal control problem

$$\int_{t_0}^{t_*} f(x(t) - y(t))dt + R(u(\cdot)) \to \min_{z, u(\cdot)} \tag{10}$$

$$\text{s.t. } \dot{x}(t) = u(t),\ x(0) = z.$$

Here $f(z)$ is some function that characterizes the deviation of $|z|$ from zero, $R(u(\cdot))$ is a regularizing term. The particular choice of the functions $f(z)$ and $R(u(\cdot))$ depends on the a priori information about the restored function $y^0(t),\ t \in [t_0, t_*]$, and the nature of the unknown noise $v(t),\ t \in [t_0, t_*]$. Most often the l_1- and l_2-norms are used as $f(z)$ and the functional $\beta \int_{t_0}^{t_*} u^2(t)dt$ is used as the regularizing term $R(u(\cdot))$ [21].

In this paper (see also [19]), we propose to use the Huber loss function as the deviation function $f(z)$ [22]. This function is a combination of l_1- and l_2- norms.

It is known that the Huber function is robust in the sense that it can reduce the influence of "wild" data points (outliers). The Huber function has the form

$$f_\gamma(z) = \begin{cases} z^2/2 \text{ if } |z| \leq \gamma, \\ \gamma|z| - \gamma^2/2 \text{ if } |z| > \gamma. \end{cases}$$

It is quadratic if the module of the deviation z is smaller than a given constant $\gamma > 0$ and has an absolute value term if the module of the deviation is greater than γ.

As it is known (see [22]), the tuning parameter γ is related with the perturbing parameter ϵ by means of the implicit equation

$$\frac{1}{1-\varepsilon} = \frac{1}{\gamma}\sqrt{\frac{2}{\pi}}\ \exp\left[-\frac{\gamma^2}{2}\right] + \operatorname{erf}\left(\frac{\gamma}{\sqrt{2}}\right).$$

The Huber function is more robust than the l_2-function in the sense that it is less sensitive to the outliers in the measurement data.

The functional

$$R_{\alpha_*,\beta_*}(u(\cdot)) := \beta_* \int_{t_0}^{t_*} u^2(t)dt + \eta_* \int_{t_0}^{t_*} \dot{u}^2(t)dt$$

can be used as a regularizing term. Here $\alpha_* \geq 0, \beta_* \geq 0$ are the weight coefficients.

Thus, in order to generate the estimate $y^*(t), t \in [t_0, t_*]$, of the unknown function $y^0(t), t \in [t_0, t_*]$, on the basis of the noisy function $y^*(t), t \in [t_0, t_*]$, we solve the following optimal control problem:

$$\int_{t_0}^{t_*} f_\gamma(x(t) - y(t))dt + \beta_* \int_{t_0}^{t_*} u^2(t)dt + \eta_* \int_{t_0}^{t_*} \dot{u}^2(t)dt, \rightarrow \min_{z,u(\cdot)}, \qquad (11)$$

$$\dot{x}(t) = u(t),\ x(0) = z.$$

Let z^0, $u^0(t), t \in [t_0, t_*]$, be an optimal solution of problem (11).

Then the function

$$y^*(t) = z^0 + \int_{t_0}^{t} u^0(\tau)d\tau,\ \ t \in [t_0, t_*],$$

is considered as an approximation of the unknown function $y^0(t), t \in [t_0, t_*]$.

To solve problem (11), we apply a method of a stage-by-stage optimization. The idea of the method is identical to that of the method described in [17–19]. The essence of the method consists in the reduction of the filtering procedure over the entire time interval $t \in [t_0, t_*]$ to the consecutive solution of p filtering problems on small time intervals $t \in [\tau_j, \tau_{j+1}]$, $j = 0, 1, ..., p-1$, where $\tau_j = t_0 + jL\Delta t$, is a time instant from the set $\{t_i, i = 0, 1, ..., M\} \subset [t_0, t_*]$, $L > 0$, $p > 0$ are integers. Here $L\Delta t$ is the length of the stage defined by the parameter L and a time stepsize $\Delta t > 0$. The time stepsize $\Delta t > 0$ is a step with which the noisy measurements were carried out (see (6)).

Given j ($j = 0, ..., p-1$), to find an approximation $y^*(t)$ of the unknown function $y^0(t)$ at the interval $t \in [\tau_j, \tau_{j+1}]$, on the base of the noisy data $y(t_i)$, $t_i \in [\tau_j, \tau_{j+1}]$, $i = jL, ..., (j+1)L$, the following optimal control problem is solved.

Problem $\mathrm{F_j}$**:** It is required to find a control $u(t), t \in [\tau_j, \tau_{j+1}]$, that solves the problem

$$\sum_{i=jL}^{(j+1)L} f_\gamma(x(t_i) - y(t_i)) + \beta_* \int_{\tau_j}^{\tau_{j+1}} u^2(t)dt + \eta_* \int_{\tau_j}^{\tau_{j+1}} \dot{u}^2(t)dt +$$
$$\bar{\eta}_*(z - y^*(\tau_j - 0))^2 \to \min_{z, u(\cdot)}$$
$$\text{s.t.}\, \dot{x}(t) = u(t), \quad x(\tau_j) = z.$$

Here, in the cost functional, we add one additional term $(z - y^*(\tau_j - 0))^2$, where $y^*(\tau_j - 0) := \lim_{t \to \tau_j, t < \tau_j} y^*(t)$, with the weighting coefficient $\bar{\eta}_* \geq 0$. This term is responsible for "continuous matching" of the functions $y^*(t), t \in [\tau_{j-1}, \tau_j]$, and $y^*(t), t \in [\tau_j, \tau_{j+1}]$, being the optimal trajectories in problems F_{j-1} and F_j respectively, at the boundary point τ_j for two adjacent subintervals. For $j = 0$, we set $y^*(\tau_j - 0) = y(t_0)$.

Let $u^0(t)$, $t \in [\tau_j, \tau_{j+1}]$, z^{0j} be an optimal solution in problem $\mathrm{F_j}$. Then we set

$$y^*(t) = z^{0j} + \int_{\tau_j}^{t} u^0(\tau)d\tau, \quad t \in [\tau_j, \tau_{j+1}],$$

and consider $y^*(t)$, $t \in [\tau_j, \tau_{j+1}]$, as a result of filtering the noisy data $y(t)$ at the j-th state.

For the numerical purposes, we consider the problem $\mathrm{F_j}$ in a class of the piecewise constant controls

$$u(t) = u_i, \ t \in [t_i, t_{i+1}], \quad i = jL, jL+1, ..., (j+1)L - 1, \tag{12}$$
$$t_i = t_0 + i\Delta t, \quad \Delta t = (t_* - t_0)/M, \quad M = (p-1)L.$$

In that case, the problem $\mathrm{F_j}$ is equivalent to a quadratic programming problem that can be easy solved by a standard quadratic programming solver.

4 Heat Flux Reconstruction by the SSO Method

The problem of reconstructing the heat flux $q(x_0, t), t \in [t_0, t_*]$, is formulated as an optimal control problem in which the role of the sought control is played by the reconstructed heat flux and the purpose of optimization is to minimize the functional of the squared deviation between the calculated states $T(x_1^*, t_i)$ of system (1) and the data $y^*(t_i)$, $i = 1, ..., M$. The optimal control problem is solved by the SSO method.

As it was already mentioned above, the idea of the method consists in reducing the single problem of reconstructing the heat flux $q(x_0, t)$ over the entire time interval $t \in [t_0, t_*]$ to the succession of p problems of reconstructing this flux on the small intervals $t \in [\tau_j, \tau_{j+1}]$, $j = 0, 1, ..., p-1$, where $\tau_j = t_0 + jL\Delta t$ and integers $L > 0$, $p > 0$, $M > 0$ are the same as in Sect. 3.

Let us describe the main steps of the method.

Given j ($j = 0, ..., p-1$), let us suppose that the heat flux $q_0(t) := q(x_0, t)$ has been restored for $t \in [t_0, \tau_j]$ and the noisy data $y(t_i)$, $t_i \in [t_0, \tau_j]$, $i = 1, ..., jL$, have been filtered. Hence we know the function $q_0^*(t), t \in [t_0, \tau_j]$, approximating the heat flux $q_0(t)$, $t \in [t_0, \tau_j]$, and the function $y^*(t), t \in [t_0, \tau_j]$. This gives us opportunity to find the solution $T^*(x, t)$, $q^*(x, t)$, $x \in [x_0, x^*]$, $t \in [t_0, \tau_j]$, of system (1)–(4) with $q_0(t)$, $t \in [t_0, \tau_j]$, replaced by $q_0^*(t)$, $t \in [t_0, \tau_j]$.

Using the known number $y^*(\tau_j - 0)$ and the functions $T^*(x, \tau_j)$, $q^*(x, \tau_j)$, $x \in [x_0, x^*]$, we consider the problem of reconstructing the heat flux $q(x_0, t)$ on the interval $t \in [\tau_j, \tau_{j+1}]$.

For this purpose, first of all, we apply the filtering procedure described in Sect. 3 to the known number $y^*(\tau_j - 0)$ and the known noisy data $y(t_i)$, $t_i \in [\tau_j, \tau_{j+1}]$, $i = jL+1, ..., (j+1)L$. To do this, we have to solve problem F_j. As a result, we obtain the estimates $y^*(t)$, $t \in [t_j, t_{j+1}]$, of the noisy measurements.

In order to reconstruct the heat flux $q(x_0, t)$, $t \in [\tau_j, \tau_{j+1}]$, from the new data $y^*(t)$, $t \in [\tau_j, \tau_{j+1}]$, and the known function $T^*(x, \tau_j - 0)$, $q^*(x, \tau_j - 0)$, $x \in [x_0, x_*]$, the following optimal control problem is solved.

Problem $\mathrm{P_j}$**:** find a control $U(t), t \in [\tau_j, \tau_{j+1}]$, which minimizes the cost functional

$$\sum_{i=jL+1}^{(j+1)L} (T(x_1^*, t_i) - y^*(t_i))^2 + \eta_j \int_{\tau_j}^{\tau_{j+1}} \left(\frac{dU(t)}{dt}\right)^2 dt + \\ \gamma_j (U^*(\tau_j - 0) - U(\tau_j + 0))^2 \to \min \tag{13}$$

on the trajectories $T(x, t)$, $q(x, t)$, $x \in [x_0, x_*]$, $t \in [\tau_j, \tau_{j+1}]$, of the system

$$\rho(T)C(T)\frac{DT}{Dt} = -\frac{\partial q}{\partial x}, \\ \tau \frac{Dq}{Dt} = -q - \lambda(T)\frac{\partial T}{\partial x}, \tag{14}$$

$$T = T(x, t), \;\; q = q(x, t), \;\; x_0 \le x \le x_*, \;\; \tau_j \le t \le \tau_{j+1},$$

$$T(x, \tau_j) = T^*(x, \tau_j - 0), \;\; q(x, \tau_j) = q^*(x, \tau_j - 0), \;\; x_0 \le x \le x_*, \tag{15}$$

$$T(x_*, t) = T_*(t), \;\; q(x_0, t) = U(t), \;\; \tau_j \le t \le \tau_{j+1}. \tag{16}$$

Here, in the cost functional (13), the second term is a Tikhonov type regulator with a weighting coefficient $\eta_j > 0$. The third term $\gamma_j(U^*(\tau_j - 0) - U(\tau_j + 0))^2$ is the penalty term (with a weighting coefficient $\gamma_j > 0$) which is responsible for matching the boundary values $U^*(\tau_j - 0)$ and $U^*(\tau_j + 0)$ of the controls obtained at the neighboring $(j-1)$-th and j-th stages; $U^*(t), T^*(x, t), q^*(x, t), t \in [\tau_{j-1}, \tau_j]$, $x \in [x_0, x_*]$, are the optimal control, the corresponding temperature and the heat flux obtained on the previous $(j-1)$-th stage. For $j = 0$ we consider $\gamma_0 = 0$ and $T^*(x, \tau_0 - 0) = T_{\mathrm{in}}(x)$,$q^*(x, \tau_0 - 0) = q_{\mathrm{in}}(x)$, $x \in [x_0, x_*]$.

Let $U^*(t)$ and $T^*(x,t)$, $q^*(x,t)$, $t \in [\tau_j, \tau_{j+1}]$, $x \in [x_0, x_*]$, be an optimal control and the corresponding trajectory in the problem $\mathrm{P_j}$. Then we set

$$q_0^*(t) = q^*(x_0, t), \quad t \in [\tau_j, \tau_{j+1}], \tag{17}$$

and consider $q_0^*(t)$, $t \in [\tau_j, \tau_{j+1}]$, as an approximation of the recoverable heat flux $q(x_0, t)$ at the j-th state.

To solve problem $\mathrm{P_j}$ numerically, the nonlinear system of partial differential equations (14)–(16) is approximated by a system of ordinary differential equations. For this purpose, let us partition the interval $[x_0, x_*]$ into N parts by the points

$$x^i = x_0 + i\Delta x, \; i = 0, 1, ..., N, \;\; \Delta x = (x_* - x_0)/N, \tag{18}$$

$$x_1^* = x_{i_*} = x_0 + i_* \Delta x.$$

Here and in what follows, without loss of generality, we consider that the point x_1^*, at which the temperature's measurements were performed, is a node of the grid (18).

Denote

$$z_1(t) = T_1(t) = T(x_1, t), \;\; z_{2i} = T_{i+1}(t) = T(x_{i+1}, t), \;\; i = \overline{1, N-2};$$
$$z_{2i-1}(t) = q_i(t) = q(x_i, t), \; i = \overline{2, N-1}; \; z_{2N-2}(t) = q_N(t) = q(x_N, t),$$

and consider a vector-function $Z(t) = (z_1(t), ..., z_{2N-2}(t))$, $t \in [\tau_j, \tau_{j+1}]$. Then the cost functional (13) takes the form

$$\sum_{i=jL+1}^{(j+1)L} (z_k(t_i) - y^*(t_i))^2 + \eta_j \int_{\tau_j}^{\tau_{j+1}} \left(\frac{dU(t)}{dt}\right)^2 dt +$$
$$\gamma_j (U^*(\tau_j - 0) - U(\tau_j + 0))^2 \to \min, \tag{19}$$

where $k = 2(i_* - 1)$ if $1 < i_* \le N - 1$, and $k = 1$ if $i_* = 1$.

If N is a rather large number, the system (14)–(16) can be approximated by the following nonlinear system of ordinary differential equations:

$$\frac{dZ(t)}{dt} = \bar{F}(Z(t), \; T_*(t), \; U(t)), \quad Z(t_0) = Z_0 = (z_1^0, ..., z_{2N-2}^0), \tag{20}$$

with

$$\begin{array}{l} z_1^0 = T^*(x_1, \tau_j), \;\; z_{2i}^0 = T^*(x_{i+1}, \tau_j), \;\; i = 1, \dots, N-2; \\ z_{2i-1}^0 = q^*(x_i, \tau_j), \;\; i = 2, \dots, N-1; \; z_{2N-2}^0 = q^*(x_N, \tau_j); \end{array} \tag{21}$$

$$\bar{F}(Z, \, T_*(t), \; U) = F(Z, U, t) = (F_i(Z, U, t), i = 1, \dots, 2(N-1)),$$

$$F_1(Z, U, t) = -\nu_1 \frac{z_2 - z_1}{\Delta x} - \frac{z_3 - U}{\Delta x C(z_1) \rho(z_1)};$$

$$F_{2i}(Z, U, t) = -\nu_{i+1} \frac{z_{2(i+1)} - z_{2i}}{\Delta x} - \frac{z_{2i+3} - z_{2i+1}}{\Delta x C(z_{2i}) \rho(z_{2i})}; \quad i = 1, \dots, N-3;$$

$$F_{2(N-2)}(Z,U,t) = -\nu_{N-1}\frac{T_*(t) - z_{2(N-2)}}{\Delta x} - \frac{z_{2N-2} - z_{2N-3}}{\Delta x C(z_{2(N-2)})\rho(z_{2(N-2)})};$$

$$F_3(Z,U,t) = -\nu_2\frac{z_3 - u}{\Delta x} - \left(z_3 + \lambda(z_2)\frac{z_2 - z_1}{\Delta x}\right)\frac{1}{\tau};$$

$$F_{2i-1}(Z,U,t) = -\nu_i\frac{z_{2i-1} - z_{2i-3}}{\Delta x} - \left(z_{2i-1} + \lambda(z_{2(i-1)})\frac{z_{2(i-1)} - z_{2(i-2)}}{\Delta x}\right)\frac{1}{\tau},$$
$$i = 3, ..., N-1;$$

$$F_{2N-2}(Z,U,t) = -\nu_N\frac{z_{2N-2} - z_{2N-3}}{\Delta x} - \left(z_{2N-2} + \lambda(T^*(t))\frac{T_*(t) - z_{2(N-2)}}{\Delta x}\right)\frac{1}{\tau},$$

where $\nu_i = \nu(x_{i-1})$, $i = 1, 2, ..., N$.

Problem (19)–(21) is an optimal control problem for the nonlinear dynamic system (20) with a $(2N-2)$−dimensional state vector $Z(t) = (z_1(t), ..., z_{2N-2}(t))$, $t \in [\tau_j, \tau_{j+1}]$, and a scalar control $U(t)$, $t \in [\tau_j, \tau_{j+1}]$. This problem has a number of specific features, which make it impossible to use the standard computing packages meant for solving "standard" optimal control problems. Therefore we will make some simplifications.

First of all, taking into account that the interval $[\tau_j, \tau_{j+1}]$ is small, we linearize system (20) on the interval $[\tau_j, \tau_{j+1}]$, having replaced $\lambda(z_i(t))$, $C(z_i(t))$, $\rho(z_i(t))$, $i = 1, ..., 2N-2$, by $\lambda(z_i^0)$, $C(z_i^0)$, $\rho(z_i^0)$, $i = 1, ..., 2N-2$. Remind that here the vector $Z(t_0) = Z_0 = (z_1^0, ..., z_{2N-2}^0)$ is defined according to (21), i.e. it is considered to be known at the moment τ_j.

Besides, we will solve this problem in a class of piecewise constant admissible controls

$$U(t) = U_i = const,\ t \in [t_i, t_{i+1}], i = jL, ..., (j+1)L - 1. \tag{22}$$

Denote the linearized problem (19)–(21) with additional conditions (22) by $\mathrm{P}_\mathrm{j}^{\mathrm{linear}}$. The problem $\mathrm{P}_\mathrm{j}^{\mathrm{linear}}$ can be easily reduced (see, for example, [17]) to a quadratic programming problem and solved by standard methods.

Let $U(t) = U^*(t), t \in [\tau_j, \tau_{j+1}]$, be an optimal control in linear quadratic problem $\mathrm{P}_\mathrm{j}^{\mathrm{linear}}$. Using this control we integrate the system of partial differential equations (14)–(16) with $U(t) = U^*(t)$, $t \in\in [\tau_j, \tau_{j+1}]$. As a result we obtain the trajectory $T^*(x,t)$, $q^*(x,t)$, $x \in [x_0, x_*]$, $t \in [\tau_j, \tau_{j+1}]$. Knowing the trajectory, we set $q_0^*(t) = q^*(x_0, t)$, $t \in [\tau_j, \tau_{j+1}]$, and consider this function as an approximation of the heat flux $q(x_0, t)$, $t \in [\tau_j, \tau_{j+1}]$, obtained at the jth stage.

Using new vector and functions

$$y^*(\tau_{j+1} - 0),\ T^*(x, \tau_{j+1} - 0),\ q^*(x, \tau_{j+1} - 0),\ x \in [x_0, x_*], \tag{23}$$

we go to the next $(j+1)$th stage whose aim is to reconstruct the heat flux $q(x_0, t), t \in [\tau_{j+1}, \tau_{j+2}]$, on the base of data (23) and the noisy temperature measurements $y(t_i)$ at the time instants $t_i \in [\tau_{j+1}, \tau_{j+2}]$, $i = (j+1)L + 1, ...,$ $(j+2)L$.

As it was shown in [17,18], for the optimal control problem P_j^{linear}, the value of index i_* (see relation (18)) is of great significance since it defines the index k of the cost functional of the problem. Notice that $i_* = N(x_1^* - x_0)/(x_* - x_0)$ is uniquely defined by the given values $x_0,\ x_1^*,\ x_*$ and a chosen parameter N.

We recall [27] that the index of the cost functional

$$\int_a^b f(Z(t), U(t), t)dt$$

is the smallest integer number k such that $\frac{\partial}{\partial U}\frac{d^k}{dt^k} f(Z(t), U(t), t) \neq 0$. Here the derivatives $\frac{d^k Z(t)}{dt^k}$ are calculated taking into account a specified system of differential equations. In the case under consideration this system coincides with the linearized system (20).

The index k characterizes the degree of the direct influence of a control $U(t)$, $t \in [\tau_j, \tau_{j+1}]$, on the cost functional. The higher the value of the index k, the weaker the influence of $U(t)$, $t \in [\tau_j, \tau_{j+1}]$, on the cost functional (or rather on its first term that is responsible for the restoration's quality) and the more "irregular" the restoration problem becomes. For the problem P_j^{linear}, the indices i_* and k are related as follows:

$$k = 2(i_* - 1) \text{ if } 1 < i_* \text{ and } \ k = 1, \text{if } i_* = 1.$$

Notice that for the problems considered in papers [17,18] the values of the indices i_* and k coincide: $k = i_*$. It illustrates once again that the identification problems considered in this paper are more difficult than the ones studied in [17,18].

The study of the problem P_j^{linear}, for large index i_* values, shows that the values of the control function $U(t) = U^{(j)}(t), t \in [\tau_j, \tau_{j+1}]$, that are situated closer to the end of the interval $[\tau_j, \tau_{j+1}]$, exert the smallest influence on the first term $\sum_{i=jL+1}^{(j+1)L}(z_k(t_i) - y^*(t_i))^2$ of the cost function: the closer control to the end of the interval, the weaker its influence. The choice of these control values is carried out mainly just for the purpose of minimization of the regularizing term $\int_{\tau_j}^{\tau_{j+1}}(dU(t)/dt)^2dt$ in the cost functional (19). It is clear that these control values will be "regular", but far from the values of the restored function.

To overcome the specified difficulties arising for large values of the index i_*, it is necessary to insert the following changes to the described above algorithm.

Let us select one more integer parameter L_b, $0 \le L_b \le L$. The value of L_b specifies the part $[\tau_j, \tau_j + L_b\Delta t]$ (called confidence interval) of the interval $[\tau_j, \tau_{j+1}]$ where the obtained control actions are supposed to be restored correctly. Only this part, $U^*(t), t \in [\tau_j, \tau_j + L_b\Delta t]$, of the obtained control function $U^*(t), t \in [\tau_j, \tau_j + L\Delta t]$, defined on the confidence interval will be used on the subsequent steps of the algorithm.

Taking into account these changes, the algorithm becomes as follows.

Step 0 (Initialization). Set $j = 0$, $\bar{\tau}_0 = t_0$, $T^*(x, \bar{\tau}_0 - 0) = T_{\text{in}}(x)$, $q^*(x, \bar{\tau}_0 - 0) = q_{\text{in}}(x)$, $x \in [x_0, x_*]$, $y^*(\bar{\tau}_0 - 0) = y(t_0)$.
Step 1. Using the known vector $y^*(\bar{\tau}_j - 0)$, apply filtering procedures to the noisy data $y(t)$, $t \in [\bar{\tau}_j, \tau^*_{j+1}]$, where $\tau^*_{j+1} = \min\{\bar{\tau}_j + L\Delta t,\ t_*\}$, and get function $y^*(t), t \in [\bar{\tau}_j, \tau^*_{j+1}]$.
Step 2. Set

$$\begin{array}{l} z_1^0 = T^*(x_1, \bar{\tau}_j),\ \ z_{2i}^0 = T^*(x_{i+1}, \bar{\tau}_j),\ \ i = 1, \dots, N-2; \\ z_{2i-1}^0 = q^*(x_i, \bar{\tau}_j),\ \ i = 2, \dots, N-1\ z_{2N-2}^0 = q^*(x_N, \bar{\tau}_j). \end{array} \tag{24}$$

Using the filtered data $y^*(t), t \in [\bar{\tau}_j, \tau^*_{j+1}]$, solve the optimal control problem $\mathrm{P}_{\mathrm{j}}^{\text{linear}}$ on the interval $[\bar{\tau}_j, \tau^*_{j+1}]$. Notice that in (19) one should replace $\tau_j,\ \tau_{j+1}$ by $\bar{\tau}_j,\ \tau^*_{j+1}$ and $\sum_{i=jL+1}^{(j+1)L}$ by $\sum_{i=m(\bar{\tau}_j)}^{m(\tau^*_{j+1})}$ where $m(\tau) \in \{0, 1, ..., M\}$ with $\tau \in \{t_j, j = 0, 1, ..., M\}$ is such integer number that $\tau = t_{m(\tau)}$. Let $U^*(t), t \in [\bar{\tau}_j, \tau^*_{j+1}]$, be the optimal control of the problem.
Step 3. Set $\bar{\tau}_{j+1} := \bar{\tau}_j + L_b \Delta t$. Integrate the nonlinear system of partial differential equations (14)–(16) on the interval $[\bar{\tau}_j, \bar{\tau}_{j+1}] \subset [\bar{\tau}_j, \tau^*_{j+1}]$ replacing $\tau_j,\ \tau_{j+1}$ and $U(t), t \in [\tau_j, \tau_{j+1}]$, by $\bar{\tau}_j,\ \bar{\tau}_{j+1}$ and $U^*(t), t \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$. This yields the trajectory $T^*(x, t)$, $q^*(x, t)$, $x \in [x_0, x_*]$, $t \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$.
Step 4. Set $q_0^*(t) = q^*(x_0, t), t \in [\bar{\tau}_j, \bar{\tau}_{j+1}]$.
Step 5. If $\bar{\tau}_{j+1} = t_*$, go to Step 7, otherwise go to Step 6.
Step 6. Set $j := j + 1$ and go to Step 1.
Step 7. The Algorithm stops the work.

The constructed function $q_0^*(t), t \in [t_0, t_*]$, is taken as the restored heat flux $q(x_0, t)$, $t \in [t_0, t_*]$ at the point x_0. The described algorithm is consistent with the approach based on the sequential estimation [1,26].

Thus, in the proposed method, the process of solving a single reconstruction problem for nonlinear system (1)–(4) on the large interval is reduced to the process of solving a succession of p optimal control problems $\mathrm{P}_{\mathrm{j}}^{\text{linear}}$ for linear systems on small intervals $t \in [\tau_j, \tau_{j+1}]$, $j = 0, ..., p-1$. It should be noted that, for a one-stage reconstruction procedure, i.e., when $p = 1$, only one problem of the optimal control is solved on the entire interval $[t_0, t_*]$. However, the dimensionality of this problem grows up as the discretization steps Δt and Δx decrease, which makes it impossible to solve this problem with high accuracy. In the proposed approach for arbitrarily small values of the steps Δt and Δx, the dimensions of the quadratic programming problems to be solved at each stage may take any prescribed values. For fixed values of dimensionality of these problems, the reduction of the discretization steps Δt and Δx results only in the increase of the number p of stages.

Note also that the small lengths of the intervals $[\tau_j, \tau_{j+1}]$, $j = 0, 1, ..., p-1$, (determined by the parameters L and M) and the stage-by-stage character of the SSO method enable the user to circumvent effectively the difficulties associated with the nonlinearity of the system. The fact that the algorithm includes the confidence interval determined by the parameter L_b allows one to reduce the difficulties associated with the irregularity of the problem for large values of the index i_*.

Besides, at each stage it is possible to analyze the quality of restoration and to correct the regularization parameters at the next stage on the basis of the results of this analysis.

5 Numerical Modeling

First of all, to evaluate the quality of approximation of the system of partial differential equations (1)–(4) by the system of ordinary differential equations (20), we solve two direct problems.

The first problem consists in constructing the functions

$$T(x,t),\ \ q(x,t),\ \ x_0 \le x \le x_*,\ \ t_0 \le t \le t_*, \tag{25}$$

that satisfy system (1) and the given initial and boundary conditions (2)–(4) with the following parameter's values

$$\begin{gathered} C(T) = 1,\ \tau = 2,\ \lambda(T)= 1,\ \rho(T)= 5,\ \nu(x) = 0.1, \\ x_0 = 0,\ x_* = 5,\ t_0 = 0,\ t_* = 50, \end{gathered} \tag{26}$$

and functions $q_{\text{in}}(x)$, $T_{\text{in}}(x)$, $x \in [x_0, x_*]$, and $q_0(t)$, $T_*(t)$, $t \in [t_0, t_*]$, that are presented in Fig. 1(a) and (b), respectively. Here the plots of the functions $q_{\text{in}}(x)$ and $q_0(t)$ are denoted by the continuous lines and the plots of the functions $T_{\text{in}}(x)$ and $T_*(t)$ are denoted by the dotes.

Notice that the initial and boundary conditions (2)–(4) satisfy the following consistency constraints:

$$T_{\text{in}}(x_*) = T_*(t_0),\ \ q_{\text{in}}(x_0) = q_0(t_0), \tag{27}$$

$$\rho(T_*(t_0))C(T_*(t_0))\left(\frac{dT_*(t_0)}{dt} + \nu(x_*)\frac{dT_{\text{in}}(x_*)}{dx}\right) = -\frac{dq_{\text{in}}(x_*)}{dx} \tag{28}$$

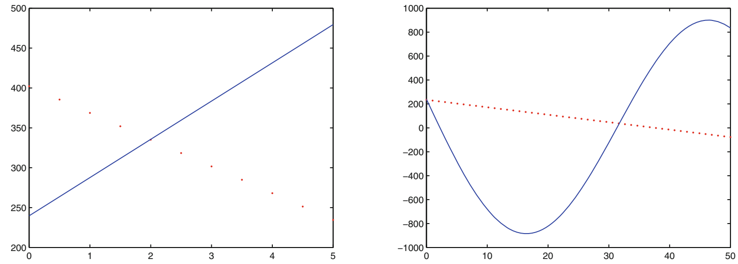
$$\tau\left(\frac{dq_0(t_0)}{dt} + \nu(x_0)\frac{dq_{\text{in}}(x_0)}{dx}\right) = -q_{\text{in}}(x_0) - \lambda(T_{\text{in}}(x_0))\frac{dT_{\text{in}}(x_0)}{dx}. \tag{29}$$

To solve the first direct problem, the standard **pdepe**-program of the computing package MATLAB was used.

For the given data, the plots of functions (25) are shown in Fig. 2.

After that, for the same set of parameters (26) and functions $q_{\text{in}}(x)$, $T_{\text{in}}(x)$, $x \in [x_0, x_*]$, $q_0(t)$, $T_*(t)$, $t \in [t_0, t_*]$, we have solved the initial value problem for system (20) in which the state vector $Z(t) = (z_1(t), ..., z_{2N-2}(t))$ (with $N = 11$) had the form

$$\begin{aligned} &z_1(t) = T_1(t) = T(x_1,t),\ \ z_{2i} = T_{i+1}(t) = T(x_{i+1},t),\ \ i = 1,\dots,N-2; \\ &z_{2i-1}(t) = q_i(t) = q(x_i,t),\ \ i = 2,\dots,N-1;\ \ z_{2N-2}(t) = q_N(t) = q(x_N,t). \end{aligned}$$



a) b)

Fig. 1. (a) Functions $q_{\text{in}}(x)$, $T_{\text{in}}(x)$, $x \in [x_0, x_*]$; (b) functions $q_0(t)$, $T_*(t)$, $t \in [t_0, t_*]$.

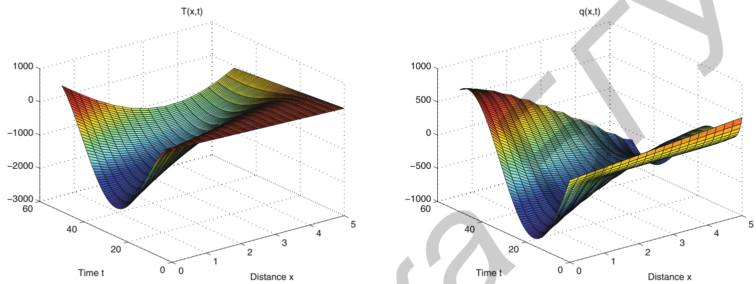


Fig. 2. Functions $T(x,t)$, $q(x,t)$, $t \in [t_0, t_*]$, $x \in [x_0, x_*]$.

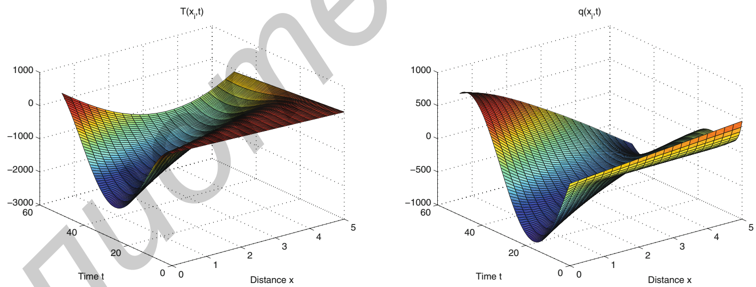


Fig. 3. Functions $T_i(t)$, $q_i(t)$, $t \in [t_0, t_*]$, $i = 1, ..., N$.

The plots of the functions

$$T_i(t) = T(x_i, t),\ q_i(t) = q(x_i, t),\ i = 1, \ldots, N,\ t_0 \le t \le t_*,$$

are presented in Fig. 3.

The carried out numerical calculations showed that for a rather large value of the parameter N, the system of the ordinary differential equations (20) well approximates the initial system of partial differential equations (1)–(4).

The main attention in the experiment was paid to solving the inverse problems, i.e. problems of reconstruction of the function $q_0(t) = q(x_0, t),\ t \in [t_0, t_*]$, on the base of the given noisy temperature measurements (6) at a given point x_1^*, $x_0 \le x_1^* \le x_*$.

Two types of functions $v(t)$, $t \in [t_0, t_*]$, modeling noises were considered:

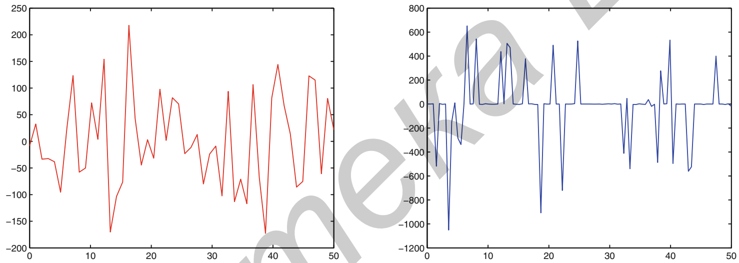
(A) noises of the type $v(t) = \sigma w(t), t \in [t_0, t_*]$, where σ is the standard value of the measurement errors deviation, and $w(t)$ is a random variable with a normal distribution, zero mean, a unit standard deviation, and uncorrelated values for various time instants;

(B) noises with outliers (spike noises).

The examples of functions $v(t) = \sigma w(t)$, $t \in [t_0, t_*]$, of types (A) and (B) are presented in Fig. 4.

As it was mentioned before, without loss of generality one can consider that the point x_1^*, at which the temperature measurements were performed, belongs to the nodes of the grid (18), namely,

$$x_1^* = x_{i_*} = x_0 + i_* \Delta x,\ \text{where } i_* \in \{1, ..., N-1\}. \tag{30}$$



a) b)

Fig. 4. (a) Examples of the function $\sigma w(t)$, $t \in [t_0, t_*]$: (a) of type (A) with $\sigma = 100$; and (b) of type (B) with $\sigma = 400$

For solving the reconstruction problem the described method of SSO was applied.

For different levels and types of noise σ and different values of the point $x_1^* = x_{i_*}$, the results of reconstruction of the function $q_0(t)$, $t \in [t_0, t_*]$, are presented in Figs. 5, 6, 7 and 8. Here the model function $q_0(t)$, $t \in [t_0, t_*]$, and functions $q_0^*(t)$, $t \in [t_0, t_*]$, obtained as a result of application of the described reconstruction method are shown. The parameter N was chosen to be equal to 11.

In the top parts of Figs. 5, 6, 7 and 8, the functions of the model heat flux $q_0(t) = q(x_0, t)$, $t \in [t_0, t_*]$, (the dashed line) and the reconstructed fluxes $q_0^*(t)$, $t \in [t_0, t_*]$, (the dot line) are represented.

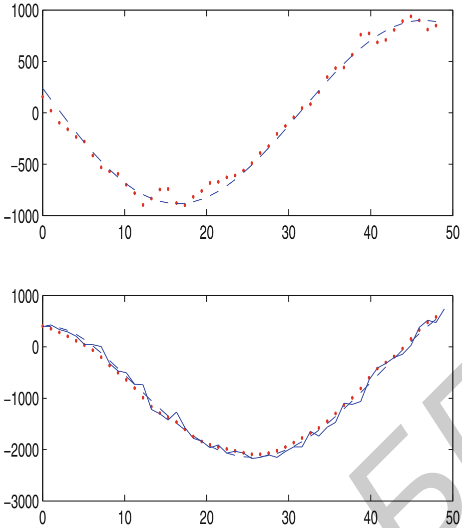


Fig. 5. The results of numerical modeling for reconstruction of heat flux with $i_* = 1$ $(L = 30, \alpha_u = 1.5, \alpha_1 = 2)$ and noise of type (A) $(\sigma = 100)$

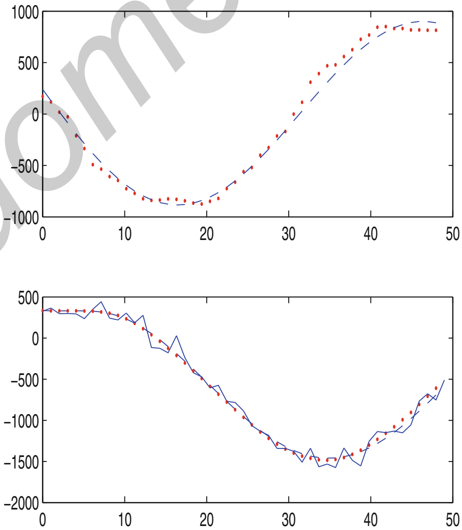


Fig. 6. The results of numerical modeling for reconstruction of heat flux with $i_* = 5$ $(L = 10, \eta_j = 4, \gamma_j = 4)$ and noise of type (A) $(\sigma = 100)$

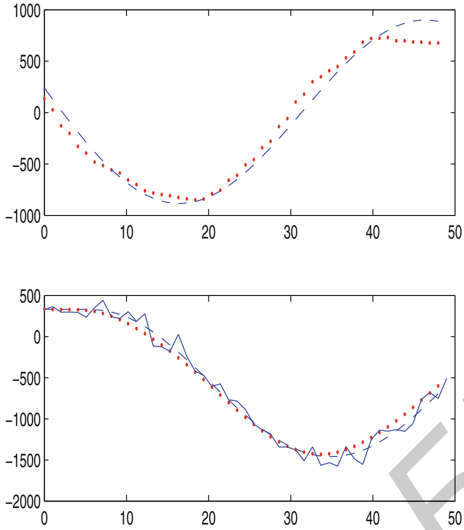


Fig. 7. The results of numerical modeling for reconstruction of heat flux with $i_* = 5$ ($L = 30, \eta_j = 1, \gamma_j = 1$) and noise of type (A) ($\sigma = 100$)

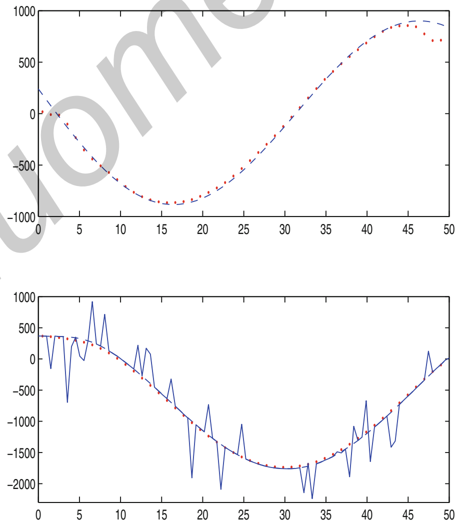


Fig. 8. The results of numerical modeling for reconstruction of heat flux with $i_* = 3$ ($L = 30, \eta_j = 0.25, \gamma_j = 0.25$) and noise of type (B) ($\sigma = 400$)

In the lower parts of these figures, the following functions are presented:

- function of temperature $T(x_1^*, t)$, $t \in [t_0, t_*]$, obtained as a result of solving the direct problem with data (26) and the functions presented on Fig. 1 (this function is denoted by the dashed lines);
- function of temperature obtained as a result of solving the direct problem with the same data in which the model function of $q_0(t)$, $t \in [t_0, t_*]$, is replaced by the function of $q_0^*(t)$, $t \in [t_0, t_*]$, constructed by the offered restoration procedure (this function is denoted by the dot line);
- and also function $y(t)$, $t \in [t_0, t_*]$, (5) (it is denoted by the continuous line).

The performed experiment shows that for systems (1)–(4) the method of stage-by-stage optimization is effective.

The analysis of the obtained results shows that the quality of reconstruction significantly depends on the value of the point x_1^*, in which the temperature's measurements are performed (i.e. on a relation of the parameters i_* and N) and on the noise level σ.

6 Conclusion

In the paper, we have developed the method of the stage-by-stage suboptimal optimization (SSO method) for solving the inverse heat conduction problems. The problem of reconstruction of the time-varying surface heat flux on the base of the temperature measurements in an internal point was studied. We considered the hyperbolic type systems taking into account the heat flux relaxation time, conductive and convective components of the heat distribution processes. Such systems generalize the studied earlier ones and are rather difficult.

The mathematical model of the heat transmission takes into account the medium nonlinearity, the speed of heat convection, and the heat flux relaxation time. For filtering the noisy input data, the method of robust estimation, based on the Tikhonov-Huber cost functional, was applied.

The numerical experiments performed have confirmed the effectiveness of the proposed approach for solving reconstruction problems on the base of both standard noise and noise with outliers.

References

1. Beck, J.V., Blackwell, B., St. Clair Jr., C.R.: Inverse Heat Conduction: Ill-Posed Problems. Wiley, New York (1985)
2. Ozisik, M.N., Orlande, H.R.B.: Inverse Heat Transfer: Fundamentals and Applications. Tailor and Francis, New York (2000)
3. Alifanov, O.M.: Inverse Heat Transfer Problems. Springer Verlag, Berlin (1994)
4. Murio, D.A.: The Mollification Method and the Numerical Solution of Ill-Posed Problems. Wiley, New York (1993)
5. Woodbury, K.A., Beck, J.V.: Estimation metrics and optimal regularization in a Tikhonov digital filter for the inverse heat conduction problem. Int. J. Heat Mass Transf. **62**, 31–39 (2013)

6. Jarny, Y., Orlande, H.R.B.: Adjoint methods. In: Orlande, H.R.B., Fudym, O., Maillet, D., Cotta, R.M. (eds.) Thermal Measurements and Inverse Techniques. CRC Press, Boca Raton (2011)
7. Borukhov, V.T.: Reconstruction of heat fluxes through differential temperature measurement by the method of inverse dynamic systems. J. Eng. Phys. Thermophys. **47**(3), 1098–1102 (1984)
8. Lee, H.-L., Chang, W.-J., Wu, S.-C., Yang, Y.-C.: An inverse problem in estimatingthe base heat flux of an annular fin based on the hyperbolic model of heat conduction. Int. Commun. Heat Mass Transf. **44**, 31–37 (2013)
9. Christov, C.I., Jordan, P.M.: Heat conduction paradox involving second sound propagation in moving media. Phys. Rev. Lett. **94**(15), 154301 (2005)
10. Papanicolaou, N.C., Christov, C.I., Jordan, P.M.: The influence of thermal relaxation on the oscillatory properties of two-gradient convection in a vertical slot. Eur. J. Mech. B. Fluids **30**, 68–75 (2011)
11. Vernotte, P.: Paradoxes in the continuous theory of the heat equation. Compt. Rend. Acad. Sci. (Paris) **246**, 3154–3155 (1958)
12. Cattaneo, C.: A form of heat conduction which eliminates the paradox of instantaneous propagation. Compt. Rend. Acad. Sci. (Paris) **247**, 431–433 (1958)
13. Luikov, A.V., Bubnov, V.A., Soloview, I.A.: On the wave solutions of heat conduction equation. Int. J. Heat Mass Transf. **19**, 245–248 (1976)
14. Chandrasekharaiah, D.S.: Hyperbolic thermoelasticity: a review of recent literature. Appl. Mech. Rev. **51**, 705–729 (1998)
15. Ván, P., Czél, B., Fülöp, T., Gyenis, G., Gróf, Á., Verha, J.: Experimenal aspects of heat conduction beyond Fourier. Thesis, 12th Joint European Thermodynamics Conference, Brescia (2013)
16. Zubair, S.M., Chaudhry, M.A.: Int. J. Heat Mass Transf. **39**(14), 3067–3074 (1996)
17. Borukhov, V.T., Kostyukova, O.I., Kurdina, M.A.: Tracking of the preset program of weighted temperatures and reconstruction of heat transfer coefficients. J. Eng. Phys. Thermophys. **83**(3), 622–631 (2010)
18. Borukhov, V.T., Kostyukova, O.I.: Identification of timedependent coefficients of heat transfer by the method of suboptimal stagebystage optimization. Int. J. Heat Mass Transf. **59**, 286–294 (2013)
19. Borukhov, V.T., Kostyukova, O.I.: Reconstruction of Heat Transfer Coefficients Using the Approach of Stage by Stage Suboptimal Optimization and Huber. Tikhonov Filtering of Input Data. Autom. Control Comput. Sci. **47**(6), 289–299 (2013)
20. Morozov, V.A.: Methods for Solving Incorrectly Posed Problems. Springer-Verlag, Heidelberg (1984)
21. Tikhonov, A.N., Arsenin, V.Y.: Solutions of Ill-Posed Problems. V.H. Winston & Sons, Washington, D.C. (1977)
22. Huber, P.J.: Robust estimation of a location parameter. Ann. Math. Stat. **35**, 73–101 (1964)
23. Huber, P.J., Ronchetti, E.M.: Robust Statistics, 2nd edn. Wiley, New York (2009)
24. Petrus, P.: Robust Huber adaptive filter. IEEE Trans. Signal Process. **47**(4), 1129–1133 (1999)
25. Binder, T., Kostina, E.: GaussNewton methods for robust parameter estimation. In: Bock, H.G., Carraro, T., Jäger, W., Körkel, S., Rannacher, R., Schlöder, J.P. (eds.) Model Based Parameter Estimation, vol. 4, pp. 55–87. Springer, Heidelberg (2013)

26. Alifanov, O.M.: Identifikatsiya protsessov teploobmena letatelnykh apparatov (vvedenie v teoriyu obratnykh zadach) (Identification of Heat Transfer Processes of Aircrafts (Introduction to the Theory of Inverse Problems)). Mashinostroenie, Moscow (1979)
27. Malanowski, K., Maurer, H.: Sensitivity analysis of optimal control problems subject to higher order state constraints. Ann. Oper. Res. **101**, 43–73 (2001). (Optimization with Data Perturbations II)