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Spin 1 Particle with Anomalous Magnetic Moment in an External Uniform Magnetic Field

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Within the matrix 10-dimensional Duffin–Kemmer (DK) formalism applied to the Shamaly–Capri field, we study the behavior of a vector particle with anomalous magnetic moment in epy presence of an external uniform magnetic field. The separation of variables in the wave equation is performed using projective operator techniques and the theory of DK-algebras. The problem is reduced to a system of 2-nd order differential equations for three independent functions, which is solved in terms of confluent hypergeometric functions. Three series of energy levels are found, two of which substantially differ from those for spin 1 particles without anomalous magnetic moment. For assigning them physical sense for all the values of the main quantum number $n = 0, 1, 2, \ldots$, one has to impose special restrictions on a parameter related to the anomalous moment. Otherwise, only some part of the energy levels corresponds to the bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in occurrence of a space scaling of arguments of the wave functions, compared to a particle without such a moment.

Also we give some details of the general theory of the Shamaly–Capri particle; in particular, we describe some features of this theory extended to General Relativity.

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1. Introduction

Commonly, we shall use only the simplest wave equations for fundamental particles of spin $0, 1/2, 1$. Meanwhile, it is known that other

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more complicated equations can be proposed for particles with such spins, these equations are based on application of extended sets of the Lorentz group representations (see [1]-[18]).

Such generalized wave equations allow to describe more complicated objects, which have, besides mass, spin, and electric charge, other electromagnetic characteristics like polarizability or anomalous magnetic moment. These additional characteristics manifest themselves explicitly in presence of external electromagnetic fields.

In particular, within this approach Petras [3] proposed a 20-component theory for spin 1/2 particle, which-after excluding 16 subsidiary components – turns out to be equivalent to the Dirac particle theory modified by the presence of the Pauli interaction term. In other words, this theory describes a spin 1/2 particle with anomalous magnetic moment. In this approach, a generalized equation for a spin 1 particle has been proposed by Shamaly–Capri [6], [7] (also see [16], [17]). The last one describes a vector particle with anomalous magnetic moment (see also some recent papers [20–22] in which scalar, spinor and vector particles are studied in presence of external fields).

In the present paper, we investigate the wave equation for spin 1 particle with anomalous magnetic moment in the presence of an external uniform magnetic field. The generalized formulas for the Landau energy levels are derived, and the corresponding wave functions are constructed. The new formulas for the energies in presence of an external magnetic field, in principle, allow to experimentally distinguish such a particle.

The restriction to the case of neutral vector boson (the uncharged spin 1/2 particle with anomalous magnetic moment) is performed.

As well, we give some details of the general theory of the Shamaly–Capri particle; in particular, we describe some features of this theory extended to General Relativity.

2. The separation of the variables

The wave equation for spin 1 particle with anomalous magnetic moment is presented in Duffin–Kemmer (DK) matrix formalism [6, 7], it includes an additional non-minimal interaction term with an external field through the electromagnetic tensor $F_{[\mu\nu]}$ and reads

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} P J_{[\mu\nu]} + M\right)\Psi = 0 \quad (1)$$

where the 10-component wave function and the 10×10 dimensional DK-matrices β_a are used:

$$\Psi = \begin{vmatrix} \Psi_\mu \\ \Psi_{[\mu\nu]} \end{vmatrix}, \qquad J_{[\mu\nu]} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu\,;$$

the matrix P stands for a projective operator separating from Ψ its vector component Ψ_μ, $D_\mu = \partial_\mu - ieA_\mu$, in Minkowski space, the metrics with imaginary unit is used: since $x_4 = ict$, λ_3 denotes an arbitrary complex number related to anomalous magnetic moment (more details are given in Sec. 7), $J_{[\mu\nu]}$ are Lorentz group generators, M is a mass term. The matrix equation (1) may be re-written in the tensor form as follows

$$D_\mu\Psi_\nu - D_\nu\Psi_\mu + M\Psi_{[\mu\nu]} = 0\,,$$
$$D_\nu\Psi_{[\mu\nu]} \pm 2\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]}\Psi_\nu + M\Psi_\mu = 0\,. \quad (2)$$

When using DK-matrices, we apply the method of generalized Kronecker's symbols (for more details see [24]). The collective indexes $A(B, C, D, \ldots)$ list the following 10 independent components of the wave function: $1, 2, 3, 4$, [23], [31], [12], [14], [24], [34]:

$$\beta_\mu = e^{\nu,[\nu\mu]} + e^{[\nu\mu],\nu}, \quad P = e^{\nu,\nu}\,,$$
$$(e^{A,B})_{CD} = \delta_{AC}\delta_{BD}\,, \quad e^{A,B}e^{C,D} = \delta_{BC}e^{A,D}\,,$$
$$\delta_{[\mu\nu],[\rho\sigma]} = \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\,,$$

the entity $(e^{A,B})_{CD}$ denotes the elements of complete matrix algebra. The main properties of DK-matrices are

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\rho + \delta_{\rho\nu}\beta_\mu\,,$$
$$[\beta_\lambda, J_{[\rho\sigma]}]_- = \delta_{\lambda\rho}\beta_\sigma - \delta_{\lambda\sigma}\beta_\rho\,.$$

We use the following representation for these matrices:

$$\beta_1 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta_2 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta_3 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta_4 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

A uniform magnetic field is specified by the relations

$$A_1 = -\frac{1}{2}Bx_2, A_2 = \frac{1}{2}Bx_1, A_{3,4} = 0,\ F_{[12]} = B.$$

The non-minimal interaction through the anomalous magnetic moment is given by the term

$$\pm\frac{ie}{M}\lambda_3\lambda_3^* F_{[\mu\nu]} P J_{[\mu\nu]} = \pm 2\frac{ie}{M}\lambda_3\lambda_3^* B P J_{[12]}\,.$$

Correspondingly, eq.(1) is written as

$$\Big[\beta_1(\partial_1 + \frac{ie}{2}Bx_2) + \beta_2(\partial_2 - \frac{ie}{2}Bx_1) + \beta_3\partial_3 + \beta_4\partial_4 \pm 2\frac{ie}{M}\lambda_3\lambda_3^* BPJ_{[12]} + M\Big]\Psi = 0\,. \quad (3)$$

Let us introduce the matrix

$$Y = iJ_{[12]} = i(\beta_1\beta_2 - \beta_2\beta_1);$$

it satisfies the minimal polynomial equation $Y(Y-1)(Y+1) = 0$, which permits us to define three projective operators:

$$P_0 + P_- + P_+ = I, \quad P_0 = 1 - Y^2,$$
$$P_+ = \frac{1}{2}Y(Y+1), \quad P_- = \frac{1}{2}Y(Y-1)$$

and to resolve the wave function into three components, $\Psi = \Psi_- + \Psi_0 + \Psi_+$:

$$\Psi_0 = P_0\Psi, \quad \Psi_+ = P_+\Psi, \quad \Psi_- = P_-\Psi.$$

By transforming (3) to cylindric coordinates

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad \tan\phi = x_2/x_1\,,$$

we get

$$\Big[\beta_1\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} + iB_0 r\sin\phi\right) + \beta_2\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - iB_0 r\cos\phi\right) + (\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M)\Big]\Psi = 0 \quad (4)$$

where we have used the shortening notation:

$$\frac{eB}{2} = B_0 \,, \quad \pm 4\frac{B_0}{M}\lambda_3\lambda_3^* = \Gamma \,. \tag{5}$$

We further act on (4) by the projective operator P_0; taking into account the identities

$$P_0\beta_3 = \beta_3 P_0, \quad P_0\beta_4 = \beta_4 P_0, \quad PY = YP \,,$$

$$P_0\beta_1 = \beta_1(1-P_0) = \beta_1(P_+ + P_-) \,,$$

$$P_0\beta_2 = \beta_2(1-P_0) = \beta_2(P_+ + P_-) \,,$$

and we get

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\Psi_0$$
$$+\left[\beta_1(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}) + \beta_2(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}) + iB_0 r\sin\phi\beta_1 - iB_0 r\cos\phi\beta_2\right]\Psi_+$$
$$+\left[\beta_1(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}) + \beta_2(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}) + iB_0 r\sin\phi\beta_1 - iB_0 r\cos\phi\beta_2\right]\Psi_- = 0$$

where we took into account the identities $YP_0 \equiv 0 \implies \Gamma Y\Psi_0 = \Gamma(YP_0)\Psi = 0$. By introducing the notation

$$\beta_+ = \frac{1}{\sqrt{2}}(\beta_1 + i\beta_2) \,, \quad \beta_- = \frac{1}{\sqrt{2}}(\beta_1 - i\beta_2) \,,$$

we can transform the previous equation to the form

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\Psi_0 + \frac{1}{\sqrt{2}}\left[e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r) + e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r)\right]\Psi_+$$
$$+\frac{1}{\sqrt{2}}\left[e^{+i\phi}\beta_-(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r) + e^{-i\phi}\beta_+(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r)\right]\Psi_- = 0 \,.$$

By making use of the projective operators

$$P_\pm = \frac{1}{2}[\beta_1\beta_1 - 2\beta_1\beta_1\beta_2\beta_2 \pm i(\beta_1\beta_2 - \beta_2\beta_1)],$$

and the commutation relations for DK-matrices, we prove the identities $\beta_- P_+ = \beta_+ P_- = 0$; so the above equation is written simpler

$$[\beta_3\partial_3 + \beta_4\partial_4 + M]\,\Psi_0 + \frac{1}{\sqrt{2}}e^{-i\phi}\beta_+\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r\right)\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r\right)\Psi_- = 0. \tag{6}$$

Now, we act on (4) by $1 - P_0 = P_+ + P_-$; this gives

$$(1-P_0)\beta_1\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi} + iB_0 r\sin\phi\right)\Psi + (1-P_0)\beta_2\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - iB_0 r\cos\phi\right)\Psi$$
$$+ (\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M)\,(\Psi_+ + \Psi_-) = 0 \,.$$

By using the identities $(1-P_0)\beta_1 = \beta_1 P_0$ and $(1-P_0)\beta_2 = \beta_2 P_0$, we transform this equation to

$$\beta_1 \left(\cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\Psi_0 + iB_0 r\sin\phi\beta_1\Psi_0 + \beta_2\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\Psi_0 - iB_0 r\cos\phi\beta_2\Psi_0$$
$$+ \left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)(\Psi_+ + \Psi_-) = 0\,, \qquad (7)$$

from which it follows that

$$\frac{1}{\sqrt{2}}\left[e^{-i\phi}\beta_+\left(\frac{\partial}{\partial r} - \frac{i}{r} - B_0 r\right) + e^{+i\phi}\beta_-\left(\frac{\partial}{\partial r} + \frac{i}{r} + B_0 r\right)\right]\Psi_0$$
$$+ \left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)(\Psi_+ + \Psi_-) = 0\,. \qquad (8)$$

Now, let us act on (8) by $\frac{1}{2}(1+Y)$. Because

$$\frac{1}{2}(1+Y)P_+ = P_+,\ \frac{1}{2}(1+Y)P_- = 0,$$

$$Y\beta_- = \beta_- P_0, \qquad Y\beta_+ = -\beta_+ P_0,$$

the above equation simplifies to

$$\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)\Psi_+$$
$$+\frac{1}{\sqrt{2}}e^{+i\phi}\beta_-\left(\frac{\partial}{\partial r} + \frac{i}{r} + B_0 r\right)\Psi_0 = 0. \qquad (9)$$

Similarly, by multiplying (8) by $\frac{1}{2}(1-Y)$ and taking into account the identities

$$\frac{1}{2}(1-Y)P_+ = 0, \quad \frac{1}{2}(1-Y)P_- = P_-,$$

$$Y\beta_- = \beta_- P_0, \qquad Y\beta_+ = -\beta_+ P_0,$$

we derive

$$\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma PY + M\right)\Psi_- + \frac{1}{\sqrt{2}}e^{-i\phi}\beta_+$$
$$\times\left(\frac{\partial}{\partial r} - \frac{i}{r} - B_0 r\right)\Psi_0 = 0. \qquad (10)$$

Now, by considering the relations

$$YP_+ = \frac{1}{2}(Y^3 + Y^2) = \frac{1}{2}(1+Y^2) = P_+,$$
$$YP_- = \frac{1}{2}(Y^3 - Y^2) = \frac{1}{2}(1-Y^2) = -P_-,$$

we transform (6), (9) and (10) to the form

$$\left[\beta_3\partial_3 + \beta_4\partial_4 + M\right]\Psi_0$$
$$+\frac{1}{\sqrt{2}}e^{-i\phi}\beta_+\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi} - B_0 r\right)\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi} + B_0 r\right)\Psi_- = 0\,,$$
$$\left(\beta_3\partial_3 + \beta_4\partial_4 + \Gamma P + M\right)\Psi_+ + \frac{1}{\sqrt{2}}e^{+i\phi}\beta_-\left(\frac{\partial}{\partial r} + \frac{i}{r} + B_0 r\right)\Psi_0 = 0\,,$$
$$\left(\beta_3\partial_3 + \beta_4\partial_4 - \Gamma P + M\right)\Psi_- + \frac{1}{\sqrt{2}}e^{-i\phi}\beta_+\left(\frac{\partial}{\partial r} - \frac{i}{r} - B_0 r\right)\Psi_0 = 0\,. \qquad (11)$$

To separate the variables, we search for three components of the wave function in the form

$$\Psi_0 = e^{ip_4x_4}e^{ip_3x_3}e^{im\phi}f_0(r)\,,$$

$$\Psi_{\pm} = e^{ip_4 x_4} e^{ip_3 x_3} e^{i(m\pm 1)\phi} f_{+}(r)\,.$$

The resulting from (11) radial equations are written in symbolic form as

$$(ip_3\beta_3 + ip_4\beta_4 + M)\, f_0 + \frac{1}{\sqrt{2}}\beta_{+}(\frac{d}{dr} + \frac{m+1}{r} - B_0 r) f_{+} + \frac{1}{\sqrt{2}}\beta_{-}(\frac{d}{dr} - \frac{m-1}{r} + B_0 r) f_{-} = 0,$$

$$(ip_3\beta_3 + ip_4\beta_4 + \Gamma P + M)\, f_{+} + \frac{1}{\sqrt{2}}\beta_{-}\left(\frac{d}{dr} - \frac{m}{r} + B_0 r\right) f_0 = 0\,,$$

$$(ip_3\beta_3 + ip_4\beta_4 - \Gamma P + M)\, f_{-} + \frac{1}{\sqrt{2}}\beta_{+}\left(\frac{d}{dr} + \frac{m}{r} - B_0 r\right) f_0 = 0.$$

3. The radial system

By using the notations

$$\hat{a}_m = \frac{1}{\sqrt{2}}(+\frac{d}{dr} + \frac{m - B_0 r^2}{r}),$$

$$\hat{b}_m = \frac{1}{\sqrt{2}}(-\frac{d}{dr} + \frac{m - B_0 r^2}{r}),\ ip_3\beta_3 + ip_4\beta_4 = i\hat{p},$$

the equations (12) are rewritten shorter as

$$(i\hat{p} + M)\, f_0 + \beta_{+}\hat{a}_{m+1} f_{+} - \beta_{-}\hat{b}_{m-1} f_{-} = 0, \quad (12)$$

$$\begin{aligned} &(i\hat{p} + \Gamma P + M)\, f_{+} - \beta_{-}\hat{b}_m f_0 = 0,\\ &(i\hat{p} - \Gamma P + M)\, f_{-} + \beta_{+}\hat{a}_m f_0 = 0. \end{aligned} \quad (13)$$

We further act on (12) by the operator $(M + \Gamma)^{-1}(M + \Gamma\bar{P})$, where $\bar{P} = 1 - P$. This yields

$$\left[\frac{1}{M+\Gamma}(M + \Gamma\bar{P}) i\hat{p} + \frac{1}{M+\Gamma}(M + \Gamma\bar{P})(M + \Gamma P)\right]$$

$$\times f_{+} - \frac{1}{M+\Gamma}(M + \Gamma\bar{P})\beta_{-}\hat{b}_m f_0 = 0.$$

We note the relation

$$\frac{(M + \Gamma\bar{P})(M + \Gamma P)}{M + \Gamma}$$

$$= \frac{M^2 + M\Gamma P + M\Gamma\bar{P} + \Gamma^2\bar{P}P}{M+\Gamma} = \frac{M^2 + M\Gamma}{M+\Gamma} = M,$$

which is valid due to the identities $P + \bar{P} = 1$, $P\bar{P} = \bar{P}P = 0$. We introduce the notations:

$$\frac{M + \Gamma\bar{P}}{M+\Gamma}\, i\hat{p} = A\,, \quad \frac{M + \Gamma\bar{P}}{M+\Gamma}\beta_{-} = \beta'_{-}\,.$$

Then the above equation transforms to

$$(A + M) f_{+} - \beta'_{-}\hat{b}_m f_0 = 0\,.$$

Similarly, we act on (13) by the operator $(M - \Gamma)^{-}(M - \Gamma\bar{P})$, $\bar{P} = 1 - P$, which yields

$$\left[\frac{1}{M-\Gamma}(M - \Gamma\bar{P}) i\hat{p} + \frac{1}{M-\Gamma}(M - \Gamma\bar{P})(M - \Gamma P)\right] f_{-} + \frac{1}{M-\Gamma}(M - \Gamma\bar{P})\beta_{+}\hat{a}_m f_0 = 0.$$

Considering the identities

$$\frac{(M-\Gamma\bar{P})(M-\Gamma P)}{M-\Gamma} = \frac{M^2 - M\Gamma P - M\Gamma\bar{P} + \Gamma^2\bar{P}P}{M-\Gamma} = \frac{M^2 - M\Gamma}{M-\Gamma} = M$$

and introducing the notations

$$\frac{(M-\Gamma\bar{P})}{M-\Gamma}\,i\hat{p} = C\,, \quad \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_+ = \beta'_+\,,$$

we present the above equation in the form

$$(C+M)f_- + \beta'_+\hat{a}_m f_0 = 0.$$

Thus, the radial system can be written as

$$(i\hat{p}+M)\,f_0 + \beta_+\hat{a}_{m+1}f_+ - \beta_-\hat{b}_{m-1}f_- = 0\,,$$
$$(A+M)f_+ - \beta'_-\hat{b}_m f_0 = 0\,,$$
$$(C+M)f_- + \beta'_+\hat{a}_m f_0 = 0\,.$$

To proceed with these equations, we introduce the matrices (note that $p^2 = p_3^2 + p_4^2$) with the properties

$$\overline{(i\hat{p}+M)}(i\hat{p}+M) = p^2 + M^2,$$
$$\overline{(A+M)}(A+M) = p^2 + M^2, \qquad (14)$$
$$\overline{(C+M)}(C+M) = p^2 + M^2\,.$$

In fact, these formulas determine the inverse matrices to within numerical factors $(p^2+M^2)^{-1}$. Then the system of radial equations can be rewritten alternatively as

$$(i\hat{p}+M)\,(p^2+M^2)f_0 + \beta_+\hat{a}_{m+1}$$
$$\times(p^2+M^2)f_+ - \beta_-\hat{b}_{m-1}(p^2+M^2)f_- = 0, \qquad (15)$$

$$(p^2+M^2)f_+ - \overline{(A+M)}\beta'_-\hat{b}_m f_0 = 0,$$

$$(p^2+M^2)f_- + \overline{(C+M)}\beta'_+\hat{a}_m f_0 = 0.$$

The first equation in (15), with the help of the other two, can be transformed into an equation for the component $f_0(r)$:

$$(i\hat{p}+M)(p^2+M^2)f_0 + \beta_+\hat{a}_{m+1}\overline{(A+M)}\beta'_-\hat{b}_m f_0$$
$$+\beta_-\hat{b}_{m-1}\overline{(C+M)}\beta'_+\hat{a}_m f_0 = 0, \quad (16)$$

while two remaining ones are not changed

$$(p^2+M^2)f_+ - \overline{(A+M)}\beta'_-\hat{b}_m f_0 = 0\,,$$
$$(p^2+M^2)f_- + \overline{(C+M)}\beta'_+\hat{a}_m f_0 = 0\,. \qquad (17)$$

In fact, the equations (17) mean that it suffices to solve (16) with respect to f_0; the other two components f_+ and f_- can be calculated by means of equations (17).

To proceed further, we need to know the explicit form of the inverse operators (14). To solve this task, we first establish the minimal polynomials for the relevant matrices. The minimal polynomial for $(i\hat{p})$ is $i\hat{p}\,[(i\hat{p})^2 + p^2] = 0$ (see in [3]). We further consider the operator A^2:

$$A^2 = \frac{1}{(M+\Gamma)^2}(iM\hat{p} + i\Gamma\bar{P}\hat{p})(iM\hat{p} + i\Gamma\bar{P}\hat{p}) = \frac{1}{(M+\Gamma)^2}[-M^2\hat{p}^2 - M\Gamma\hat{p}\bar{P}\hat{p} - M\Gamma\bar{P}\hat{p}^2 - \Gamma^2\bar{P}\hat{p}\bar{P}\hat{p}]\,.$$

Due to the identities

$$\begin{aligned} \beta_\mu = P\beta_\mu + \beta_\mu P = \bar{P}\beta_\mu + \beta_\mu \bar{P}, \\ \beta_\mu P = P\beta_\mu, \bar{P}\beta_\mu = \beta_\mu \bar{P}, \\ P\beta_\mu P = \bar{P}\beta_\mu \bar{P} = 0, \\ \beta_\mu\beta_\nu P = P\beta_\mu\beta_\nu, \beta_\mu\beta_\nu \bar{P} = \bar{P}\beta_\mu\beta_\nu, \\ P + \bar{P} = 1, P\bar{P} = \bar{P}P = 0, \end{aligned}$$

we find

$$A^2 = \frac{1}{(M+\Gamma)^2}(-M^2\hat{p}^2 - M\Gamma\hat{p}^2) = -\frac{M\hat{p}^2}{M+\Gamma}\,.$$

Thus, we get the minimal polynomial for A as

$$A^3 = -\frac{M}{(M+\Gamma)^2}(M+\Gamma\bar{P})(i\hat{p})\hat{p}^2$$

$$= -\frac{Mp^2}{(M+\Gamma)}\frac{(M+\Gamma\bar{P})}{M+\Gamma}(i\hat{p}) = -\frac{Mp^2}{M+\Gamma}A.$$

Similarly, we find

$$C^3 = -\frac{M\hat{p}^2}{M-\Gamma}C\,.$$

Therefore, the necessary inverse operators must be quadratic with respect to the relevant matrices. They are given by the formulas:

$$\begin{aligned} \overline{(M+i\hat{p})} &= \frac{1}{M}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right], \\ \overline{(A+M)} &= \frac{p^2+M^2}{M}\left[1 - \frac{M+\Gamma}{p^2+M^2+M\Gamma}A \right. \\ &\left. + \frac{M+\Gamma}{M(p^2+M^2+M\Gamma)}A^2\right], \\ \overline{(C+M)} &= \frac{p^2+M^2}{M}\left[1 - \frac{M-\Gamma}{p^2+M^2-M\Gamma}C \right. \\ &\left. + \frac{M-\Gamma}{M(p^2+M^2-M\Gamma)}C^2\right]. \end{aligned}$$

Let us turn back to the equation for f_0, rewritten in the form

$$(p^2+M^2)^2 f_0 + \overline{(M+i\hat{p})}\beta_+\hat{a}_{m+1}\overline{(A+M)}$$

$$\times\beta'_-\hat{b}_m f_0 + \overline{(M+i\hat{p})}\beta_-\hat{b}_{m-1}\overline{(C+M)}\beta'_+\hat{a}_m f_0 = 0.$$

Taking into account the explicit form for the inverse operators, we get

$$\begin{aligned} (p^2+M^2)f_0 + \frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right]\beta_+\hat{a}_{m+1} \\ + \left[1 - \frac{M+\Gamma}{p^2+M^2+M\Gamma}A + \frac{M+\Gamma}{M(p^2+M^2+M\Gamma)}A^2\right]\beta'_-\hat{b}_m f_0 + \frac{1}{M^2}[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)]\beta_-\hat{b}_{m-1} \\ + [1 - \frac{M-\Gamma}{p^2+M^2-M\Gamma}C + \frac{M-\Gamma}{M(p^2+M^2-M\Gamma)}C^2\,]\,\beta'_+\hat{a}_m f_0 = 0\,. \end{aligned}$$

Now, by using the formulas

$$\begin{aligned} A = \frac{M+\Gamma\bar{P}}{M+\Gamma}\,i\hat{p}, A^2 = -\frac{M\hat{p}^2}{M+\Gamma}, \\ C = \frac{M-\Gamma\bar{P}}{M-\Gamma}i\hat{p}, C^2 = -\frac{M\hat{p}^2}{M-\Gamma}, \\ \beta'_- = \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_-, \beta'_+ = \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_+, \end{aligned}$$

we transform the above equation into following one

$$
\begin{aligned}
&(p^2+M^2)f_0 + \frac{1}{M^2}\left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\right]\,\beta_+ \\
&\times [1 - \frac{M+\Gamma\bar{P}}{p^2+M^2+M\Gamma}\,i\hat{p} + \frac{(i\hat{p})^2}{p^2+M^2+M\Gamma}\,]\,\frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_-\,\hat{a}_{m+1}\hat{b}_m f_0 \\
&+\frac{1}{M^2}\,[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\,]\beta_- \\
&\times [1 - \frac{M-\Gamma\bar{P}}{p^2+M^2-M\Gamma}\,i\hat{p} + \frac{(i\hat{p})^2}{p^2+M^2-M\Gamma}\,]\,\frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_+\,\hat{b}_{m-1}\hat{a}_m f_0 = 0\,.
\end{aligned}
$$

After some manipulation with the use of identity $\hat{p}\beta_+\hat{p} = \hat{p}\beta_-\hat{p} = 0$, this equation can be presented in a different way as

$$
\begin{aligned}
&\{\,(p^2+M^2) + \hat{a}_{m+1}\hat{b}_m \frac{1}{M^2(p^2+M^2+M\Gamma)}\,\frac{1}{M+\Gamma} \\
&\times[\,(p^2+M^2+M\Gamma)(i\hat{p})^2\beta_+ - M(p^2+M^2+M\Gamma)i\hat{p}\beta_+ \\
&+(p^2+M^2)(p^2+M^2+M\Gamma)\beta_+ - (p^2+M^2)\beta_+ i\hat{p}(M+\Gamma P) \\
&+(p^2+M^2)\beta_+(i\hat{p})^2]\,(M+\Gamma\bar{P})\beta_- + \hat{b}_{m-1}\hat{a}_m \frac{1}{M^2(p^2+M^2-M\Gamma)}\,\frac{1}{M-\Gamma} \\
&\times[\,(p^2+M^2-M\Gamma)(i\hat{p})^2\beta_- - M(p^2+M^2-M\Gamma)i\hat{p}\beta_- \\
&+(p^2+M^2)(p^2+M^2-M\Gamma)\beta_- - (p^2+M^2)\beta_- i\hat{p}(M-\Gamma P) \\
&+(p^2+M^2)\beta_-(i\hat{p})^2]\,(M-\Gamma\bar{P})\beta_+\,\}f_0 = 0. \qquad (18)
\end{aligned}
$$

Now we take into account the explicit form of f_0, $i\hat{p}$, and matrices $\beta_+, \beta_-, \bar{P}$. Then, we derive four equations:

$$
\begin{aligned}
&(p^2+M^2)f_3 + \hat{a}_{m+1}\hat{b}_m \frac{1}{M^2(p^2+M^2+M\Gamma)}\,\frac{1}{M+\Gamma} \\
&\times\{(p^2+M^2)(p^2+M^2+M\Gamma)(M+\Gamma)f_3 - p_4(M+\Gamma)(p^2+M^2+M\Gamma)(p_4f_3 - p_3f_4) \\
&-p_3(p^2+M^2)(M+\Gamma)(p_3f_3+p_4f_4) + p_3M(M+\Gamma)(p^2+M^2)f_{12}\} \\
&+\hat{b}_{m-1}\hat{a}_m \frac{1}{M^2(p^2+M^2-M\Gamma)}\,\frac{1}{M-\Gamma}\{\,(p^2+M^2)(p^2+M^2-M\Gamma)(M-\Gamma)f_3 \\
&-p_4(M-\Gamma)(p^2+M^2-M\Gamma)(p_4f_3 - p_3f_4) \\
&-p_3(p^2+M^2)(M-\Gamma)(p_3f_3+p_4f_4) - p_3M(M-\Gamma)(p^2+M^2)f_{12}\} = 0\,, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&(p^2+M^2)f_4 + \hat{a}_{m+1}\hat{b}_m \frac{1}{M^2(p^2+M^2+M\Gamma)}\,\frac{1}{M+\Gamma} \\
&\times\{\,(p^2+M^2)(p^2+M^2+M\Gamma)(M+\Gamma)f_4 + p_3(M+\Gamma)(p^2+M^2+M\Gamma)(p_4f_3 - p_3f_4) \\
&-p_4(p^2+M^2)(M+\Gamma)(p_3f_3+p_4f_4) + p_4M(M+\Gamma)(p^2+M^2)f_{12}\} \\
&+\hat{b}_{m-1}\hat{a}_m \frac{1}{M^2(p^2+M^2-M\Gamma)}\,\frac{1}{M-\Gamma}\{(p^2+M^2)(p^2+M^2-M\Gamma)(M-\Gamma)f_4 \\
&+p_3(M-\Gamma)(p^2+M^2-M\Gamma)(p_4f_3 - p_3f_4) \\
&-p_4(p^2+M^2)(M-\Gamma)(p_3f_3+p_4f_4) - p_4M(M-\Gamma)(p^2+M^2)f_{12}\} = 0\,, \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
&(p^2+M^2)f_{12}+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\\
&\times\{M(p^2+M^2)(p^2+M^2+M\Gamma)f_{12}-M(p^2+M^2)p^2f_{12}-M(M+\Gamma)(p^2+M^2)(p_3f_3+p_4f_4)\}\\
&+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\{M(p^2+M^2)(p^2+M^2-M\Gamma)f_{12}-M(p^2+M^2)p^2f_{12}\\
&+M(M-\Gamma)(p^2+M^2)(p_4f_3+p_3f_4)\}=0\,, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&(p^2+M^2)f_{34}+\hat{a}_{m+1}\hat{b}_m\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}\{-iM(p^2+M^2+M\Gamma)(M+\Gamma)(p_4f_3-p_3f_4)\}\\
&+\hat{b}_{m-1}\hat{a}_m\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}\{-iM(p^2+M^2-M\Gamma)(M-\Gamma)(p_4f_3-p_3f_4)\}=0, \qquad (22)
\end{aligned}
$$

$$
(p^2+M^2)f_{34}-\frac{i}{M}(\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m)(p_4f_3-p_3f_4)=0\,. \qquad (23)
$$

The equations (19) and (20) may be simplified to

$$
\begin{aligned}
&(p^2+M^2)f_3+\frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\{(p^2+M^2)f_3-p_4(p_4f_3-p_3f_4)\\
&-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_3(p_3f_3+p_4f_4)+\frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_3f_{12}\}+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\Big\{(p^2+M^2)f_3-p_4(p_4f_3-p_3f_4)\\
&-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_3(p_3f_3+p_4f_4)-\frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_3f_{12}\Big\}=0,
\end{aligned}
$$

$$
\begin{aligned}
&(p^2+M^2)f_4+\frac{\hat{a}_{m+1}\hat{b}_m}{M^2}\Big\{(p^2+M^2)f_4+p_3(p_4f_3-p_3f_4)-\frac{(p^2+M^2)}{p^2+M^2+M\Gamma}p_4(p_3f_3+p_4f_4)\\
&+\frac{M(p^2+M^2)}{p^2+M^2+M\Gamma}p_4f_{12}\Big\}+\frac{\hat{b}_{m-1}\hat{a}_m}{M^2}\\
&\times\Big\{(p^2+M^2)f_4+p_3(p_4f_3-p_3f_4)-\frac{(p^2+M^2)}{p^2+M^2-M\Gamma}p_4(p_3f_3+p_4f_4)-\frac{M(p^2+M^2)}{p^2+M^2-M\Gamma}p_4f_{12}\Big\}=0\,.
\end{aligned}
$$

By multiplying the first equation by p_4, and the second one by $-p_3$, and then summing these two results, we find

$$
\Big[\hat{a}_{m+1}\hat{b}_m+\hat{b}_{m-1}\hat{a}_m+p^2+M^2\Big]\times(p_4f_3-p_3f_4)=0. \qquad (24)
$$

By taking into consideration (23), we obtain

$$
f_{34}=-\frac{i}{M}\,(p_4f_3-p_3f_4)\,. \qquad (25)
$$

We further consider (21), which can be simplified to the form

$$\left[\, (\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{(p^2+M^2)^2 - M^2\Gamma^2}{p^2+M^2} + \frac{2M\Gamma B_0}{p^2+M^2} \, \right] \, f_{12}$$
$$+ \left[\, \frac{\Gamma}{p^2+M^2}(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{2B_0}{M} \, \right] \, (p_3 f_3 + p_4 f_4) = 0 \qquad (26)$$

where the following identity has been used: $\hat{a}_{m+1}\hat{b}_m - \hat{b}_{m-1}\hat{a}_m = -2B_0$.

Now, we turn again to (19) and (20). By multiplying the first relation by p_3, and the second one by p_4, and summing the results, we find

$$(p_3 f_3 + p_4 f_4) + \hat{a}_{m+1}\hat{b}_m \frac{1}{M(p^2 +^2 + M\Gamma)}[(M+\Gamma)(p_3 f_3 + p_4 f_4) + p^2 f_{12}]$$
$$+\hat{b}_{m-1}\hat{a}_m \frac{1}{M(p^2+M^2-M\Gamma)}[(M-\Gamma)(p_3 f_3 + p_4 f_4)] - p^2 f_{12}] = 0\,.$$

Thus, we have found two equations for $(p_3 f_3 + p_4 f_4)$ and f_{12}:

$$\left[\, (\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{(p^2+M^2)^2 - M^2\Gamma^2}{p^2+M^2} + \frac{2M\Gamma B_0}{p^2+M^2} \, \right] f_{12}$$
$$+ \left[\, \frac{\Gamma}{p^2+M^2}(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{2B_0}{M} \right] (p_3 f_3 + p_4 f_4) = 0\,;$$

$$[\,(p^2+M^2)^2 - M^2\Gamma^2\,](p_3 f_3 + p_4 f_4) + (p^2+M^2-\Gamma^2)(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m)(p_3 f_3 + p_4 f_4)$$
$$-\frac{2B_0\Gamma p^2}{M}(p_3 f_3 + p_4 f_4) - p^2 \left[\, \Gamma(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m) + \frac{2B_0(p^2+M^2)}{M} \, \right] f_{12} = 0.$$

These equations may be reduced to such a form, that the 2-nd order operator $(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m)$ acts on a single function:

$$\left(\frac{2B_0}{M} - \Gamma\right)(p_3 f_3 + p_4 f_4) + \Big[\hat{a}_{m+1}\hat{b}_m$$
$$+\hat{b}_{m-1}\hat{a}_m + p^2 + M^2 - \Gamma^2 - \frac{2B_0\Gamma}{M}\Big] f_{12} = 0,$$
$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2](p_3 f_3 + p_4 f_4)$$
$$+ \left(p^2\Gamma - \frac{2B_0 p^2}{M}\right) f_{12} = 0.$$

Thus, the final form of the equations for the four functions f_3, f_4, f_{12}, f_{34} has the following relatively simple structure:

$$f_{34} = -\frac{i}{M}\,(p_4 f_3 - p_3 f_4)\,, \qquad (27)$$

$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2]$$
$$\times(p_4 f_3 - p_3 f_4) = 0, \qquad (28)$$

$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2](p_3 f_3 + p_4 f_4)$$
$$= -p^2\left(\Gamma - \frac{2B_0}{M}\right) f_{12}, \qquad (29)$$

$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2] f_{12} = \Gamma$$
$$\times\left(\Gamma - \frac{2B_0}{M}\right) f_{12} + (\Gamma - \frac{2B_0}{M})(p_3 f_3 + p_4 f_4) \qquad (30)$$

4. Solving the radial equations, the energy spectra

The analysis of (27) and (28) can be now clearly done. The second sub-system (29)–(30) is solved through diagonalizing the mixing matrix. To this goal, let us introduce the new functions

$$\Phi_1 = (p_3 f_3 + p_4 f_4) + \lambda_1 f_{12}, \qquad \Phi_2 = (p_3 f_3 + p_4 f_4) + \lambda_2 f_{12} \tag{31}$$

where λ_1, λ_2 stand for the roots of the equation $\lambda^2 - \lambda\Gamma + p^2 = 0$:

$$\lambda_1 = \frac{1}{2}\,(\Gamma + \sqrt{\Gamma^2 - 4p^2}\,),$$

$$\lambda_2 = \frac{1}{2}\,(\Gamma - \sqrt{\Gamma^2 - 4p^2}\,).$$

So we get two separate equations:

$$\left(\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 + \lambda'_{1,2}\right)\Phi_{1,2} = 0$$

where $\lambda'_1 = (\frac{2B_0}{M} - \Gamma)\lambda_1$, and $\lambda'_2 = (\frac{2B_0}{M} - \Gamma)\lambda_2$. The radial equations read

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \epsilon^2 - M^2 - p_3^2 - \lambda'_{1,2} - \frac{(m - B_0 r^2)^2}{r^2}\right)\Phi_{1,2} = 0.$$

In variable $x = |B_0| r^2$, the the equation for Φ_1 takes the form

$$\left[4|B_0|\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right) - \frac{|B_0|(m - xB_0/|B_0|)^2}{x} + \epsilon^2 - M^2 - p_3^2 - \lambda'_1\right]\Phi_1 = 0.$$

First, let it be $B_0 = -|B_0|$; then we have

$$\left[x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{(m+x)^2}{4x} + \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|}\right]\Phi_1 = 0.$$

With the substitution $\Phi_1 = x^A e^{-Cx}\bar{\Phi}_1$, $A = |m|/2$, $c = \frac{1}{2}$, we get

$$\left[x\frac{d^2}{dx^2} + (|m| + 1 - x)\frac{d}{dx} - \left(\frac{|m| + m + 1}{2} - \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|}\right)\right]\bar{\Phi}_1 = 0\,.$$

This is a confluent hypergeometric equation; to get polynomial solutions we must impose the condition

$$\frac{|m| + m + 1}{2} - \frac{\epsilon^2 - M^2 - p_3^2 - \lambda'_1}{4|B_0|} = -n\,;$$

whence it follows $\epsilon^2 - M^2 - p_3^2 - \lambda'_1 = 2|B_0|(m + |m| + 1 + 2n)$. Hence, the two energy spectra are

$$\Phi_{1,}, \qquad \epsilon_1^2 - M^2 - p_3^2 = 2|B_0|(m + |m| + 1 + 2n) + \lambda'_{1,2}\,.$$

using the simplifying notations

$$2|B_0|(m + |m| + 1 + 2n) = N\,,$$

$$-p^2 = \epsilon^2 - p_3^2 = E > 0,\ \frac{2B_0}{M} - \Gamma = x,$$

$$\lambda'_1 = \frac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E}),\ \lambda'_2 = \frac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E}).$$

the formulas for energy levels read

$$\Phi_1,\ B_0 = -|B_0|,\ E - M^2 = N + \frac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E}),$$

$$\Phi_2,\ B_0 = -|B_0|,\ E - M^2 = N + \frac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E}).$$

We solve these equations for E:

$$2E - 2M^2 - 2N - x\Gamma = \pm x\sqrt{\Gamma^2 + 4E} \quad \Longrightarrow \quad z \equiv 2N + 2M^2 + x\Gamma,$$

$$E^2 - E(z + x^2) + \frac{z^2 - x^2\Gamma^2}{4} = 0\,;$$

and the roots are

$$E_1 = \frac{z + x^2}{2} + \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)},$$
$$E_2 = \frac{z + x^2}{2} - \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}. \tag{32}$$

To have both E_1 and E_2 real-valued and positive (such that these refer to physical energy levels), we require

$$z^2 - x^2\Gamma^2 > 0, \quad z + x^2 > 0,$$

$$(z + x^2)^2 - (z^2 - x^2\Gamma^2) > 0\,.$$

We consider the first inequality

$$z^2 - x^2\Gamma^2 = (z - x\Gamma)(z + x\Gamma) > 0$$

$$\Longrightarrow \quad (2N + 2M^2)(2N + 2M^2 + 2x\Gamma) > 0\,;$$

this holds true if we impose the following restriction (we remind that $B_0 = -|B_0| < 0$):

$$x\Gamma > 0 \quad \Longleftrightarrow \quad \left(\frac{2|B_0|}{M} + \Gamma\right)\Gamma < 0$$
$$\Longleftrightarrow \quad -\frac{2|B_0|}{M} < \Gamma < 0\,. \qquad (33)$$

The second inequality $z+x^2 = (2N+2M^2)+x\Gamma+x^2 > 0$ is valid due to (33). The third inequality $2zx^2 + x^4 + x^2\Gamma^2 > 0$ is valid due to

$$z = 2N + 2M^2 + x\Gamma,\ x\Gamma > 0.$$

Thus, we get one simple restriction on the parameter Γ:

$$B_0 = -|B_0|, \qquad -\frac{2|B_0|}{M} < \Gamma < 0\,, \qquad (34)$$

which ensures that both spectra are physical (real and positive) for all the values of quantum numbers. In the case under consideration, $B_0 = -|B_0| < 0$, from (5) it follows

$$\Gamma = \pm 4\frac{-|B_0|}{M}\lambda_3\lambda_3^*\,;$$

therefore we have the only case when the upper sign is related to $\Gamma < 0$.

Similar results can be obtained for the case of the opposed orientation of the magnetic field, $B_0 = +|B_0|$:

$$\Phi_{1,2},\ \epsilon_1^2 - M^2 - p_3^2 = 2|B_0|(-m+|m|+1+2n) + \lambda'_{1,2}\,.$$

With the similar notation

$$2|B_0|(-m + |m| + 1 + 2n) = N,$$
$$-p^2 = \epsilon^2 - p_3^2 = E > 0,\ \frac{2B_0}{M} - \Gamma = x,$$
$$\lambda'_1 = \frac{x}{2}(\Gamma + \sqrt{\Gamma^2 + 4E}),\ \lambda'_2 = \frac{x}{2}(\Gamma - \sqrt{\Gamma^2 + 4E}),$$

we derive the formulas for the energies:

$$E_1 = \frac{z + x^2}{2} + \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}\,,$$

$$E_2 = \frac{z + x^2}{2} - \frac{1}{2}\sqrt{(z + x^2)^2 - (z^2 - x^2\Gamma^2)}.$$

In order to have energy values positive and real-valued, we must impose the following restrictions

$$z + x^2 > 0,\ z^2 - x^2\Gamma^2 > 0,\ (z + x^2)^2 - (z^2 - x^2\Gamma^2) > 0\,.$$

From the inequality

$$z^2 - x^2\Gamma^2 = (z - x\Gamma)(z + x\Gamma) > 0$$

$$\Longrightarrow \quad (2N + 2M^2)(2N + 2M^2 + 2x\Gamma) > 0$$

we get the main restriction (we remind that $B_0 = +|B_0| < 0$):

$$x\Gamma > 0 \iff (-\frac{2|B_0|}{M} + \Gamma)\Gamma > 0 \iff \Gamma < 0.$$

We note that the possibility of positive values $\Gamma > 0$, $\Gamma > 2|B_0|/M$ is ignored, because in this case the admissible region for Γ does not contain the close to zero values. The two remaining inequalities are valid as well:

$$z + x^2 = (2N + 2M^2) + x\Gamma + x^2 > 0,$$

$$2zx^2 + x^4 + x^2\Gamma^2 > 0$$

$$(z = 2N + 2M^2 + x\Gamma,\ x\Gamma > 0)\,.$$

Let us summarize the main results of the Sections 2–4.

Three series of the energy levels have been found; two of them substantially differ from those for spin 1 particles without anomalous magnetic moment.

The formula (32) and its restriction (34) provide us with two series for the energy levels (we remember the formal change $m \Longrightarrow -m$, when inverting the orientation of the magnetic field) in both cases $B_0 = -|B_0|$, and $B_0 = +|B_0|$. To assign to the energies E_1 and E_2 physical sense for all values of the main quantum number $n = 0, 1, 2, \ldots$, one must impose special restrictions – which are explicitly formulated – on the values of the anomalous magnetic moment. Without these restrictions, only some part of the energy levels correspond to bound states.

The third series of the energy levels (see (28)) has the form:

$$B_0 = -|B_0| :$$

$$E_3 = \epsilon^2 - M^2 - p_3^2 = 2|B_0|(\, m + |m| + 1 + 2n) \,,$$

$$B_0 = +|B_0| :$$

$$E_3 = \epsilon^2 - M^2 - p_3^2 = 2|B_0|(-m + |m| + 1 + 2n) \,;$$

in these states the anomalous magnetic moment does not manifest itself at all.

5. Neutral particles with anomalous magnetic moment

The case of a neutral vector boson is of particular interest; now the radial system for f_3, f_4, f_{12}, f_{34} becomes more simple and reads:

$$f_{34} = -\frac{i}{M}(p_4 f_3 - p_3 f_4),$$
$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2] \times (p_4 f_3 - p_3 f_4) = 0, \qquad (35)$$

$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2] \times (p_3 f_3 + p_4 f_4) = -p^2 \Gamma f_{12}, \qquad (36)$$
$$[\hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2] f_{12} = \Gamma^2 f_{12} + \Gamma(p_3 f_3 + p_4 f_4). \qquad (37)$$

Solving (35) is a trivial task. The system (36)–(37) can be solved trough the diagonalization of the mixing matrix. Let us introduce the notation

$$\Delta = \frac{1}{\Gamma} \, [\; \hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 \;] \,,$$

$$(p_3 f_3 + p_4 f_4) = \Phi_1, \quad f_{12} = \Phi_2 \,;$$

then the system (36)–(37) reads in the matrix form as follows

$$\Delta \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} = \begin{vmatrix} 0 & -p^2 \\ 1 & \Gamma \end{vmatrix} \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix}$$

$$\Longrightarrow \; \Delta \, S \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} = S \begin{vmatrix} 0 & -p^2 \\ 1 & \Gamma \end{vmatrix} S^{-1} S \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix}.$$

Requiring

$$S \begin{vmatrix} 0 & -p^2 \\ 1 & \Gamma \end{vmatrix} S^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}, \qquad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix},$$

we derive two sub-systems:

$$\begin{vmatrix} -\lambda_1 & 1 \\ -p^2 & (\Gamma - \lambda_1) \end{vmatrix} \begin{vmatrix} s_{11} \\ s_{12} \end{vmatrix} = 0,$$

$$\begin{vmatrix} -\lambda_2 & 1 \\ -p^2 & (\Gamma - \lambda_2) \end{vmatrix} \begin{vmatrix} s_{21} \\ s_{22} \end{vmatrix} = 0$$

We use the solutions of the form:

$$\lambda_1 = \frac{1}{2}(\Gamma + \sqrt{\Gamma^2 - 4p^2}), \quad s_{11} = 1, \quad s_{12} = \lambda_1 \,;$$
$$\lambda_2 = \frac{1}{2}(\Gamma - \sqrt{\Gamma^2 - 4p^2}) \,, \; s_{21} = 1, \; s_{22} = \lambda_2. \; (38)$$

Thus, for the functions Φ_1 and Φ_2, we get the *separated* equations

$$(\; \hat{a}_{m+1}\hat{b}_m + \hat{b}_{m-1}\hat{a}_m + p^2 + M^2 - \Gamma\lambda_{1,2} \;) \; \Phi_{1,2} = 0 \,.$$

In the explicit form, these read

$$\epsilon^2 - M^2 - p_3^2 + \Gamma\lambda_{1,2} \equiv D \,,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + D - \frac{m^2}{r^2} \right) \; \Phi_{1,2} = 0 \,.$$

Let us search for solutions of the form $\Phi = r^A e^{Br} f(r)$; for $f(r)$, we derive

$$\frac{d^2 f}{dr^2} + (\frac{2A+1}{r} + 2B) \; \frac{df}{dr}$$

$$+\left(\frac{A^2-m^2}{r^2}+\frac{2AB+B}{r}+B^2+D\right)\ f=0\,.$$

By imposing the following restrictions:

$$A^2-m^2\equiv 0 \quad\Longrightarrow\quad A=\pm\mid m\mid\,;$$

$$B^2=-D \quad\Longrightarrow\quad B=\pm i\sqrt{D}\,,$$

the above equation is simplified to

$$r\frac{d^2f}{dr^2}+(2A+1+2Br)\,\frac{df}{dr}+(2AB+B)\,f=0.$$

If we take the positive case $A=+\mid m\mid$, then the solutions are vanishing near the point $r=0$. Moreover, from physical considerations, we must require the parameter D to be positive, in order to agree with the correspondence principle:

$$\Gamma=0 \quad\Longrightarrow\quad D\to D_0=\epsilon^2-M^2-p_3^2>0\,.$$

Without loss of generality, let us assume that $B=+i\sqrt{D}$. In new variable, the above equation reads as a confluent hypergeometric equation

$$2Br=-x,$$
$$x\frac{d^2f}{dx^2}+(2A+1-x)\,\frac{df}{dx}-\left(A+\frac{1}{2}\right)f=0,$$
$$F''+(c-x)F'-aF=0,$$
$$a=A+1/2,\quad c=2A+1=2a$$

where $x=-2Br=-2i\sqrt{M^2-p_3^2+\Gamma\lambda_{1,2}}$. Thus, for a neutral particle, no bound states exist, and the qualitative manifestation of the anomalous magnetic moment is mainly revealed by appearing of space scaling of the arguments of the wave functions, in comparison with the case of particles without the magnetic moment. Formally, we have two sorts of states depending on the sign of Γ: $\lambda_{1,2}$,

$$x=-2Br=-2i\sqrt{M^2-p_3^2+\Gamma\frac{1}{2}(\Gamma\pm\sqrt{\Gamma^2-4p^2})}\,.$$

There exists a third type of states in which the parameter Γ does not manifest itself at all (see (35)):

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\epsilon^2-M^2-p_3^2-\frac{m^2}{r^2}\right)$$

$$\times(p_4f_3-p_3f_4)=0,$$

for these states, the solutions depend on non-modified argument x: $x=-2Br=-2i\sqrt{M^2-p_3^2}$.

6. Shamaly–Capri theory and General Relativity

Now, let us show that in Minkowski space, the Shamaly–Capri 20-component model for the spin 1 particle in the absence of an external electromagnetic field is reduced to ordinary DK 10-component theory. We start with a free particle wave equation

$$(\,i\,\Gamma^a\partial_a\,-\,m)\,\Psi(x)=0 \qquad (39)$$

where the 20-component wave function includes the tensors, scalar and vector Φ, Φ_a and additionally includes antisymmetric and irreducible symmetric tensors $\Phi_{[ab]}$, $\Phi_{(ab)}$; note that the matrices Γ^a are 20×20-dimensional ones:

$$\Gamma^a=-i(\lambda_1e^{4,a}-\lambda_1^*e^{a,4}+\lambda_2g_{kn}\,e^{n,[ka]}$$
$$-\lambda_2^*g_{kn}e^{[ka],n}-\lambda_3\,g_{kn}\,e^{n,(ka)}-\lambda_3^*g_{kn}e^{(ka),n}, \qquad (40)$$

$(*)$ stands for complex conjugation, $(g_{ab})=\text{diag}(+1,-1,-1,-1)$; note that in this section it is more convenient in local Minkowski space to use metric without complex unit. In (40), numerical parameters λ_i obey the following set of restrictions (see [18]):

$$\lambda_1\lambda_1^*-\frac{3}{2}\lambda_3\lambda_3^*=0\,,\ \lambda_2\lambda_2^*-\lambda_3\lambda_3^*=1\,. \qquad (41)$$

We determine the explicit form of the matrices Γ^a by using the basic elements of the relevant matrix algebra $e^{A,B}$:

$$(e^{\,A,B})_C^{\ D}=\delta_C^A\,g^{B,D},\ e^{A,B}e^{C,D}=g^{B,C}\,e^{A,D}\,,$$

$$A,\,B,\ldots=0,\,a,\,[ab],\,(ab)$$

where δ_A^B is the generalized Kronecker symbol. The symbols with upper indexes $g^{A,B}$ are derived from δ_B^A with the help of the Minkowski metric tensor. We use the following Kronecker symbols:

$$\delta_{[cd]}^{[ab]} = \delta_c^a\,\delta_d^b \,-\, \delta_d^a\,\delta_c^b\,,\; g^{[ab],[cd]} = g^{ac}\,g^{bd} \,-\, g^{ad}\,g^{bc},$$

$$\delta_{(cd)}^{(ab)} = \delta_c^a\,\delta_d^b \,+\, \delta_d^a\,\delta_c^b \,-\, \frac{1}{2}\,g^{ab}\,g_{cd}\,,$$

$$g^{(ab),(cd)} = g^{ac}\,g^{bd} + g^{ad}\,g^{bc} - \frac{1}{2}g^{ab}\,g^{cd},$$

and also the generators J^{ab} for the Lorentz group representation

$$J^{ab} = (e^{a,b} - e^{b,a}) + g_{kn}(e^{[ak],[bn]} - e^{[bk],[an]})$$

$$+g_{kn}(e^{(ak),(bn)} - e^{(bk),(an)})\,.$$

Let us transform now (39) to its tensor form with respect to $\Phi, \Phi_a, \Phi_{[ab]}, \Phi_{(ab)}$:

$$\begin{gathered}\lambda_1\,\partial^a\Phi_a = m\,\Phi,\\ -\lambda_1^*\partial_b\Phi + \lambda_2\partial^a\Phi_{[ba]} - \lambda_3\partial^a\Phi_{(ba)} = m\Phi_b,\\ \lambda_2^*(\partial_b\Phi_a - \partial_a\Phi_b) = m\Phi_{[ba]},\\ -\lambda_3^*(\partial_b\Phi_a + \partial_a\Phi_b - \frac{1}{2}g_{ab}\partial^c\Phi_c) = m\Phi_{(ab)}.\end{gathered} \quad (42)$$

From the first and fourth equations in (42), by considering the relations (41), we obtain

$$-\,\lambda_1^*\,\partial_b\Phi \,-\, \lambda_3\,\partial^a\Phi_{(ba)} = \frac{\lambda_3\lambda_3^*}{\lambda_2^*}\,\partial^a\Phi_{[ab]}\,.$$

Then (see (42) and (41)) we get $\frac{1}{\lambda_2^*}\,\partial^a\Phi_{[ba]} = m\,\Phi_b$. Defining now $\Psi_a = \lambda_2^*\Phi_a\,,\; \Psi_{[ab]} = \Phi_{[ab]}\,,$ we obtain the ordinary Proca tensor equations

$$\partial^b\Psi_{[ab]} = m\Psi_a\,, \quad \partial_a\Psi_b - \partial_b\Psi_a = m\Psi_{[ab]}\,. \quad (43)$$

The last equation can be represented in DK matrix form as

$$(\,i\,\beta^a\partial_a - m)\Psi = 0\,, \quad \Psi = \left|\begin{array}{c}\Psi_a\\ \Psi_{[ab]}\end{array}\right|, \qquad (44)$$
$$\beta^a = -ig_{bc}(e^{c,[ba]} - e^{[ba],c})\,.$$

So, the equations (39) and (44) are equivalent from physical standpoint, because their solutions must be unambiguously mutually related.

The generalization of (39) to the case of arbitrary curved space-time with the metric $g_{\alpha\beta}(x)$ and any relevant $e^{\mu}_{(a)}(x)$, may be performed in accordance with the tetrad method of Tetrode–Weyl–Fock–Ivanenko [23]. Such an equation has the form

$$\begin{gathered}[\,i\Gamma^\mu(\partial_\mu + B_\mu) \,-\, m\,]\,\Psi = 0\,,\\ (\,i\Gamma^a\partial_{(a)} \,+\, \frac{i}{2}\Gamma^a J^{cd}\gamma_{cda} \,-\, m\,)\,\Psi = 0\end{gathered} \qquad (45)$$

where the notation is used:

$$\Gamma^\mu = \Gamma^a e^{\mu}_{(a)}\,,\; B_\mu = \frac{1}{2}J^{ab}e^{\nu}_{(a)}\nabla_\mu e_{(b)\nu},$$

$$\partial_{(a)} = e^{\mu}_{(a)}\partial_\mu\,,\; \gamma_{abc} = -(\nabla_\beta e_{(a)\alpha})e^{\alpha}_{(b)}e^{\beta}_{(c)},$$

∇_μ represents the covariant derivative, while γ_{abc} stands for the Ricci rotation coefficient.

The general covariant matrix wave equation (45) may be transformed to the tetrad tensor form

$$\begin{aligned}&\lambda_1\,(\partial^{(a)} \,+\, \gamma^{ba}{}_{b})\,\Phi_a \,= m\,\Phi\,,\\ &-\lambda_1^*\,\partial_{(r)}\Phi \,+\, \lambda_2\,(\,\partial^{(a)}\Phi_{[ra]} + \gamma_r{}^{bc}\Phi_{[bc]} \,+\, \gamma^{bc}{}_{c}\Phi_{[br]}\,)\\ &-\lambda_3\,(\,\partial^{(a)}\Phi_{(ra)} \,+\, \gamma_r{}^{dc}\Phi_{(dc)} + \gamma^{dc}{}_{c}\Phi_{(dr)}\,) \,= m\,\Phi_r\,,\\ &\lambda_2^*\,(\,\partial_{(r)}\Phi_s \,-\, \partial_{(s)}\Phi_r + \gamma^d_{rs}\Phi_d - \gamma^d_{sr}\Phi_d\,)\\ &= m\Phi_{[rs]},\\ &-\lambda_3^*\,[\,(\,\partial_{(r)}\Phi_s \,+\, \partial_{(s)}\Phi_r \,+\, \gamma_r{}^d{}_s\Phi_d \,+\, \gamma^d_{sr}\Phi_d)\\ &-\frac{1}{2}g_{rs}\,(\,\partial^{(a)}\Phi_a \,+\, \gamma^{ad}_a\Phi_d\,)\,] \,= m\,\Phi_{(rs)}\,.\end{aligned} \qquad (46)$$

Let us eliminate the components, and obtain the equation for the main components Φ_a and $\Phi_{[cd]}$. To this end, from the first and fourth equations in (46) we express Φ and $\Phi_{(ad)}$ and substitute the results into the second one. Due to the conditions (41) and the third equation in (46), we get

$$-\lambda_1^* \partial_{(r)}\Phi - \lambda_3 [\partial^{(a)}\Phi_{(ra)} + \gamma_r{}^{da}\Phi_{(da)} + \gamma^{ad}_{\ \ a}\Phi_{(dr)}] = -\frac{\lambda_3\lambda_3^*}{\lambda_2^*} [\partial^{(a)}\Phi_{[ra]} + \gamma_r{}^{bc}\Phi_{[bc]} + \gamma^{bc}_{\ \ c}\Phi_{[br]}]$$

$$= -\frac{2\lambda_3\lambda_3^*}{m} [\gamma^{b\ \ ,(a)}_{\ ar}\Phi_b + \gamma^{ba}_{\ \ b,(r)}\Phi_a - \gamma^{ab}_{\ \ r}\partial_{(a)}\Phi_b - \gamma_r{}^{ab}\partial_{(a)}\Phi_b + \gamma_r{}^{ab}\gamma^d_{\ ba}\Phi_d - \gamma^{ab}_{\ \ b}\gamma^c_{\ ar}\Phi_c].$$

Hence, we obtain

$$\frac{1}{\lambda_2^*}(\partial^{(a)}\Phi_{[ra]} + \gamma_r{}^{bc}\Phi_{[bc]} + \gamma^{bc}_{\ \ c}\Phi_{[br]})$$
$$-\frac{2\lambda_3\lambda_3^*}{m} [\gamma^{b\ \ ,(a)}_{\ ar}\Phi_b + \gamma^{ba}_{\ \ b,(r)}\Phi_a - \gamma^{ab}_{\ \ r}\partial_{(a)}\Phi_b - \gamma_r{}^{ab}\partial_{(a)}\Phi_b + \gamma^{ab}_{\ \ r}\gamma^d_{\ ba}\Phi_d - \gamma^{ab}_{\ \ b}\gamma^c_{\ ar}\Phi_c] = m\Phi_r. \quad (47)$$

It remains to transform the third equation in (46) and Eq. (47) to the above introduced variables Ψ_a and Ψ_{ab}. In the end, we derive the tetrad generalized Proca system:

$$\partial^{(a)}\Psi_{[ra]} + \gamma_r{}^{bc}\Psi_{[bc]} + \gamma^{bc}_{\ \ c}\Psi_{[br]}$$
$$-2\lambda_3\lambda_3^* m^{-1}[\gamma^{b\ \ ,(a)}_{\ ar}\Psi_b + \gamma^{ba}_{\ \ b,(r)}\Psi_a - \gamma^{ab}_{\ \ r}\partial_{(a)}\Psi_b - \gamma_r{}^{ab}\partial_{(a)}\Psi_b + \gamma_r{}^{ab}\gamma^d_{\ ba}\Psi_d - \gamma^{ab}_{\ \ b}\gamma^c_{\ ar}\Psi_c] = m\Psi_r,$$

$$\partial_{(a)}\Psi_b - \partial_{(b)}\Psi_a + \gamma^d_{\ ab}\Psi_d - \gamma^d_{\ ba}\Psi_d = m\Psi_{[ab]}. \quad (48)$$

The term proportional to $\frac{2\lambda_3\lambda_3^*}{m}$ determines an additional interaction term for a generalized vector particle with the gravitational field.

If we take into account the tetrad form of the Riemann and Ricci tensors through the Ricci rotation coefficients:

$$R_{abcd} = -\gamma_{abc,(d)} + \gamma_{abd,(c)} + \gamma_{akc}\gamma^k_{\ bd} + \gamma_{abn}\gamma^n_{\ cd} - \gamma_{akd}\gamma^k_{\ bc} - \gamma_{abn}\gamma^n_{\ dc},$$
$$R^b_r = R^{ab}_{\ \ ra} = -\gamma^{ab}_{\ \ r,(a)} + \gamma^{ab}_{\ \ a,(r)} + \gamma_{rna}\gamma^{ban} - \gamma^a_{\ ka}\gamma^{kb}_r,$$

we finally get

$$\partial^a\Psi_{[ra]} + \gamma_r{}^{bc}\Psi_{[bc]} + \gamma^{bc}_{\ \ c}\Psi_{[br]} - \frac{2\lambda_3\lambda_3^*}{m}$$

$$[R^b_{\ r}\Psi_b - \gamma^{ab}_{\ \ r}\partial_{(a)}\Psi_b - \gamma_r{}^{ab}\partial_{(a)}\Psi_b] - m\Psi_r = 0.$$

Like in (43), the system (48) can be represented in the matrix DK form:

$$\Big\{ i\beta^a\partial_{(a)} + \frac{i}{2}\beta^a J^{cd}_{(0)}\gamma_{cda} - \frac{\lambda_3\lambda_3^*}{m}[(\gamma^a_{\ bk} - \gamma^a_{\ kb}) \times (e^{b,k} - e^{k,b})\partial_{(a)} + R_{bk}(e^{b,k} + e^{k,b})] - m \Big\}\Psi = 0.$$

It can be readily proved that the tetrad system (46) can be transformed to the generally covariant tensor form (below we use the notation $D_\alpha = \nabla_\alpha - ieA_a(x)$):

$$\lambda_1 D_\alpha\Phi^\alpha = m\Phi, \quad (49)$$
$$-\lambda_1^* D_\beta\Phi + \lambda_2 D^\alpha\Phi_{[\beta\alpha]} - \lambda_3 D^\alpha\Phi_{(\alpha\beta)} = m\Phi_\beta,$$
$$\lambda_2^*(D_\alpha\Phi_\beta - D_\beta\Phi_\alpha) = m\Phi_{[\alpha\beta]}, \quad (50)$$
$$-\lambda_3^*[D_\alpha\Phi_\beta + D_\beta\Phi_\alpha - \frac{1}{2}g_{\alpha\beta}\nabla^\rho\Phi_\rho] = m\Phi_{(\alpha\beta)}. \quad (51)$$

The relations between the tetrad and the tensor components are:

$$\Phi_\alpha = e^{(a)}_\alpha\Phi_a, \quad \Phi_{[\alpha\beta]} = e^{(a)}_\alpha e^{(b)}_\beta\Phi_{[ab]},$$

$$\Phi_{(\alpha\beta)} = e^{(a)}_\alpha e^{(b)}_\beta\Phi_{(ab)}.$$

As for (46), the system (51) can be reduced to the minimal form

$$\frac{1}{\lambda_2^*}D^\alpha\Phi_{[\beta\alpha]} + \frac{2\lambda_3\lambda_3^*}{m}[D_\alpha, D_\beta]_-\Phi^\alpha = m\Phi_\beta,$$

$$\lambda_2^* \left(D_\alpha \Phi_\beta \, - \, D_\beta \Phi_\alpha \right) \, = m \, \Phi_{[\alpha\beta]} \, .$$

or, alternatively, to

$$D^\alpha \Psi_{[\beta\alpha]} \, + \, \frac{2\lambda_3 \lambda_3^*}{m} \, [D_\alpha, D_\beta]_- \, \Psi^\alpha = m \, \Psi_\beta \, , \qquad D_\alpha \Psi_\beta - D_\beta \Psi_\alpha = m \, \Psi_{[\alpha\beta]} \, . \tag{52}$$

Taking into account that $[D_\alpha, D_\beta]_- \, \Psi^\alpha = (-ieF_{\alpha\beta} + R_{\alpha\beta})\Psi^\alpha$, we conclude that the parameter $\frac{\lambda_3 \lambda_3^*}{m}$ in (52) determines both the *anomalous magnetic moment* of a spin 1 particle and *the additional interaction term with non-Euclidean space-time background* through the Ricci tensor $R_{\alpha\beta}$.

7. Conclusions

By applying the matrix 10-dimensional Duffin–Kemmer formalism to the Shamaly–Capri field, the behavior of a vector particle with anomalous magnetic moment is studied in the presence of an external uniform magnetic field. The problem is reduced to a system of 2-nd order differential equations for three independent functions, these equation are solved in terms of confluent hypergeometric functions. Three series of the energy levels are found; two of them substantially differ from those for spin 1 particle without anomalous magnetic moment. To assign them physical sense for all values of the main quantum number $n = 0, 1, 2, ...$ one has to impose special restrictions on a parameter related to the anomalous moment. Otherwise, the energy levels corresponds only partially to the bound states. The neutral spin 1 particle is considered as well. In this case no bound states exist in the system, and the main qualitative manifestation of the anomalous magnetic moment consists in occurrence of space scaling of the arguments of the wave functions, in comparison with a particle which has no such a moment. Some features of theory of the Shamaly–Capri particle within General Relativity are given.

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