The Simplest Problem of Fuzzy Dynamical Systems Optimal Control

Paluh B.V., Dzyuba S.M., Egereva I.A., Emelyanova I.I.
Tver State Technical University
Tver, Russia
Email: Pboris@tvstu.ru
Email: sdzyuba@mail.ru
Email: Irina.egereva@gmail.com
Email: Emelyanova-123@yandax.ru

***Abstract*—Abstract–The paper explores the question of the terminal state control of fuzzy dynamical systems, characterized by classical fuzzy relations. A solution of the problem is traced to the functional equation solution such as Bellman equation.**

Keywords*—*fuzzy dynamical systems, terminal state control, Bellman equation.

I. Introduce

The notion of fuzziness nowadays is widely used for processes description of technical, economical and other nature. This comes from the fact that this notion allows to operate simply and naturally on qualitative information which is in many cases mainly responsible for ambivalence (see, eg. [1, 2]).

Initially, the methods of fuzzy set theory have been directed to use logical methods of decision-making, based on the compositional inference rule (see., eg, [1]). Subsequently the methods of the dynamic programming theory and fuzzy sets theory was used to develop control problem-solving techniques of deterministic and stochastic systems with fuzzy objectives and restrictions (see., eg, [2]).

It enabled to form a general theory of mathematical programming and the theory of decision-making with fuzzy objectives and restrictions (see, eg, [3] - [5]). Further development of fuzzy dynamic programming can be found, for example, in [6] overview, where, in particular, the problem of deterministic and stochastic systems with the fuzzy end time and infinite horizon control are considered.

Let's now note that simultaneously with the development of the theory of dynamic and mathematical fuzzy relationship have been widely used in the control theory and theory of decision-making under fuzzy initial information. For example, in [3] and [7] papers there have been considered and studied in details the fuzzy preference relations of great importance to the problems of decision-making in fuzzy systems (including active ones). Let's also mention [8] paper, where on the basis of fuzzy relations questions of the theory of approximate reasoning in fuzzy control are considered.

The main goal of this paper lies in the further development of the [2]-[6] results. All the constructions are actually based on the almost apparent modification of classical composition operation of fuzzy sets shown below in par. II.

II. Fuzzy Sets Compositions

In considering the control problem further we'll need some important capabilities of fuzzy sets compositions.

Let X, Y and Z be certain sets. Let's assume that at a $X \times Y$ set a fuzzy relation A with membership function μ_A, is defined and at a $Y \times Z$ set a fuzzy relation B with membership function μ_B is defined. Therefore theis defined. Therefore the $A \circ B$ composition of fuzzy set A and B is the fuzzy relation in $X \times Z$ space with the membership function

$$\mu_{A \circ B}(x, z) = \sup_{y \in Y} \min[\mu_A(x, y), \mu_B(y, z)] \tag{1}$$

(see, eg. [1]).

Let's now assume that in X space the fuzzy set R with membership function μ_R s defined. Therefore the fuzzy relation μ_A induces the fuzzy set $R \circ A$ n the Y spase. In accordance with (1) the membership function $\mu_{R \circ A}$ of $R \circ A$ set is given by an equation

$$\mu_{R \circ A}(y) = \sup_{x \in X} \min[\mu_R(x), \mu_A(x, y)].$$

These compositions of fuzzy relations are widely used in the theory of fuzzy sets for the construction of compositional inference rules (see [1]). For the behavioral research of fuzzy dynamical systems we'll use the supplementary rule of fuzzy sets composition.

Let's assume that at X set the fuzzy relation S with membership function μ_S is defined. Further, let's assume that in X space G set with membership function μ_G s also defined. Therefore we can determine the $S \circ G$ composition of fuzzy sets S and G and following (1) the membership function $\mu_{S \circ G}$ of $S \circ G$ set will be defined by equation

$$\mu_{S \circ G}(x_1) = \sup_{x_2 \in X} \min[\mu_S(x_1, x_2), \mu_G(x_2)]. \tag{2}$$

One can readily see that $S \circ G$ composition allows to determinate the membership degree of element of X set to fuzzy G set with fuzzy relation S. Specifically for each $x_1 \in X$ the membership degree of $\mu_{S \circ G}(x_1)$ of x_1 the fuzzy set G is defined by equation (2).

III. Simplest Problem Of Optimal Control

Let X and U be certain compact metric spaces. Let's consider control system when X is state space and U is control space.

Let's assume that evolution of system state is characterized by the fuzzy relation S representing fuzzy set S in $X \times U \times X$ pace with membership function μ_S. Provided that the initial state $x_0 \in X$ is defined. As a result of choosing of $u_0 \in U$ control the system goes into some new state x_1 which was earlier unknown. It is only known that with u_0 and x_0 fixed, x_0, u_0 and x_1 variables are related by the fuzzy relation S with membership function $\mu_S(x_0, u_0, x_1)$. In other words with x_0 and u_0 fixed at point of time n=0 the state x_1 can be defined only by value of membership function $\mu_S(x_0, u_0, x_1)$. However at point of time n=1 we can observe exact value of state x_1.

In a similar way if at some point of time n the state x_n is known than a result of choosing of u_n control we can estimate the state x_{n+1} by fuzzy relation S with membership function $\mu_S(x_n, u_n, x_{n+1})$. Moreover, at the next point of time $n+1$ the state x_{n+1} becomes known.

Let's consider that the control aim is characterized by fuzzy goal set G in X space with membership function μ_G. Let's also assume that both functions μ_S and μ_G are continuous in the range of their definition.

Now let's assume that time N of end of system work is defined. The control problem is to search the sequence

$$u_0, u_1, \ldots, u_{N-1} \tag{3}$$

of points of U set maximizing the membership degree of x_0 states to fuzzy set G with fuzzy relations with membership functions

$$\mu_S(x_0, u_0, x_1), \mu_S(x_1, u_1, x_2), \ldots, \mu_S(x_{N-1}, u_{N-1}, x_N).$$

Therefore the fuzzy set G is the control aim and the problem consists in searching the control sequences (3) providing the maximal membership degree of the state x_0 to the fuzzy set G with that the evolution of system state is described be the composition of fuzzy sets S and G.

By equation

$$D_N = \underbrace{S \circ \ldots \circ S}_{N} \circ G$$

let's put for consideration the fuzzy set D_N being conditional for variables (3) in the X space with membership function μ_{D_N} satisfying the equation

$$\mu_{D_N}(x_0 \,|\, u_0, u_1, \ldots, u_{N-1}) =$$
$$= \max_{x_1, x_2, \ldots, x_N} \min[\mu_S(x_0, u_0, x_1), \mu_S(x_1, u_1, x_2), \ldots,$$
$$\mu_S(x_{N-1}, u_{N-1}, x_N), \mu_G(x_N)].$$

Therefore according to equation (2) $\mu_{D_N}(x_0 \,|\, u_0, u_1, \ldots, u_{N-1})$ he values of function μ_{D_N} have the form of the membership degree of the state x_0 to G set with the use of any fixed sequence of control of (3) kind.

Let's set

$$\mu_N(x_0) = \max_{u_0, u_1, \ldots, u_{N-1}} \mu_{D_N}(x_0 \,|\, u_0, u_1, \ldots, u_{N-1}). \tag{4}$$

Following [1] let's consider the initial task in the context of task family where x_0 and N are variable values. Therefore with $N = 0$ the required membership degree x_0 to G et with the fuzzy relation S is prescribed by the equation

$$\mu_0(x_0) = \mu_G(x_0). \tag{5}$$

Function μ_0 s continuous by convention over all of the intervals at X set. Moreover because of continuity of functions it is easy to note that for each function f which is defined and continuous over all of the intervals at X and possesses values at the interval [0,1], the function

$$g(x_0, u_0, x_1) = \min[\mu_S(x_0, u_0, x_1), f(x_1)]$$

is continuous over all of the intervals. But X and U spaces are compact. Therefore, the function

$$h(x_0) = \sup_{u_0, x_1} \min[\mu_S(x_0, u_0, x_1), f(x_1)] =$$
$$\max_{u_0, x_1} \min[\mu_S(x_0, u_0, x_1), f(x_1)]$$

is continuous over all of the intervals at X set. Provided that

$$\max_{u_0, u_1, \ldots, u_N} \mu_{D_{N+1}}(x_0 \,|\, u_0, u_1, \ldots, u_N) =$$
$$= \max_{u_0, x_1} \min[\mu_S(x_0, u_0, x_1),$$
$$\max_{u_1, u_2, \ldots, u_N} \mu_{D_N}(x_1 \,|\, u_1, u_2, \ldots, u_N)]$$

(see, eg. [9]). Then by virtue of (4) for certain N the equation

$$\mu_{N+1}(x_0) = \max_{u_0, x_1} \min[\mu_S(x_0, u_0, x_1), \mu_N(x_1)], \tag{6}$$

is executed where $\mu_{N+1}(x_0)$ is the maximal membership degree of the statex_0 to the G set with the relation S and the condition where end of system work time is equal to $N+1$, and $\mu_N(x_1)$ is the maximal membership degree of the state x_1 to the G set with relation S and the condition where end of system work time is equal to N.

One can readily see that recurrence relationship (6) with the condition (5) is similar to Bellman's functional equation for classical problems of dynamic programming. This relationship interprets the control u_0 as function of time N and the state x_0, i.e.

$$u_0 = u_0^*(x_0, N), \quad N = 1, 2, 3, \ldots \tag{7}$$

IV. Asymptotic properties of relationship (6)

In many practical cases it is appropriate to replace control law (7) by autonomous law

$$u = u^*(x) \tag{8}$$

(see, eg. [3]). In order to understand the availability of getting such a law let's study asymptotic properties of relationship (6).

Let's set as an set closure operation. Let $C(X, [0, 1])$ be a space of continuous functions defined at X set and possessed value at interval [0,1]. For certain function $\varphi \in C(X, [0, 1])$ let's assume

$$A\varphi = \max_{u_0, x_1} \min[\mu_S(x, u_0, x_1), \varphi(x_1)],$$

where A is an operator mapping the space $C(X, [0, 1])$ into itself. Therefore the following sentences are correct.

Proposition 1. Let's assume that the X space is finite. Therefore the set

$$\Omega(\mu_0) = \bigcap_{N \geq 0} \left(\bigcup_{k \geq N} A^k \mu_0 \right)$$

isn't empty, it is compact in the topology of simple convergence and is invariant. In such a case the equation

$$\lim_{k \to +\infty} A^k \mu_0 = \Omega(\mu_0). \tag{9}$$

Proposition 2. Let M be a set of functions

$$\mu_0, \mu_1, \ldots, \mu_N, \ldots \tag{10}$$

Therefore if the X space is finite then the set

$$\Omega(M) = \bigcap_{N \geq 0} A^N M \tag{11}$$

isn't empty, is compact in the topology of simple convergence and is invariant. In such a case the equation

$$\lim_{N \to +\infty} A^N M = \Omega(M). \tag{12}$$

The assumption of the X space finiteness in each specific case requires justification. However let's note that in our case the X space is initially considered as compact. Accordingly it is separable, i.e. in X here is dense set being countable everywhere. Thus there is a countable -netcovering the X space. But in virtue of the X space compactness out of each of its countable covering the finite subcover can be chosen. In other words there is a finite -net overing the X space.

Thus in general case the X space can be approximated by a finite set to a high accuracy. Thus the conditions of sentences 1 and 2 are shown with prescribed accuracy . In addition the compact space approximation by its certain finite part is justified in many practical situations for modeling of fuzzy systems (see, eg. [1]). For this reason sentences 1 and 2 set asymptotic properties of relationship (6) applicable for practice.

V. Asymptotic Autonomous Control Law

Asymptotic properties of relationship (6) set by sentences 1 and 2 prevent from thinking directly of the optimal autonomous control law existence without any additional requirements.

Actually we can speak about the existence of such a law only with sequence convergence (10). In this case according to A operator continuity in some cases the function μ defined at the X set by the equation

$$\mu(x) = \lim_{N \to +\infty} \mu_N(x), \tag{13}$$

is a continuous solution of the equation

$$\mu(x_0) = \max_{u_0, x_1} \min[\mu_S(x_0, u_0, x_1), \mu(x_1)]. \tag{14}$$

For the purpose that the equation (14) is followed by existence of the equation (14) continuous solution it is sufficient the convergence (13) is uniform at Xset and control laws (7) are continuous. Again the necessary and sufficient condition of uniform convergence in the equation (13) as is known lies in the fact that the set (10) is equicontinuous (see, eg. [9]). Then in the case under consideration the check of equicontinuity of the set (10) is rather difficult in virtue of representation of A operator.

Let's note that the situation is extremely rare where at the X set the equation is simply (even nonuniformly) satisfied (see, eg. [9]). The situation becomes complicated by the fact that even if the equation (14) has a unique solution it doesn't mean that the control law (8) corresponding to this solution will be unique. Thus we have to speak in the majority of practical situations only about the existence of suboptimal autonomous control law.

For the development of such law let's assume that the X set is finite. Therefore according to the sentence 2 the closure M of the set (10) isn't empty, is compact and invariant. But each compact, invariant set contains compact minimum set (see attachment 2). Thus under conditions of sentence 2 the set $\Omega(M)$ contains the compact minimum set . In this case by the finiteness of the X set one can find such finite system of compact, invariant set

$$\Omega(M) \supset M_1 \supset M_2 \supset \ldots \supset M_N, \tag{15}$$

that equation

$$\mathcal{M} = \Omega(M) \cap M_1 \cap \ldots \cap M_{N_M}, \tag{16}$$

is satisfied where N is a certain sufficiently great positive integer (see, eg. [9]).

If the system (15) is built then according to the equation (16) the set is also built. Let μ be arbitrary function of the set. Therefore by virtue of the fact that is a minimum set the function

$$A\mu = \max_{u_0, x_1} \min[\mu_S(x, u_0, x_1), \mu(x_1)] \qquad (17)$$

belongs to and v.v. (see [10]).

One can readily see that maximum in the equation (17) is attained with the use of the certain control law

$$u = u_\mu(x). \qquad (18)$$

Moreover by equation (13 and (14) it is easy to note that if the optimal law (8) exists, the law is the same as law (18).

Thus a certain kind of control law (18) corresponds to each function $\mu \in$. Any of these laws in general case is only suboptimal. However by sentences 1 and 2 the equation (17) not only sets the existence of such suboptimal laws but provides a procedure of its construction.

ACKNOWLEDGMENT

The research was supported by the Russian Foundation for Basic Research (project nos. 17-07-01339).

REFERENCES

[1] L. Zadeh, *The concept of linguistic variable and its application to the adoption of the approximate solutions*. 0.5em minus 0.4emMoscow, Russia: Mir, 1973.

[2] R. Bellman and L. Zadeh, *Decision-making in vague terms*, Questions analysis and decision-making procedures. Moscow, Russia: Mir, 1976, p. 172–215.

[3] S. A. Orlovsky, *Decision-making problems with fuzzy initial information*. Moscow, Russia: Nauka, 1981.

[4] K. Negoytse, *The application of systems theory to control problems*. Moscow, Russia: Mir, 1981.

[5] L.A. Zadeh, J. Kacprzyk (Eds.) *Fuzzy logic for the management of uncertainty*. Wiley, New York, 1992.

[6] J. Kacprzyk, A.O. Esogbue, *Fuzzy dynamic programming: Main developments and applications*. Fuzzy Sets and Systems, 1996, vol 81, p. 31–45.

[7] D.A. Novikov and S. N. Petrakov, *Course theory of active systems*. Moscow, Russia: SINTEG, 1999.

[8] A.P. Ryzhov, *Elements of the theory of fuzzy sets and measure the fuzziness*. Moscow, Russia: Dialog – MSU, 1998.

[9] A.P. Afanas'ev, and Dzyuba, S.M., *Ustoichivost' po Puassonu v dinamicheskikh i nepreryvnykh periodicheskikh sistemakh (Poisson Stability in Dynamical and Continuous Periodic Systems)*. Moscow, Russia: LKI, 2007.

[10] J. Hale,*Theory of Functional Differential Equations, New York: Springer, 1977. Translated under the title Teoriya funktsional'no-differentsial'nykh uravnenii*. Moscow: Mir, 1984.

ПРОСТЕЙШАЯ ЗАДАЧА УПРАВЛЕНИЯ НЕЧЕТКИМИ ДИНАМИЧЕСКИМИ СИСТЕМАМИ

Палюх Б.В., Дзюба С.М.,
Егерева И.А., Емельянова И.И.

В работе рассмотрена задача управления конечным состоянием нечетких динамических систем. Эволюция состояний рассматриваемых систем характеризуется классическими нечеткими отношениями. Решение задачи сведено к решению функционального уравнения типа уравнения Беллмана. На основе современных методов общей теории динамических систем изучены асимптотические свойства решений полученного функционального уравнения. Изучена проблема существования и построения субоптимального автономного закона управления с обратной связью.