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TORSION FREE AFFINE CONNECTIONS ON THREE-DIMENSIONAL HOMOGENEOUS SPACES

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Abstract. The purpose of the work is the classification of three-dimensional homogeneous spaces with torsion-free invariant affine connections only. In the case considered in the work, a t-equivalence class contains only one space, i.e., invariant affine connections with coinciding geodesics do not exist. The local classification of homogeneous spaces is equivalent to the description of effective pairs of Lie algebras. In this work we use the algebraic approach for description of connections, methods of the theory of Lie groups, Lie algebras and homogeneous spaces.

Keywords: invariant connection, homogeneous space, transformation group, torsion tensor, holonomy algebra.

1. Introduction

Invariant connections on (mainly, reductive) homogeneous space have been studied independently by P. K. Rashevskii [1, 2], M. Kurita [3], E. B. Vinberg [4, 5] and S. Kobayashi, K. Nomizu [6, 7]. Nguyen van Hai studied existence conditions of invariant affine connection on (not necessarily reductive) homogeneous space. His result in [8] generalizes some results of K. Nomizu and is connected with a problem of studying affine connection which supposes transitive group of affine transformations. B. Opozda, O. Kowalski, Z. Vlasek, T. Arias-Marco classify all torsion-free [9, 10] (arbitrary torsion [11]) locally homogeneous connections on two-dimensional manifolds. B. Dubrov, B. Komrakov, Y. Tchempkovsky [12] describe

Mozhey, N.P., Torsion free affine connections on three-dimensional homogeneous spaces.

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maximal affine pairs (three-dimensional locally homogeneous space, affine connection) whose symmetry group is transitive and at least five-dimensional.

In this work we consider only spaces, not supposing connections with a nonzero torsion tensor. Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$ (for example, [13]). In the study of homogeneous spaces, it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only effective actions of $\overline{G}$ on M. Since we are interested in only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can put into correspondence the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$; that is, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$; a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective. An *isotropic $\mathfrak{g}$-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective.

We have divided the solution of the problem of classification all three-dimensional isotropically-faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ into the following parts. We classified (up to isomorphism) faithful three-dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classified (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic. All of these pairs are described in [17].

Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [7] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$, then this pair is isotropy-faithful [6]. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas:

$$R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}),\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)] - \Lambda([x_1, x_2]);$$

$$T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m},\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto \Lambda(x_1)(x_2+\mathfrak{g}) - \Lambda(x_2)(x_1+\mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}.$$

We restate the theorem of Wang on the holonomy algebra of an invariant connection: the Lie algebra of the holonomy group of the invariant connection defined by $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(3, \mathbb{R})$ on $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given by $V + [\Lambda(\bar{\mathfrak{g}}), V] + [\Lambda(\bar{\mathfrak{g}}), [\Lambda(\bar{\mathfrak{g}}), V]] + \dots$, where V is the subspace spanned by $\{[\Lambda(x), \Lambda(y)] - \Lambda([x, y]) | x, y \in \bar{\mathfrak{g}}\}$. Let $a_{\bar{\mathfrak{g}}}$ be the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ generated by $\{\Lambda(x); x \in \bar{\mathfrak{g}}\}$. Originally, $a_{\bar{\mathfrak{g}}}$ was introduced as such in the Riemannian case by B. Kostant [14], and has been used by A. Lichnerowicz [15] and H. Wang [16] under more general circumstances. The basic properties of $a_{\bar{\mathfrak{g}}}$ are given by the following: let $\mathfrak{h}^*$ be the Lie algebra of the holonomy group, then $\mathfrak{h}^* \subset a_{\bar{\mathfrak{g}}} \subset \mathbb{N}(\mathfrak{h}^*)$, where $N(\mathfrak{h}^*)$ is the normalizer of $\mathfrak{h}^*$ in $\mathfrak{gl}(3, \mathbb{R})$. We shall say that a invariant connection is *normal* if $\mathfrak{h}^* = a_{\bar{\mathfrak{g}}}$.

As it is known, the set of all affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$ forms an affine space assosiated with the vector space $(\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$. Really, let Λ_1, Λ_2 be two affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then $\Lambda_1 - \Lambda_2$ equals 0 on $\mathfrak{g}$ and, therefore, can be identified with a mapping $\Delta\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$. Since the mappings Λ_1 and Λ_2 are $\mathfrak{g}$-invariant, it follows that Δ is also $\mathfrak{g}$-invariant. But the set of all $\mathfrak{g}$-invariant mappings from $\mathfrak{m}$ to $\mathfrak{gl}(\mathfrak{m})$ is canonically isomorphic to $(\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$. On the other hand, if Λ is any affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $\Delta\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is $\mathfrak{g}$-invariant, then $\Lambda + \Delta$ is also an affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ (here we identify Δ with the mapping $\bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ which is equal to 0 on $\mathfrak{g}$).

We say that two connections Λ_1, Λ_2 are *t-equivalent* if $\Lambda_1 - \Lambda_2 \in (\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$. This definition has the following geometrical meaning: two invariant affine connections on a homogeneous space are t-equivalent if and only if their geodesics coincide (see [6]). It follows immediately from the definition that the t-equivalence classes of affine connections are just parallel affine subspaces in the set of all affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to the subspace $(\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}} \subset (\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$. Each t-equivalence class of affine connections on $\bar{\mathfrak{g}}$ contains a unique torsion-free connection. Really, let Λ in the set of all affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$. Let $T \in (\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$ be the torsion tensor of Λ. Consider the affine connection $\Lambda' = \Lambda + 1/2T$. Then its torsion tensor T' has the form: $T'\colon (x_1+\mathfrak{g}) \wedge (x_2+g) \mapsto [x_1, x_2]_{\mathfrak{m}} - \Lambda'(x_1)(x_2+\mathfrak{g}) + \Lambda'(x_2)(x_1+\mathfrak{g}) = T(x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) - \frac{1}{2}T(x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) + \frac{1}{2}T(x_2+\mathfrak{g}) \wedge (x_1+\mathfrak{g}) = 0$, for all $x_1, x_2 \in \bar{\mathfrak{g}}$. The uniqueness of the torson-free connection can be proved in the same way. Picking out the unique torsion-free connection subordinate to a family of parametrized geodesics is known as absorption of torsion, and it is one of the stages of Cartan's equivalence method.

In the case considered in the work, a t-equivalence class contains only one space, i.e. there do not exist affine connections with coincide geodesics.

Also, in particular, if M is a symmetric space, then K. Nomizu showed that there is a torsion-free affine connection on M whose curvature is parallel. Conversely, a manifold with such a connection is locally symmetric (i.e., its universal cover is a symmetric space). Such manifolds can also be described as those affine manifolds whose geodesic symmetries are all globally defined by affine diffeomorphisms, generalizing the Riemannian and pseudo-Riemannian case.

We define $(\bar{\mathfrak{g}}, \mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$. Here by $\{e_1, ..., e_n\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n = \dim \bar{\mathfrak{g}}$), by small Greek letters (λ, μ and others) we denote parameters of $\bar{\mathfrak{g}}$ (as described in [17]). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1, ..., e_{n-3}$. Let $\{u_1 = e_{n-2}, u_2 = e_{n-1}, u_3 = e_n\}$ be a basis of $\mathfrak{m}$. We describe the affine connection by $\Lambda(u_1)$, $\Lambda(u_2)$, $\Lambda(u_3)$, the curvature tensor R by $R(u_1, u_2)$, $R(u_1, u_3)$, $R(u_2, u_3)$ and the torsion tensor T by $T(u_1, u_2)$, $T(u_1, u_3)$, $T(u_2, u_3)$. We say that the affine connection is *trivial* if $\Lambda(u_1) = \Lambda(u_2) = \Lambda(u_3) = 0$. To refer to the pair we use the notation $d.n.m$, where d is the dimension of the subalgebra, n is the number of the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$, m is the number of $(\bar{\mathfrak{g}}, \mathfrak{g})$ in [17].

The description of three-dimensional isotropically-faithful pairs with torsion-free invariant affine connections only can be divided into the following parts:

Part I: Classification of all pairs that allow nontrivial affine connections;

a) the curvature tensor is only zero;

– $\bar{\mathfrak{g}}$ is nonsolvable (in Theorem 1, $\mathfrak{g}$ is nonsolvable; in Theorem 2, $\mathfrak{g}$ is solvable);

– $\bar{\mathfrak{g}}$ is solvable (Theorem 3);

b) the curvature tensor is not only zero;
– $\bar{\mathfrak{g}}$ is solvable (Theorem 4);
– $\bar{\mathfrak{g}}$ is nonsolvable (in this case $\mathfrak{g}$ is solvable for all pairs, Theorem 5);
Part II: Classification of all pairs with only trivial affine connections.
a) the curvature tensor is only zero;
– $\bar{\mathfrak{g}}$ is nonsolvable (in Theorem 6, $\mathfrak{g}$ is semisimple; in Theorem 7, $\mathfrak{g}$ is not semisimple, the commutant of the radical of $\bar{\mathfrak{g}}$ is noncommutative; in Theorem 8, $\mathfrak{g}$ is not semisimple, the commutant of the radical of $\bar{\mathfrak{g}}$ is commutative);
– $\bar{\mathfrak{g}}$ is solvable (Theorem 9);
b) the curvature tensor is not only zero;
– $\bar{\mathfrak{g}}$ is nonsolvable (in Theorem 10, the radical of $\bar{\mathfrak{g}}$ is commutative; in Theorem 11, the radical of $\bar{\mathfrak{g}}$ is noncommutative);
– $\bar{\mathfrak{g}}$ is solvable (Theorem 12).
Three-dimensional isotropically-faithful pairs with torsion-free invariant affine connections only, except presented in Theorems 1-12, do not exist.

2. Pairs of Lie algebras that allow nontrivial affine connections

2.1. **The curvature tensor is zero for all connections.**

2.1.1. *Transformation group is nonsolvable.* In this case semisimple transformation groups do not exist.

Theorem 1. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are only zero, $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are nonsolvable, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the following pair:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
6.3.2	$\{\{u_1\}, \{-4e_1, 2e_6, -2u_2, -4e_2, 2e_5, -2u_3, -4e_3, -e_1-3e_4-u_1\}\}$

Pair	Levi decomposition $\mathfrak{g}$
6.3.2	$\{\{e_4, 2e_5, 2e_6\}, \{-4e_1+2e_5, -4e_2+2e_6, -4e_3\}\}$

Pair	Connection
6.3.2	$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

Proof. For the subalgebras $\mathfrak{g}$ of $\mathfrak{gl}(3, \mathbb{R})$ in [17], we find the isotropy-faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and choose the pairs that allow nontrivial affine connections such that the curvature and torsion tensors are zero for all connections and $\mathfrak{g}$ is nonsolvable. For example, let isotropic representation has the form 6.3, let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1 and e_4.

Since $\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_4$, $\mathfrak{g}^{(2,0)}(\mathfrak{h}) = \mathbb{R}e_2$, $\mathfrak{g}^{(-2,0)}(\mathfrak{h}) = \mathbb{R}e_3$, $U^{(0,0)}(\mathfrak{h}) = \mathbb{R}u_1$, $\mathfrak{g}^{(-1,-1)}(\mathfrak{h}) = \mathbb{R}e_5$, $U^{(1,1)}(\mathfrak{h}) = \mathbb{R}u_2$, $\mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_6$, $U^{(-1,1)}(\mathfrak{h}) = \mathbb{R}u_3$ we have $[u_1, u_2] = \alpha_2 u_2$, $[u_1, u_3] = \beta_3 u_3$, $[u_2, u_3] = 0$, $\alpha_2, \beta_3 \in \mathbb{R}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	0	$-e_5$	e_6	0	u_2	$-u_3$
e_2	$-2e_2$	0	e_1	0	$-e_6$	0	0	0	u_2
e_3	$2e_3$	$-e_1$	0	0	0	$-e_5$	0	u_3	0
e_4	0	0	0	0	$-e_5$	$-e_6$	0	u_2	u_3
e_5	e_5	e_6	0	e_5	0	0	0	pe_1+A	$2pe_3$
e_6	$-e_6$	0	e_5	e_6	0	0	0	$2pe_2$	$-pe_1+A$
u_1	0	0	0	0	0	0	0	0	0
u_2	$-u_2$	0	$-u_3$	$-u_2$	$-pe_1-A$	$-2pe_2$	0	0	0
u_3	u_3	$-u_2$	0	$-u_3$	$-2pe_3$	pe_1-A	0	0	0

,

where $A{=}3pe_4{+}u_1$, $p \in \mathbb{R}$. If $p{=}0$ then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair 6.3.1. If $p \neq 0$ then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, i.e. 6.3.2, by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where $\pi(e_i){=}e_i$, $i{=}\overline{1,6}$, $\pi(u_j){=}(1/p)u_j$, $j{=}\overline{1,3}$.

Since the Lie algebra 6.3.1 is reductive, and 6.3.2 is nonreductive, we see that the pairs 6.3.1 and 6.3.2 are not equivalent.

If $(\bar{\mathfrak{g}}, \mathfrak{g})$ is 6.3.1 then $\Lambda|_{\mathfrak{g}}$ is the isotropic representation of $\mathfrak{g}$. Let

$$\Lambda(u_1){=}\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix}, \Lambda(u_2){=}\begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix}, \Lambda(u_3){=}\begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix},$$

$p_{i,j},\ q_{i,j},\ r_{i,j} \in \mathbb{R}$ $(i, j = \overline{1,3})$. Λ is $\mathfrak{g}$-invariant $\Rightarrow$ $[\Lambda(e_2),\Lambda(u_1)]=\Lambda([e_2,u_1])$ $\Rightarrow$ $[\Lambda(e_2),\Lambda(u_1)]=0$, $p_{3,1}=p_{3,2}=p_{1,2}=0$, $p_{3,3}=p_{2,2}$. $[\Lambda(e_1),\Lambda(u_1)]=\Lambda([e_1,u_1])$ $\Rightarrow$ $p_{1,3}=p_{2,1}=p_{2,3}=0$. $[\Lambda(e_5),\Lambda(u_1)]=\Lambda([e_5,u_1])$ $\Rightarrow$ $p_{2,2}=p_{1,1}$. If $[\Lambda(e_2),\Lambda(u_2)]=0$ then $q_{3,1}=q_{3,2}=q_{1,2}=0$, $q_{3,3}=q_{2,2}$. $[\Lambda(e_1),\Lambda(u_2)]=\Lambda(u_2)$, $q_{1,1}=q_{2,2}=q_{2,3}=0$. $[\Lambda(e_3),\Lambda(u_2)]{=}\Lambda(u_3)$, $r_{1,1}{=}r_{1,3}{=}r_{2,1}{=}r_{2,2}{=}r_{2,3}{=}r_{3,2}{=}r_{3,3}{=}0$, $r_{3,1}{=}q_{2,1}$, $r_{1,2}{=}{-}q_{1,3}$. If $[\Lambda(e_4),\Lambda(u_2)]{=}\Lambda(u_2)$ then $r_{1,2}{=}0$. $[\Lambda(e_5),\Lambda(u_2)]{=}\Lambda(u_1)$, $p_{1,1}{=}r_{3,1}{=}0$. In this case the affine connection is trivial, the curvature and torsion tensors are zero.

If $(\bar{\mathfrak{g}}, \mathfrak{g})$ is 6.3.2 then $[\Lambda(e_2),\Lambda(u_1)]{=}\Lambda([e_2,u_1])$ $\Rightarrow$ $[\Lambda(e_2),\Lambda(u_1)]{=}0$, $p_{3,1}{=}p_{3,2}{=}p_{1,2}{=}0$, $p_{3,3}{=}p_{2,2}$. $[\Lambda(e_1),\Lambda(u_1)]{=}\Lambda([e_1,u_1])$ $\Rightarrow$ $p_{1,3}{=}p_{2,1}{=}p_{2,3}{=}0$. $[\Lambda(e_5),\Lambda(u_1)]{=}\Lambda([e_5,u_1])$ $\Rightarrow$ $p_{2,2}=p_{1,1}$. If $[\Lambda(e_2),\Lambda(u_2)]=0$ then $q_{3,1}=q_{3,2}=q_{1,2}=0$, $q_{3,3}=q_{2,2}$. $[\Lambda(e_1),\Lambda(u_2)]=\Lambda(u_2)$, $q_{1,1}=q_{2,2}=q_{2,3}=0$. $[\Lambda(e_3),\Lambda(u_2)]=\Lambda(u_3)$, $r_{1,1}=r_{1,3}=r_{2,1}=r_{2,2}{=}r_{2,3}{=}r_{3,2}{=}r_{3,3}{=}0$, $r_{3,1}{=}q_{2,1}$, $r_{1,2}{=}{-}q_{1,3}$. If $[\Lambda(e_4),\Lambda(u_2)]{=}\Lambda(u_2)$ then $r_{1,2}{=}0$. $[\Lambda(e_5),\Lambda(u_2)]=\Lambda(u_1)+\Lambda(e_1)+3\Lambda(e_4)$, $p_{1,1}=r_{3,1}=-2$, affine connection has the form presented in the theorem, the curvature and torsion tensors are zero.

In other cases are similarly. □

Theorem 2. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are only zero, $\bar{\mathfrak{g}}$ is nonsolvable and $\mathfrak{g}$ is solvable, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
5.9.2	$\{\{2u_1, e_5, -2e_2+u_2, e_3, e_1\}, \{-4e_4, u_1+4u_3, e_5+4u_2\}\}$
4.19.2	$\{\{2e_1+u_2, e_2, -u_1, e_4\}, \{-(1/2)u_1-u_2, 2e_3+e_4, 2u_3\}\}$
4.21.11, $\mu\neq 0,1,1/2$	$\{\{-\mu^3 e_3, -2\mu^2 e_2-2\mu^2 u_1, u_1, (21/\mu)e_1+u_2\},$ $\{-4\mu^3 e_3, \mu^4 e_2+\mu^4 u_1+4\mu^3 u_3, -\mu^3 e_4+4\mu^2 u_2\}\}$
3.6.2	$\{\{u_2, -e_3+u_1, e_2\}, \{e_1, u_1, u_3\}\}$
3.12.2	$\{\{-u_1, -e_3, -2e_1+u_2\}, \{-(1/2)u_1-u_2, -2e_2-e_3, -2u_3\}\}$
3.13.6, $\mu\neq 0, 1, -1, 1/2$	$\{\{-(21/\mu)e_1+u_2, (1-\mu)\mu u_1, -\mu e_3\},$ $\{(1-\mu)\mu^2/(2(-1+\mu))u_1-\mu^2 u_2, -2\mu e_2-\mu e_3, -2\mu u_3)\}\}$
3.28.2	$\{\{u_2, -e_3, -u_1\}, \{-2e_1-u_2, -2e_2, -2u_3\}\}$
2.8.7, $\lambda\neq 0, 1, -1, 1/2$	$\{\{u_2, -(1/\lambda)e_2+u_1\}, \{\lambda^3 e_1, \lambda^2 u_1, \lambda^3 u_3\}\}$

Pair	Connection
5.9.2 3.12.2 3.13.6, $\mu\neq 0, 1, -1, 1/2$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
4.19.2.	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
4.21.11, $\mu\neq 0, 1, 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
3.6.2 2.8.7, $\lambda\neq 0, 1, -1, 1/2$	$\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}$
3.28.2	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

The proof is just as above.

Remark. For the sake of simplicity, instead of the notation for parameters $\mu\neq 0$ and $\mu\neq 1$ and $\mu\neq 1/2$ we use the following notation: $\mu\neq$0,1,1/2; instead μ=0 or μ=1 or μ=1/2 we use μ=0,1,1/2.

In this case, if $\mathfrak{g}$ is solvable then the radical of $\bar{\mathfrak{g}}$ is noncommutative, the commutant of the radical of $\bar{\mathfrak{g}}$ is commutative for all pairs.

2.1.2. *Transformation group is solvable.* In this case, $\mathfrak{g}$ is solvable too. Let curvature tensor be zero (then holonomy algebra is zero too).

Theorem 3. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are only zero, and $\bar{\mathfrak{g}}$ is solvable, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

4.8.1 (λ=0, μ=1/2), 4.11.1 (μ=0, λ=1/2), 4.11.5, 3.7.1 (λ=1/2), 3.8.1 (λ=1/2, μ= 0), 3.14.1 ($\mu\neq 0, 2$), 3.19.17, 3.20.25 ($\mu\neq 0$), 3.20.26 ($\lambda\neq 1/3, 1/4$), 2.1.1 (λ=1/2), 2.8.1 (λ=1/2), 2.9.1 (λ=1/2 ($\mu\neq 0, -1/2, 1/4$); $\lambda=2\mu$ ($\mu\neq 0, 1/4, 1/3$)); 2.19.5, 2.21.1 (λ=3/4), 1.2.1 ($\mu=2\lambda$ ($\lambda\neq 1/3, 1/4$); $\mu=\lambda/2$ ($\lambda\neq -2$); λ=1/2 ($\mu\neq 1/2$)), 1.7.1 (λ=1/2).

Pair	Connection
4.11.1, μ=0, λ=1/2 2.21.1, λ=3/4	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4.8.1, λ=0, μ=1/2 3.8.1, λ=1/2, μ=0 2.8.1, λ=1/2 2.9.1,λ=2μ($\mu\neq$0,1/4,1/3,1/2) 1.2.1, μ=λ/2 ($\lambda\neq-2$)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$
4.11.5 3.19.17 3.20.25, $\mu \neq 0$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3.7.1, $\lambda = 1/2$ 2.1.1, λ=1/2 1.2.1, $\lambda = 1/2$ ($\mu \neq 1/2, 1$) 1.7.1, λ=1/2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3.14.1, $\mu \neq 0, 2$ 2.9.1, $\lambda = 1, \mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & r_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$
2.9.1,λ=1/2($\mu\neq$0,−1/2,1/4)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3.20.26, $\lambda \neq 1/3, 1/4$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.19.5	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
1.2.1, $\mu = 2\lambda$ ($\lambda \neq 1/3, 1/4$)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
1.2.1, $\lambda = 1/2, \mu = 1$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & q_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2.2. **The curvature tensor is not zero for some connections.**

2.2.1. *Transformation group is solvable.* In this case, $\mathfrak{g}$ is solvable too.

Theorem 4. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature tensor is not only zero, the torsion tensor is only zero, and* $\bar{\mathfrak{g}}$ *is solvable, then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

3.20.26 ($\lambda = 1/4$), 3.20.27, 2.9.1 (λ=1/2, μ=1/4), 2.9.3 ($\mu = 1/4$), 1.2.1 (λ= 1/4, μ=1/2).

Pair	Connection
3.20.26, $\lambda = 1/4$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&r_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}$
3.20.27	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.1, λ=1/2, μ=1/4 2.9.3, $\mu = 1/4$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,2}&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&r_{2,3}\\0&0&0 \end{pmatrix}$
1.2.1, λ=1/4, μ=1/2	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&q_{3,2}&0 \end{pmatrix}, \begin{pmatrix} 0&0&r_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}$

Pair	Curvature tensor
3.20.26, $\lambda = 1/4$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-r_{1,3}&0\\0&0&0\\0&0&0 \end{pmatrix}$
3.20.27	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&-1\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.1, λ=1/2, μ=1/4	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,2}r_{2,3}\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.3, $\mu = 1/4$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,2}r_{2,3}-1\\0&0&0\\0&0&0 \end{pmatrix}$
1.2.1, λ=1/4, μ=1/2	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-r_{1,3}q_{3,2}&0\\0&0&0\\0&0&0 \end{pmatrix}$

Pair		Holonomy algebra
3.20.26, $\lambda = 1/4$	$r_{1,3} \neq 0$	$\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{1,3} = 0$	is equal to zero
3.20.27		$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.9.1, $\lambda = 1/2, \mu = 1/4$	$q_{1,2}r_{2,3} \neq 0$	$\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$q_{1,2}r_{2,3} = 0$	is equal to zero
2.9.3, $\mu = 1/4$	$q_{1,2}r_{2,3} \neq 1$	$\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$q_{1,2}r_{2,3} = 1$	is equal to zero
1.2.1, $\lambda = 1/4, \mu = 1/2$	$r_{1,3}q_{3,2} \neq 0$	$\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{1,3}q_{3,2} = 0$	is equal to zero

2.2.2. *Transformation group is nonsolvable.* In this case, $\bar{\mathfrak{g}}$ is not semisimple and $\mathfrak{g}$ is solvable for all pairs.

Theorem 5. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature tensor is not only zero, the torsion tensor is only zero, and* $\bar{\mathfrak{g}}$ *is nonsolvable, then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
4.21.11, μ=1/2	$\{\{(-1/2)e_2, (-1/2)u_1, (-1/8)e_4, 4e_1+u_2, u_1\},$ $\{(-1/2)e_3, (1/16)e_2+(1/16)u_1+(1/2)u_3, (-1/8)e_4+u_2\}\}$
3.13.6, μ=1/2	$\{-42e_1 + u_2, u_1, e_3\}, \{u_1 + 2u_2, 2e_2 + e_3, u_3\}\}$
2.8.7, λ=1/2	$\{\{u_2, -2e_2 + u_1\}, (1/8)e_1, (1/4)u_1, (1/8)u_3\}\}$

Pair	Connection
4.21.11, μ=1/2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
3.13.6, μ=1/2	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
2.8.7, λ=1/2	$\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r_{2,3} \\ -1/2 & 0 & 0 \end{pmatrix}$

Pair	Curvature tensor
4.21.11, $\mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3.13.6, $\mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -3r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.8.7, $\lambda = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3r_{2,3}/2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Pair		Holonomy algebra $\bar{\mathfrak{g}}$
4.21.11, $\mu = 1/2$ 3.13.6, $\mu = 1/2$	$r_{1,3} \neq 0$	$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{1,3} = 0$	is equal to zero
2.8.7, $\lambda = 1/2$	$r_{2,3} \neq 0$	$\begin{pmatrix} 0 & 0 & 0 \\ p_1 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{2,3} = 0$	is equal to zero

3. Pairs of Lie algebras with only trivial affine connections

That means $\Lambda(u_1) = \Lambda(u_2) = \Lambda(u_3) = 0$. The torsion tensor is zero for all connections here.

3.1. **The curvature tensor is zero for all connections.**

3.1.1. *Transformation group is nonsolvable.* In this case semisimple transformation groups are not exist.

Theorem 6. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *has only trivial affine connection, the curvature and torsion tensors are only zero,* $\bar{\mathfrak{g}}$ *is nonsolvable,* $\mathfrak{g}$ *is semisimple, then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to the following pair:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
8.1.1	$\{\{-u_3, -u_2, u_1\}, \{2e_3, e_4, -2e_5-2u_2, -e_6, -e_7-u_3, e_8, e_1+u_1, e_1+e_2+u_1\}\}$

Proof. If, for example, $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 8.1.1 [17], $\Lambda|_{\mathfrak{g}}=\lambda$ then Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_1), \Lambda(u_1)]=\Lambda([e_1, u_1]) \Rightarrow [\Lambda(e_1), \Lambda(u_1)]=\Lambda(u_1)$ and $p_{1,1}=p_{1,2}=p_{2,1}=p_{2,2}=p_{2,3}=p_{3,1}=p_{3,3}=0$. If $[\Lambda(e_2), \Lambda(u_1)]=\Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)]=0$, then $p_{1,3}=p_{3,2}=0$. If $[\Lambda(e_5), \Lambda(u_1)]=\Lambda([e_5, u_1])$ then $[\Lambda(e_5), \Lambda(u_1)]=\Lambda(u_2)$, $q_{1,1}=q_{1,2}=q_{1,3}=q_{2,1}=q_{2,2}=q_{2,3}=q_{3,1}=q_{3,2}=q_{3,3}=0$. If $[\Lambda(e_7), \Lambda(u_1)]=\Lambda(u_3)$ then $r_{1,1}=r_{1,2}=r_{1,3}=r_{2,1}=r_{2,2}=r_{2,3}=r_{3,1}=r_{3,2}=r_{3,3}=0$. We have $\Lambda(u_1)=\Lambda(u_2)=\Lambda(u_3)=0$, the curvature and torsion tensors are zero. In other cases, $\mathfrak{g}$ is not semisimple. □

Theorem 7. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *has only trivial affine connection, the curvature and torsion tensors are only zero,* $\bar{\mathfrak{g}}$ *is nonsolvable,* $\mathfrak{g}$ *is not semisimple, and the commutant of the radical of* $\bar{\mathfrak{g}}$ *is noncommutative, then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
7.2.1	$\{\{u_3, u_2, u_1, 2e_4{+}e_5, e_3, e_2, e_1\}, \{(1/2)e_2{-}e_5, {-}e_3{+}2e_6, 2e_7\}\}$
6.2.1	$\{\{u_3, e_1, (8\lambda-8)e_4, (-8\lambda+8)e_6, (4\lambda-4)u_1, (4\lambda-4)u_2\},$ $\{32e_3, -32e_5+(8-8\lambda)u_2, 16e_2+(4\lambda-4)u_1\}\}$
6.3.1	$\{\{u_3, u_2, u_1, e_6, e_5, e_4\}, \{-4e_1+2e_5, -4e_2+2e_6, -4e_3\}\}$
6.4.1, $\lambda \neq 1/2$	$\{\{u_3, u_2, u_1, e_6, e_5, e_1\}, \{-4e_2+(2-2\lambda)e_5,$ $-4e_3+(2-2\lambda)e_6, -4e_4\}\}$

Pair	Levi decomposition $\mathfrak{g}$
7.2.1	$\{\{e_2, -e_3, e_1, 2e_4+e_5\}, \{(1/2)e_2-e_5, -e_3+2e_6, 2e_7\}\}$
6.2.1	$\{\{e_1, (-\lambda+1)e_4, (2\lambda-2)e_6\}, \{-4e_2+(2\lambda-2)e_4, -4e_3,$ $-4e_5+(2\lambda-2)e_6\}\}$
6.3.1	$\{\{2e_5, 2e_6, e_4\}, \{-4e_1+2e_5, -4e_2+2e_6, -4e_3\}\}$
6.4.1, $\lambda \neq 1/2$	$\{\{e_5, e_6, e_1\}, \{-4e_2+(2-2\lambda)e_5, -4e_3+(2-2\lambda)e_6, -4e_4\}\}$

The proof is just as above.

Theorem 8. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has only trivial affine connection, the curvature and torsion tensors are only zero, $\bar{\mathfrak{g}}$ is nonsolvable, $\mathfrak{g}$ is not semisimple, and the commutant of the radical of $\bar{\mathfrak{g}}$ is commutative, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
9.1.1	$\{\{e_9, -u_3, -2u_2, -2u_1\}, \{-4e_1-8u_1, -2e_4, 2e_8, -4e_3, 2e_6,$ $-e_1-e_2-2u_1, 2e_7+4u_3, -4e_5-8u_2\}\}$
7.1.1	$\{\{-8e_7, u_3, e_3, e_1, 4u_2, 4u_1, 8e_6\}, \{32e_4+16e_6, -32e_5,$ $16e_2-8e_7\}\}$
5.1.1	$\{\{u_1, e_3, e_1+e_2, -u_2, -u_1\}, \{4e_4, -4e_5+u_2, 4e_1-4e_2-u_1\}\}$
4.2.1, $\lambda \neq 1/2$	$\{\{u_3, e_1, 8\lambda u_1, -8\lambda u_2\}, \{32e_3{+}16\lambda u_1, -32e_4, 16e_2{-}8\lambda u_2\}\}$
4.3.1	$\{\{-u_1, u_3, u_2, e_1\}, \{-e_2+u_1, -e_3, -e_4+u_1\}\}$
4.5.1	$\{\{u_1, u_3, -u_2, e_1\}, \{e_2-u_1, e_3, e_4-u_2\}\}$

Pair	Levi decomposition $\mathfrak{g}$
9.1.1	$\{\{u_3\}, \{-4e_1, -2e_4, 2e_8, -4e_3, 2e_6, -e_1-e_2, 2e_7, -4e_5\}\}$
7.1.1	$\{\{e_1, e_3, -e_6, 2e_7\}, \{-4e_2+2e_6, -4e_4, -4e_5+2e_7\}\}$
5.1.1	$\{\{e_3, e_1+e_2\}, \{-e_1+e_2, -2e_4, -2e_5\}\}$
4.2.1, $\lambda \neq 1/2$	$\{\{e_1\}, \{-4e_2, -4e_3, -4e_4\}\}$
4.3.1	$\{\{e_1\}, \{-e_2, -e_3, -e_4\}\}$
4.5.1	$\{\{e_1\}, \{e_2, e_3, e_4\}\}$

The proof just as above.

3.1.2. *Transformation group is solvable.* Then the stationary subgroup is solvable too.

Theorem 9. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has only trivial affine connection, the curvature and torsion tensors are only zero, $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are solvable, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

6.5.1, 5.4.1, 5.5.1, 5.6.1, 5.7.1, 5.8.1, 5.9.1, 5.10.1 ($\lambda^2+\mu^2 \neq 0$, $(\lambda-1)^2+(\mu+1)^2 \neq 0$), 4.4.1, 4.6.1, 4.7.1, 4.8.1 ($(\lambda+1)^2+(\mu-1)^2 \neq 0$, $\lambda^2+(\mu-1/2)^2 \neq 0$, $\lambda^2+\mu^2 \neq 0$), 4.9.1 ($\lambda^2+\mu^2 \neq 0$), 4.11.1 ($\lambda^2+\mu^2 \neq 0$, $(\mu+1)^2+(\lambda-1)^2 \neq 0$, $\mu^2+(\lambda-1/2)^2 \neq 0$), 4.12.1, 4.13.1, 4.14.1 ($(\mu-2)^2+\lambda^2 \neq 0$), 4.15.1, 4.16.1, 4.17.1, 4.18.1, 4.19.1, 4.20.1 ($\lambda \neq 0, -1$), 4.21.1 ($\mu \neq 0, \mu \neq 1-\lambda$), 4.22.1, 3.1.1, 3.2.1, 3.6.1, 3.7.1 ($\lambda \neq 0, 1/2$), 3.8.1 ($\lambda^2+\mu^2 \neq 0$, $(\lambda-1/2)^2+\mu^2 \neq 0$, $(\lambda+1)^2+(\mu-1)^2 \neq 0$, $(\lambda-1)^2+(\mu+1)^2 \neq 0$), 3.9.1, 3.10.1, 3.11.1, 3.12.1, 3.13.1 ($\mu \neq 0, \mu \neq 1-\lambda, \mu \neq \lambda - 1$), 3.16.1, 3.17.1 ($\lambda \neq 0$), 3.18.1, 3.19.1 ($\lambda \neq 0, -1$), 3.20.1 ($\lambda \neq 0$, $\mu \neq 0, \mu \neq 1-\lambda$), 3.21.1 ($\lambda \neq 0$), 3.22.1 ($\lambda \neq 2\mu$), 3.23.1 ($\lambda \neq 2/3, 1/2$), 3.24.1, 3.26.1, 3.27.1 ($\lambda \neq 0, 1/2$), 3.28.1, 3.29.1 ($\mu \neq 0$), 3.30.1, 3.31.1, 2.1.1 ($\lambda \neq 0, 1/2$), 2.2.1 ($(\lambda-1)^2 + (\mu-1)^2 \neq 0$), 2.3.1, 2.4.1 ($\lambda^2 + \mu^2 \neq 0$, $\lambda^2 + (\mu-2)^2 \neq 0$), 2.5.1, 2.6.1, 2.8.1 ($\lambda \neq 0, 1/2, 1, -1$), 2.9.1 ($\lambda \neq 1/2$, $\lambda \neq 0$, $\lambda \neq 1-\mu$, $\lambda \neq 2\mu$, $\lambda \neq \mu+1$, $\mu \neq 0$), 2.10.1, 2.11.1, 2.12.1, 2.14.1, 2.16.1 ($\lambda \neq 0, 1/2$), 2.19.1 ($\lambda \neq 0$), 2.21.1 ($\lambda \neq 0, 1/2, 2/3, 3/4$), 2.22.1, 1.2.1 ($\mu \neq \lambda+1, \mu \neq 2\lambda, \mu \neq 1-\lambda, \mu \neq \lambda/2$, $\lambda \neq 1/2$), 1.4.1 ($\mu \neq 2\lambda$), 1.7.1 ($\lambda \neq 0, 2, 1/2$), 1.9.1.

Remark. In the cases 5.10.1 ($\lambda = 1/2, \mu = 0$), 4.21.1 ($\mu = 1/2$), 3.8.1 ($\lambda = 0, \mu = 1/2$), 3.13.1 ($\mu = 1/2$, $\mu = \lambda/2$), 3.20.1 ($\lambda = 1/2$ or $\mu = 1/2$), 3.23.1 ($\lambda = 3/4$), 3.29.1 ($\mu = 1/2$), 2.9.1 ($\lambda \neq 1/2$, $\lambda \neq 0$, $\lambda \neq 1-\mu$, $\lambda \neq 2\mu$, $\lambda \neq \mu+1$, $\mu \neq 0$), 2.19.1 ($\lambda = 1/2$) the connection is trivial after basis replacement.

Proof. If, for example, $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 5.10.1 ($\lambda=1/2, \mu=0$) then $\Lambda|_{\mathfrak{g}}$ is the isotropic representation of $\mathfrak{g}$. Λ is $\mathfrak{g}$-invariant $\Rightarrow$ $[\Lambda(e_4), \Lambda(u_1)]=\Lambda([e_4, u_1]) \Rightarrow [\Lambda(e_4), \Lambda(u_1)]=0$. We have $p_{3,1}=p_{3,2}=p_{2,1}=0$, $p_{3,3}=p_{1,1}$. $[\Lambda(e_5), \Lambda(u_1)]=\Lambda([e_5, u_1]) \Rightarrow [\Lambda(e_5), \Lambda(u_1)]=0 \Rightarrow p_{1,2}=0$, $p_{1,1}=p_{2,2}$. $[\Lambda(e_3), \Lambda(u_1)]=\Lambda([e_3, u_1]) \Rightarrow [\Lambda(e_3), \Lambda(u_1)]=0$ then $p_{2,3}=0$. If $[\Lambda(e_1), \Lambda(u_1)]=\Lambda(u_1)$ then $p_{1,1}=p_{1,3}=0$. $[\Lambda(e_4), \Lambda(u_2)]=0$, $q_{3,1}=q_{3,2}=q_{2,1}=0$, $q_{3,3}=q_{1,1}$. $[\Lambda(e_5), \Lambda(u_2)]=0$, we have $q_{1,2}=0$, $q_{1,1}=q_{2,2}$. If $[\Lambda(e_3), \Lambda(u_2)]=\Lambda(u_1)$ then $q_{2,3}=0$. $[\Lambda(e_1), \Lambda(u_2)]=0$, $q_{1,3}=0$. $[\Lambda(e_2), \Lambda(u_2)]=\Lambda(u_2)$, $q_{1,1}=0$. If $[\Lambda(e_4), \Lambda(u_3)]=\Lambda(u_1)$ then $r_{3,1}=r_{3,2}=r_{2,1}=0$, $r_{3,3}=r_{1,1}$. $[\Lambda(e_5), \Lambda(u_3)]=\Lambda(u_2)$, $r_{1,2}=0$, $r_{1,1}=r_{2,2}$. $[\Lambda(e_3), \Lambda(u_3)]=0$, $r_{2,3}=0$. If $[\Lambda(e_1), \Lambda(u_3)]=(1/2)\Lambda(u_3)$ then $r_{1,1}=0$, the affine connection there exists and has the form

$$\Lambda(u_1) = \Lambda(u_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \Lambda(u_3) = \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

if $r_{1,3} = 0$ then the connection is trivial; if $r_{1,3} \neq 0$ then the affine connection is equivalent to the trivial connection by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where $\pi(e_i)=e_i$, $i=\overline{1,5}$, $\pi(u_1)=(1/r_{1,3})u_1$, $\pi(u_2)=(1/r_{1,3})u_2$, $\pi(u_3)=(1/r_{1,3})u_3 - e_4$, the curvature and torsion tensors are equal to zero.

Similarly we obtain the results in the other cases. □

3.2. The curvature tensor is not zero for some connections.

3.2.1. *Transformation group is nonsolvable.*

Theorem 10. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has only trivial affine connection, the curvature tensor is not only zero, the torsion tensor is only zero, $\bar{\mathfrak{g}}$ is nonsolvable and the radical of $\bar{\mathfrak{g}}$ is commutative, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the following pair:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
2.9.12	$\{\{u_1, -e_2\}, \{-2u_2, 2u_1 + 2u_3, -e_1 - e_2\}\}$

Pair	Curvature tensor
2.9.12	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Pair	Holonomy algebra
2.9.12	$\begin{pmatrix} p_2 & 0 & p_1 \\ 0 & -2p_2 & 0 \\ 0 & 0 & 2p_2 \end{pmatrix}$

Proof. Let, for example, $(\bar{\mathfrak{g}},\mathfrak{g})$ is the three-dimensional homogeneous space 2.9.12 [17], Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_2),\Lambda(u_1)]=\Lambda([e_2,u_1]) \Rightarrow [\Lambda(e_2),\Lambda(u_1)]=0$, we have $p_{3,1}=p_{3,2}=p_{2,1}=p_{3,1}=0, p_{3,3}=p_{1,1}$. $[\Lambda(e_1),\Lambda(u_1)]=\Lambda([e_1,u_1]) \Rightarrow [\Lambda(e_1),\Lambda(u_1)]=\Lambda(u_1)$, $p_{1,1}=p_{1,2}=p_{1,3}=p_{2,2}=p_{2,3}=0$. If $[\Lambda(e_2),\Lambda(u_2)]=\Lambda([e_2,u_2])$ then $[\Lambda(e_2),\Lambda(u_2)]=0$, $q_{3,1}=q_{3,2}=q_{2,1}=0$, $q_{3,3}=q_{1,1}$. If $[\Lambda(e_1),\Lambda(u_2)]=\lambda\Lambda(u_2)$ then $q_{1,1}=q_{1,2}=q_{1,3}=q_{2,2}=q_{2,3}=0$. $[\Lambda(e_1),\Lambda(u_3)]=\mu\Lambda(u_3)$, $r_{1,1}=r_{1,2}=r_{1,3}=r_{2,1}=r_{2,2}=r_{2,3}=r_{3,1}=r_{3,2}=r_{3,3}=0$. $[\Lambda(e_2),\Lambda(u_3)]=\Lambda(u_1)$, $\Lambda(u_1)=\Lambda(u_2)=\Lambda(u_3)=0$. $V=\{[\Lambda(x),\Lambda(y)]-\Lambda([x,y]) | x,y\in\bar{\mathfrak{g}}\}$, $\Lambda(\bar{\mathfrak{g}})=\Lambda(\mathfrak{g})$ and $[\Lambda(\bar{\mathfrak{g}}),V]=[\Lambda(\mathfrak{g}),V]=V$, $\Lambda(\mathfrak{g})$ is equal to V. In this case, $\mathfrak{a}_{\bar{\mathfrak{g}}}=\Lambda(\mathfrak{g})$ and the holonomy algebra $\mathfrak{h}^*=\mathfrak{a}_{\bar{\mathfrak{g}}}$, i.e. the connection is normal. □

Theorem 11. *If the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has only trivial affine connection, the curvature tensor is not only zero, the torsion tensor is only zero, $\bar{\mathfrak{g}}$ is nonsolvable and the radical of $\bar{\mathfrak{g}}$ is not commutative, then $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition
4.11.2	$\{\{-e_4,e_1,e_3,u_1\},\{-e_2-e_4,-u_2,u_1-u_3\}\}$
4.13.2	$\{\{e_1,-e_2,e_3,u_1\},\{e_4-e_2,u_2,u_1+u_3\}\}$
4.13.3	$\{\{e_1,e_2,-e_3,u_1\},\{-e_2-e_4,-u_2,u_1-u_3\}\}$
3.8.8	$\{\{-e_3,-u_1,e_1\},\{e_1-2e_2+(1/2)e_3,-2u_2,-u_1-2u_3\}\}$
2.1.2	$\{\{u_3,e_2\},\{u_1,-u_2,e_1\}\}$
2.3.2	$\{\{u_3,e_2\},\{-u_2,u_1,e_1\}\}$
2.3.3	$\{\{u_3,e_2\},\{-u_2,u_1,-e_1\}\}$

Pair	Curvature tensor
4.11.2	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4.13.2	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
4.13.3	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
3.8.8	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
2.1.2	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.3.2 2.3.3	$\begin{pmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Pair	Holonomy algebra	Pair	Holonomy algebra
4.11.2	$\begin{pmatrix} 0 & p_2 & p_1 \\ 0 & -p_3 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$	4.13.2 4.13.3	$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & -p_3 \\ 0 & p_3 & 0 \end{pmatrix}$
3.8.8	$\begin{pmatrix} p_2 & 0 & -p_1 \\ 0 & -2p_2 & 0 \\ 0 & 0 & 2p_2 \end{pmatrix}$	2.1.2	$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & -p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.3.2	$\begin{pmatrix} 0 & -p_1 & 0 \\ p_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2.3.3	$\begin{pmatrix} 0 & -p_1 & 0 \\ p_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The proof is just as above.

3.2.2. *Transformation group is solvable.*

Theorem 12. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *has only trivial affine connection, the curvature tensor is not only zero, the torsion tensor is only zero,* $\bar{\mathfrak{g}}$ *and* $\mathfrak{g}$ *are solvable, then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

5.10.2, 4.8.10, 4.11.4, 4.20.2, 4.21.2 ($\lambda \neq 1$), 3.8.9, 3.13.2 ($\mu \neq 0$), 3.13.4, 3.14.2, 3.19.16, 3.20.4 ($\lambda \neq 0$), 3.20.5 ($\mu \neq 1/2$), 3.23.2, 3.27.2, 2.8.6, 2.9.3 ($\mu \neq 0, 1/2, 1/4$), 2.16.2.

Pair	Curvature tensor
5.10.2, 4.11.4, 4.20.2, 4.21.2($\lambda \neq 1$), 3.8.9, 3.13.2($\mu \neq 0$), 3.13.4, 3.14.2, 3.19.16, 3.20.5($\mu \neq 1/2$), 3.23.2, 3.27.2, 2.8.6, 2.9.3($\mu \neq 0, 1/2, 1/4$), 2.16.2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4.8.10, 3.13.4	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3.20.4 ($\lambda \neq 0$)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Pair	Holonomy algebra
5.10.2, 4.11.4, 4.20.2, 4.21.2($\lambda \neq 1$), 3.8.9, 3.13.2($\mu \neq 0$), 3.14.2, 3.19.16, 3.20.5($\mu \neq 1/2$), 3.23.2, 3.27.2, 2.8.6, 2.9.3($\mu \neq 0, 1/2, 1/4$), 2.16.2	$\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4.8.10, 3.13.4	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_1 \\ 0 & 0 & 0 \end{pmatrix}$
3.20.4 ($\lambda \neq 0$)	$\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Remark. In the cases 4.21.2 ($\lambda = 0$), 3.13.2 ($\mu \neq 0$), 3.13.4, 3.20.4 ($\lambda \neq 0$), 3.20.5 ($\mu \neq 1/2$) the connection is trivial after basis replacement.

Proof. Just as earlier, in case, for example, 3.13.2 ($\mu \neq 0$) we have

$$\Lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \Lambda(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \Lambda(e_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_3), \Lambda(u_1)] = \Lambda([e_3, u_1]) \Rightarrow [\Lambda(e_3), \Lambda(u_1)] = 0$; hence $p_{3,1} = p_{3,2} = 0, p_{3,3} = p_{1,1}, p_{2,1} = 0$. From $[\Lambda(e_2), \Lambda(u_1)] = \Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)] = 0$ we have $p_{1,2} = 0$, $p_{1,1} = p_{2,2}$. $[\Lambda(e_1), \Lambda(u_1)] = \Lambda(u_1)$; then $p_{1,1} = p_{2,3} = p_{1,3} = 0$. If $[\Lambda(e_3), \Lambda(u_2)] = 0$ then $q_{3,1} = q_{3,2} = q_{2,1} = 0$, $q_{3,3} = q_{1,1}$. $[\Lambda(e_2), \Lambda(u_2)] = 0$, $q_{1,2} = 0$, $q_{1,1} = q_{2,2}$. $[\Lambda(e_1), \Lambda(u_2)] = (1 - 2\mu)\Lambda(u_2)$, we have $(-1 + 2\mu)q_{1,1} = q_{1,3} = q_{2,3} = 0$. If $[\Lambda(e_3), \Lambda(u_3)] = \Lambda(u_1)$ then $r_{3,1} = r_{3,2} = r_{2,1} = 0$, $r_{3,3} = r_{1,1}$. $[\Lambda(e_2), \Lambda(u_3)] = \Lambda(u_2)$, $q_{1,1} = r_{1,2} = 0$, $r_{1,1} = r_{2,2}$. $[\Lambda(e_1), \Lambda(u_3)] = \mu\Lambda(u_3)$, we have $r_{1,1} = r_{1,3}(1 - 2\mu) = r_{2,3}(1 - 4\mu) = 0$, i.e. $r_{1,3} = 0$ $(\mu \neq 1/2)$, $r_{2,3} = 0$ $(\mu \neq 1/4)$; in the cases $\mu = 1/2, 1/4$ after basis replacement the parameters are equal to zero too.

Curvature tensor $R(u_1, u_2) = R(u_1, u_3) = 0$,

$$R(u_2, u_3) = [\Lambda(u_2), \Lambda(u_3)] - \Lambda([u_2, u_3]) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Torsion tensor is equal to zero.

Holonomy algebra coincides with the algebra generated by

$$V = \{[\Lambda(x), \Lambda(y)] - \Lambda([x, y]) | x, y \in \bar{\mathfrak{g}}\} \;\Rightarrow\; \mathfrak{h}^* = \begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Really, $[\Lambda(\bar{\mathfrak{g}}), V] \subset V$, hence $\mathfrak{h}^* = V$. The connection is not normal $(\Lambda(\bar{\mathfrak{g}}) \supset \Lambda(\mathfrak{g})$, $\dim a_{\bar{\mathfrak{g}}} \geq 3$ and $\mathfrak{h}^* \neq a_{\bar{\mathfrak{g}}})$.

Similarly we obtain the results in the other cases. □

We describe all invariant affine connections with only zero torsion tensor on three-dimensional homogeneous spaces together with their curvature tensors and holonomy algebras. In this work we use the algebraic approach for description of connections, methods of the theory of Lie groups, Lie algebras and homogeneous spaces.

References

[1] P. K. Rashevskii, *On the geometry of homogeneous spaces*, Dokl. Akad. Nauk, SSSR (N. S.), **80** (1951), 169–171. MR0044183

[2] P. K. Rashevskii, *On the geometry of homogeneous spaces*, Trudy Sem. Vektor Tenzor Analiz.,**9** (1952), 49–74. MR0053586

[3] M. Kurita, *On the vector in homogeneous spaces*, Nagoya Math. J., **5** (1953), 1–33. MR0059052

[4] E. B. Vinberg, *On invariant linear connections*, Dokl. Akad. Nauk, SSSR, **128** (1959), 653–654. MR0110073

[5] E. B. Vinberg, *Invariant linear connections in a homogeneous space*, Trudy Moscow Mat. Obsc., **9** (1960), 191–210. MR0176418

[6] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, John Wiley and Sons, New York, **1** (1963); **2** (1969). MR0152974, MR0238225.

[7] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math., **76** (1954), 33–65. MR0059050

[8] Nguyen van Hai, *Construction de l'algebre de Lie des transformations infinitesimales affines sur un espace homogene a connexion lineaire invariante*, Comptes Rendus de l'Academie des Sciences, **263** (1966), 876–879. MR0206869

[9] B. Opozda, *A classification of locally homogeneous connections on 2-dimensional manifolds*, Differ. Geom. Appl., **21** (2004), 173–198. MR2073824

[10] O. Kowalski, B. Opozda, Z. Vlasek, *A classification of locally homogeneous connections on 2-dimensional manifolds via group-theoretical approach*, Cent. Eur. J. Math., **2** (2004), 87–102. MR2041671

[11] T. Arias-Marco, O. Kowalski, *Classification of locally homogeneous affine connections with arbitrary torsion on 2-dimensional manifolds*, Monatsh. Math., **153** (2008), 1–18. MR2366132

[12] B. Dubrov, B. Komrakov, Y. Tchempkovsky, *Invariant affine connections on three-dimensional homogeneous spaces*, Preprint series: Pure mathematics, **6** (1996). http://urn.nb.no/URN:NBN:no-47353

[13] A.L. Onischik, *Topology tranzitive Lie groups of transformations*, M.: Phiz.-math. lit., 1995.

[14] B. Kostant, *Holonomy and the Lie algebra of infinitesimal motions of a Riemannian manifolds*, Trans. Amer. Math. Soc., **80** (1955), 528–542. MR0084825

[15] A. Lichnerowicz, *Geometrie des Groupes de Transformations* Dunod, Paris, 1958. MR0124009

[16] H.C. Wang, *On invariant connections over a principal fibre bundle*, Nagoya Math. J., **13** (1958), 1–19. MR0107276

[17] N.P. Mozhey, *Three-dimensional isotropy-faithful homogeneous spaces and connections on them*, Izd-vo Kazan. un-ta, Kazan', 2015.

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