MINIMUM DISTANCE FROM POINT TO LINEAR VARIETY IN EUCLIDEAN SPACE OF THE TWO-DIMENSIONAL MATRICES

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Abstract

This work relates to the problem of linear approximation of multidimensional statistical data. Instead of the approach of regression analysis, we want to use another approach which is to minimize of the sum of the squares of the perpendicular distances from the system of points to the approximating plane. We receive the formula of minimum distance from point to linear variety in Euclidean space of the two-dimensional matrices as a first step in solving the problem.

1 Introduction

The approximation of statistical data by linear regression function minimizes the sum of the squares of deviations between observations of endogenous variables and variables predicted by regression function [1, 3, 7]. The another approach is to minimize of the sum of the squares of the perpendicular distances from the system of points to the approximating plane. This approach was considered in works [2, 5], however hasn't got the wide illumination in statistical literature. We want to apply this approach to matrix statistical data. We solve the first part of this problem. We give the formula of minimum distance from point to linear variety in Euclidean space of the two-dimensional matrices. Unlike the works [2, 5] we receive a new independent multidimensional-matrix solution of the problem.

2 Linear varieties in matrix arithmetical space

Let us denote $R_{[n_1n_2]}$ the linear space of $(n_1 \times n_2)$ -matrices with real elements and operations of addition and multiplication on the real numbers and let us call it arithmetical matrix linear space. Any element $X \in R_{[n_1n_2]}$ let us call a vector or point in $R_{[n_1n_2]}$. The system of vectors $\{X_1, X_2, ..., X_m\}$ we will call linear dependent if there are the real numbers $\alpha_1, \alpha_2, ..., \alpha_m$ such that at least one of them not equal zero and $\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_m X_m = 0$. If this equation is possible only when $\alpha_1 = 0, \alpha_2 = 0, ..., \alpha_m = 0$, then system of vectors is called linear independent.

We define also the linear varieties in parametric form in $R_{[n_1n_2]}$:

$$X = C_0 + t_1 C_1 + t_2 C_2 + ... + t_{n_1n_2-r_1r_2} C_{n_1n_2-r_1r_2}, \quad (1)$$

where $C_0 = (c_{i_1,i_2,0})$, $C_1 = (c_{i_1,i_2,1})$, $C_2 = (c_{i_1,i_2,2})$, , $C_{n_1n_2-r_1r_2} = (c_{i_1,i_2,n_1n_2-r_1r_2})$, $i_1 = \overline{1, n_1}$, $i_2 = \overline{1, n_2}$, – linear independent $(n_1 \times n_2)$ -matrices in $R_{[n_1n_2]}$, $t_1, t_2, ..., t_{n_1n_2-r_1r_2}$

– scalar real parameters. By analogy with vector space R^m we will call the variety (1) $(n_1n_2 - r_1r_2)$-dimensional plane in $R_{[n_1n_2]}$, and matrices C_1 , C_2 , , $C_{n_1n_2-r_1r_2}$ – direction matrices of this plane [6].

Relationship between r_1 and r_2 can by any in the framework of inequality $1 \leq r_1r_2 \leq n_1n_2$, but more easy to interpretation is case when $r_1 = n_1$, $1 \leq r_2 \leq n_2$.

For the case $r_1 = n_1$, $1 \leq r_2 \leq n_2$ we receive a new form of linear variety (1). We rewrite (1) in form

$$X = C_0 +^{0,2} (CT), \tag{2}$$

where

$$C = (c_{i_1,i_2,i'_1,i'_2}) = ((c_{i_1,i_2})_{i'_1,i'_2}) = (\tilde{C}_{i'_1,i'_2}), \quad i_1, i'_1 = \overline{1,n_1}, \quad i_2, i'_2 = \overline{1, n_2 - r_2}, \tag{3}$$

is four-dimensional matrix with sections $C_1 = \tilde{C}_{1,1}$, $C_2 = \tilde{C}_{1,2}$, , $C_{n_1n_2-r_1r_2} = \tilde{C}_{n_1,(n_2-r_2)}$, and $T = (t_{i'_1,i'_2})$, $i'_1 = \overline{1,n_1}$, $i'_2 = \overline{1, n_2 - r_2}$, – $(n_1 \times (n_2 - r_2))$-matrix, that contains the parameters t_1, t_2, , $t_{n_1n_2-n_1r_2}$ as its elements, ${}^{0,2}(CT)$ is $(0,2)$-convolute product of matrices C and T [4]. We present the matrices X , C_0 in (2) in form of the block matrices: $X = [X_{n_2-r_2}, X_{r_2}]$, $C_0 = [C_{n_2-r_2,0}, C_{r_2,0}]$, where

$$X_{n_2-r_2} = (x_{i_1,i_2}), \quad C_{n_2-r_2,0} = (c_{i_1,i_2,0}), \quad i_1 = \overline{1,n_1}, \quad i_2 = \overline{1, n_2 - r_2},$$

$$X_{r_2} = (x_{i_1,i_2}), \quad C_{r_2,0} = (c_{i_1,i_2,0}), \quad i_1 = \overline{1,n_1}, \quad i_2 = \overline{n_2 - r_2 + 1, n_2}.$$

The block $X_{n_2-r_2}$ is matrix, that contains the first $n_2 - r_2$ columns of matrix X, and block X_{r_2} is matrix, that contains the last X_{r_2} columns of X. We present also the matrix C in form of the block matrix $C = \{C_{n_2-r_2}, C_{r_2}\}$, and its blocks we define as follows:

$$\overline{C}_{n_2-r_2} = (c_{i_1,i_2,i'_1,i'_2}), \quad i_1, i'_1 = \overline{1,n_1}, \quad i_2, i'_2 = \overline{1, n_2 - r_2},$$

$$\overline{C}_{r_2} = (c_{i_1,i_2,i'_1,i'_2}), \quad i_1, i'_1 = \overline{1,n_1}, \quad i_2, i'_2 = \overline{n_2 - r_2 + 1, n_2}.$$

Now we can write two equations instead of equation (2):

$$\begin{cases} X_{n_2-r_2} = \overline{C}_{n_2-r_2,0} +^{0,2} (\overline{C}_{n_2-r_2}T), \\ X_{r_2} = \overline{C}_{r_2,0} +^{0,2} (\overline{C}_{r_2}T). \end{cases} \tag{4}$$

Because the matrices C_1 , C_2, , $C_{n_1n_2-r_1r_2}$ are linear independent, the matrix $\overline{C}_{n_2-r_2}$ is not singular, and we can get the matrix T from first equation of system (4):

$$T =^{0,2} (\overline{C}^{-1}_{n_2-r_2} (X_{n_2-r_2} - \overline{C}_{n_2-r_2,0})),$$

where $\overline{C}^{-1}_{n_2-r_2}$ is the $(0,2)$-inverse matrix to the matrix $\overline{C}_{n_2-r_2}$. Substitution this solution to the second equation of system (4) gives

$$X_{r_2} = \overline{C}_{r_2,0} +^{0,2} (\overline{C}_{r_2} {}^{0,2}(\overline{C}^{-1}_{n_2-r_2} (X_{n_2-r_2} - \overline{C}_{n_2-r_2,0}))). \tag{5}$$

The last expression shows that in the case of $(n_1n_2 - n_1r_2)$-dimensional plane in $R_{[n_1n_2]}$ the block X_{r_2} of the matrix X is linear expressed via its block $X_{n_2-r_2}$. The expression

(5) gives this dependence in explicit form for the second block X_{r_2} of matrix X. By analogy with a vector space R^m we can call the variety (1) when $n_1n_2 - n_1r_2 = 0$ ($r_2 = n_2$) as point in $R_{[n_1n_2]}$. When $n_1n_2 - n_1r_2 = n_1$ ($r_2 = n_2 - 1$) the linear variety (1) means that $n_2 - 1$ sections of matrix X (last its columns) linear depends on one its section (first column). When $n_1n_2 - n_1r_2 = n_1(n_2 - 1)$ ($r_2 = 1$) the linear variety (1) means that one its column (last column) linear depends on all previous its columns.

3 Distance from point to linear variety in Euclidean space of the two-dimensional matrices

We denote $E_{[n_1n_2]}$ Euclidean space of the two-dimensional $(n_1 \times n_2)$-matrices with the scalar product

$$(X, Y) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} x_{i_1,i_2} y_{i_1,i_2} =^{0,2} (XY), \quad X, Y \in E_{[n_1n_2]}. \tag{6}$$

We call orthogonal a vectors X and Y from $E_{[n_1n_2]}$, if $(X, Y) =^{0,2} (XY) = 0$, and we call normalized a vector $X \in E_{[n_1n_2]}$, if $(X, X) =^{0,2} (XX) = 1$. We call orthonormal the system of vectors $X_1, X_2, ..., X_m \in E_{[n_1n_2]}$, if this vectors are pairwise orthogonal and each of them has single length, i.e. if

$$(X_i, X_j) =^{0,2} (X_i X_j) = \delta_{i,j},$$

$\delta_{i,j}$ – the Kronecker symbol.

Let $\xi = (\xi_{i_1,i_2})$, $i_1 = \overline{1, n_1}$, $i_2 = \overline{1, n_2}$, – matrix from $E_{[n_1n_2]}$. We formulate the task of finding the minimum distance from point $\xi \in E_{[n_1n_2]}$ to linear variety (1). In accordance with the scalar product (6) the square of distance is determined by formula

$$\rho^2(\xi, X) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} (\xi_{i_1,i_2} - x_{i_1,i_2})^2 =^{0,2} (\xi - X)^2.$$

If we use in this formula the expression (2) for X, then we receive the optimization task:

$$\rho^2(\xi, X) =^{0,2} (\xi - X)^2 =^{0,2} (\xi - C_0 -^{0,2} (CT))^2 \to \min_T. \tag{7}$$

Now we go to the solving the task (7). We note, that we can write the variety (1) in form

$$X = C_0 +^{0,2} (TC^{T_1}),$$

where C^{T_1} is transposed matrix C in accordance with substitution $T_1 = \begin{pmatrix} i, j, k, l \\ k, l, i, j \end{pmatrix}$ [4]. Then the task (7) get form

$$\rho^2(\xi, X) =^{0,2} (\xi - X)^2 =^{0,2} ((\overset{o}{\xi} -^{0,2}(TC^{T_1}))(\overset{o}{\xi} -^{0,2}(CT))) \to \min_T, \tag{8}$$

where $\overset{o}{\xi} = \xi - C_0$. Because in (8)

$$\rho^2(\xi, X) = {}^{0,2}(\overset{o}{\xi}\overset{o}{\xi}) - 2^{0,2}({}^{0,2}(\overset{o}{\xi}\, C)T) + {}^{0,2}(T^{0,2}({}^{0,2}(C^{T_1}C)T)), \tag{9}$$

then necessary conditions for a minimum are next equation

$$\frac{d}{dT}\rho^2(\xi, X) = -2^{0,2}(\overset{o}{\xi}\, C) + 2^{0,2}({}^{0,2}(C^{T_1}C)T) = 0.$$

From this equation we get

$$T = {}^{0,2}({}^{0,2}(C^{T_1}C)^{-1\,0,2}(\overset{o}{\xi}\, C)),$$

where ${}^{0,2}(C^{T_1}C)^{-1}$ is matrix $(0,2)$-inverse to the matrix ${}^{0,2}(C^{T_1}C)$. If we substitute this solution to the expression (9), then we get the square of minimum distance:

$$\rho^2_{min}(\xi, X) = {}^{0,2}(\overset{o}{\xi}\overset{o}{\xi}) - {}^{0,2}(\overset{o}{\eta}\ {}^{0,2}(\overset{o}{\eta}\ {}^{0,2}(C^{T_1}C)^{-1})), \tag{10}$$

where

$$\overset{o}{\eta} = {}^{0,2}(C^{T_1}\ \overset{o}{\xi}).$$

We have proved the following theorem.

Theorem 1 *Let $E_{[n_1 n_2]}$ is Euclidean space of the two-dimensional $(n_1 \times n_2)$-matrices with the scalar product (6) and ξ is point in $E_{[n_1 n_2]}$. The square of distance from point ξ to the linear variety (2) in $E_{[n_1 n_2]}$ is defined by expression (10).*

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