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Analytical properties of solutions to a class of third-order nonlinear dynamical dissipative systems with no chaotic behaviour

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Abstract. The singular analysis was performed for solutions of one class of third-order nonlinear dynamical systems with no chaotic behaviour. It is established that all systems of the given class (except one) are not the systems of Painlevé-type.

1. Introduction

One of the important events in classical nonlinear physics of mid-twentieth century was the realization that nonlinear deterministic equations (systems) could have chaotic solutions that exhibited both a sensitive dependence on initial conditions and long-term unpredictability. An interesting and remaining to be solved is the problem of identifying minimum necessary conditions for chaos. In [1], computer simulation was used to obtain 19 third-order dynamic systems with complex chaotic behavior, which are algebraically simpler than the known Lorentz and Rössler ones. The distinctive difference of the above Sprott systems is that their right-hand parts contain 6 components with one quadratic nonlinearity or 5 components with two quadratic nonlinearities. In [2], the class (including four families) of third-order nonlinear dynamical dissipative systems with no chaotic behaviour was found

$$\dot{x} = y^2 - x, \;\; \dot{y} = z, \;\; \dot{z} = x. \tag{1.1}$$

$$\dot{x} = y^2 + z, \;\; \dot{y} = x, \;\; \dot{z} = -z. \tag{1.2}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x, \;\; \dot{z} = y. \tag{1.3}$$

$$\dot{x} = y^2, \;\; \dot{y} = x + z, \;\; \dot{z} = -z. \tag{1.4}$$

$$\dot{x} = y^2, \;\; \dot{y} = z - y, \;\; \dot{z} = x. \tag{1.5}$$

$$\dot{x} = y^2, \;\; \dot{y} = z, \;\; \dot{z} = x - z. \tag{1.6}$$

$$\dot{x} = yz, \;\; \dot{y} = x, \;\; \dot{z} = x - z. \tag{1.7}$$

$$\dot{x} = yz, \;\; \dot{y} = x, \;\; \dot{z} = y - z. \tag{1.8}$$

$$\dot{x} = y^2 + yz, \;\; \dot{y} = x, \;\; \dot{z} = -z. \tag{2.1}$$

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$$\dot{x} = y^2 + z^2, \;\; \dot{y} = x, \;\; \dot{z} = -z. \tag{2.2}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = xz, \;\; \dot{z} = \varepsilon y. \tag{2.3}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = xz, \;\; \dot{z} = kz. \tag{2.4}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = z^2, \;\; \dot{z} = x. \tag{2.5}$$

$$\dot{x} = y^2 + y, \;\; \dot{y} = xz, \;\; \dot{z} = -z. \tag{2.6}$$

$$\dot{x} = y^2 + z, \;\; \dot{y} = x^2, \;\; \dot{z} = -z. \tag{2.7}$$

$$\dot{x} = y^2 + z, \;\; \dot{y} = xz, \;\; \dot{z} = -z. \tag{2.8}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x^2, \;\; \dot{z} = \varepsilon x. \tag{2.9}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x^2, \;\; \dot{z} = y. \tag{2.10}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x^2, \;\; \dot{z} = kz. \tag{2.11}$$

$$\dot{x} = yz - x, \;\; \dot{y} = \varepsilon xz, \;\; \dot{z} = y. \tag{2.12}$$

$$\dot{x} = yz - x, \;\; \dot{y} = z^2, \;\; \dot{z} = \varepsilon x. \tag{2.13}$$

$$\dot{x} = \varepsilon y - x, \;\; \dot{y} = xz, \;\; \dot{z} = x^2. \tag{2.14}$$

$$\dot{x} = \varepsilon y - x, \;\; \dot{y} = xz, \;\; \dot{z} = y^2. \tag{2.15}$$

$$\dot{x} = y - x, \;\; \dot{y} = z^2, \;\; \dot{z} = x^2. \tag{2.16}$$

$$\dot{x} = y - x, \;\; \dot{y} = z^2, \;\; \dot{z} = xy. \tag{2.17}$$

$$\dot{x} = x^2 + yz, \;\; \dot{y} = -2xy, \;\; \dot{z} = -z. \tag{3.1}$$

$$\dot{x} = y^2 + yz, \;\; \dot{y} = x^2, \;\; \dot{z} = -z. \tag{3.2}$$

$$\dot{x} = y^2 + yz, \;\; \dot{y} = \varepsilon xz, \;\; \dot{z} = -z. \tag{3.3}$$

$$\dot{x} = y^2 + \varepsilon z^2, \;\; \dot{y} = x^2, \;\; \dot{z} = -z. \tag{3.4}$$

$$\dot{x} = y^2 + \varepsilon z^2, \;\; \dot{y} = xz, \;\; \dot{z} = -z. \tag{3.5}$$

$$\dot{x} = xy - x, \;\; \dot{y} = xz, \;\; \dot{z} = -yz. \tag{3.6}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = xz, \;\; \dot{z} = x^2. \tag{3.7}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = xz, \;\; \dot{z} = y^2. \tag{3.8}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = z^2, \;\; \dot{z} = x^2. \tag{3.9}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = z^2, \;\; \dot{z} = xy. \tag{3.10}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x^2, \;\; \dot{z} = \varepsilon xy. \tag{3.11}$$

$$\dot{x} = yz - x, \;\; \dot{y} = x^2, \;\; \dot{z} = y^2. \tag{3.12}$$

$$\dot{x} = yz - x, \;\; \dot{y} = \varepsilon xz, \;\; \dot{z} = y^2. \tag{3.13}$$

$$\dot{x} = y^2 + yz, \;\; \dot{y} = -y, \;\; \dot{z} = x^2. \tag{3.14}$$

$$\dot{x} = \varepsilon + y^2, \;\; \dot{y} = xz, \;\; \dot{z} = -z. \tag{4.1}$$

$$\dot{x} = 1 + yz, \;\; \dot{y} = x^2, \;\; \dot{z} = -z. \tag{4.2}$$

$$\dot{x} = 1 - x, \;\; \dot{y} = xz, \;\; \dot{z} = y^2. \tag{4.3}$$

$$\dot{x} = y^2 - x, \;\; \dot{y} = xz, \;\; \dot{z} = 1. \tag{4.4}$$

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$$\dot x = yz - x, \;\; \dot y = x^2, \;\; \dot z = 1 \tag{4.5}$$

with unknown functions x, y, z of the independent variable t, where k is a parameter ($k < 1$) and $\varepsilon^2 = 1$.

Each of the above systems is, as noted, dissipative and contains four components in the right-hand part. Right-hand parts of the systems (1.1)–(1.8), (2.1)–(2.17), (3.1)–(3.14) contain one, two, three quadratic nonlinearities correspondingly. Right-hand parts of the systems (4.1) – (4.5) contain one constant and two quadratic nonlinearities.

Assuming the independent variable t to be complex, let us determine whether the general solution of the above systems has no moving critical singular points, that is, whether the so called Painlevé property is fulfilled for them. Systems (equations) with the Painlevé property are called systems (equations) of the Painlevé-type or P-type.

1. The system (1.1) is equivalent to equation

$$\dddot y = y^2 - \ddot y. \tag{1}$$

The system (1.2) is equivalent to equation

$$\ddot y = y^2 + Ce^{-t}, \tag{2}$$

where C is an arbitrary constant.

The system (1.3) has a first integral $x + y - \frac{z^2}{2} = H$ (H is an arbitrary constant) and so is equivalent to equation

$$\ddot z + \dot z - \frac{z^2}{2} = H. \tag{3}$$

The system (1.4) also is equivalent to equation (2). Each of the systems (1.5), (1.6) is equivalent to equation (1). The system (1.8) is equivalent to equation

$$\dddot z + \ddot z - z\dot z - z^2 = 0. \tag{4}$$

Note, that the equations (2), (3) are not included in the list of equations in [3], general solutions of which have no movable critical singular points. The equation (4) is not included in [4] in the list of equations $\dddot u = P(t, u, \dot u, \ddot u)$, where P is a polynomial in $u, \dot u, \ddot u$ with analytic in t coefficients, the general solutions are free from movable critical singular points. Because of this the system (1.8) is not the system of Painlevé-type.

The system (1.7) is equivalent to equation

$$y\dddot y + y\ddot y - \dot y\ddot y - \dot y\ddot y - y^2\dot y = 0. \tag{5}$$

To clarify the question of whether the system (1.7) is the system of Painlevé-type we use the Painlevé test. The formal Painlevé test refers to any algorithm, which verifies the fulfillment of the conditions required for the existence in differential equation (system) the Painlevé property. The system (1.7) study was conducted according to the same scheme as in [5].

Theorem 1 Systems (1.2), (1.4) are not the systems of P-type. However, componet z of the data systems does not have movable critical singular points at all.

Theorem 2 Systems (1.1), (1.3), (1.5), (1.6), (1.8) are not the systems of P-type.

Theorem 3 The system (1.7) is not the system of P-type. The validity of this assertion follows from the fact that for the system (1.7) is not fulfilled the condition of the 3-rd step in Painlevé test, namely: the system for determining constant solutions of the system (1.7) in formal Laurent expansions is not joint.

Corollary 1. Equation (5) is not a Painlevé-type equation.

2. Systems (2.1), (2.2), (2.4), (2.6)–(2.8), (2.11) due to the fact that the unknown function z is explicit (and does not have movable critical singular points) are reduced to nonlinear nonautonomous equations of the second order, the solutions of which according to [3] do not possess the Painlevé property. Thus, the following is true

Theorem 4 None of the systems (2.1), (2.2), (2.4), (2.6)—(2.8), (2.11) is of the Painlevé-type. At the same time, a component z of these systems has no movable critical singular points at all.

Theorem 5 None of the systems (2.5), (2.10), (2.16), (2.17) is of the Painlevé-type.

The proof of this theorem follows from the fact that each of these systems fails the first step of Painlevé test.

Systems (2.3), (2.9), (2.12), (2.15) are equivalent to equations

$$z\dddot{z} = \dot{z}\ddot{z} - z\ddot{z} + \varepsilon\dot{z}^2z^2, \tag{6}$$

$$z\dddot{z} = \dot{z}\ddot{z} - z\ddot{z} + \varepsilon\dot{z}^2z^2 + \dot{z}^2, \tag{7}$$

$$z\dddot{z} = \dot{z}\ddot{z} - z\ddot{z} + \varepsilon\dot{z}z^3, \tag{8}$$

$$x\dddot{x} = \dot{x}\ddot{x} - x\ddot{x} + \varepsilon x^2(\dot{x} + x)^2 + \dot{x}^2. \tag{9}$$

The system (2.13) and the system (2.14) are equiavalent to the equation

$$w\dddot{w} = \dot{w}\ddot{w} - w\ddot{w} + \dot{w}^2 + \varepsilon w^4 \tag{10}$$

from z and x correspondingly.

Theorem 6 None of the systems (2.3), (2.9), (2.12)–(2.15) is of the Painlevé-type. Applying to systems (2.3), (2.9), (2.13)–(2.15) the Painlevé-test shows, that for them the condition of the first step of the given test is not fulfilled. As for the system (2.12), for it the condition of the third step of the test is not fulfilled: the system for determining constant solutions of the system (2.12) in formal Laurent expansions is not joint.

Corollary 2. None of the equations (6) – (10) is of the Painlevé-type.

3. Analysis of the systems (3.1), (3.3), (3.5) shows, that each of them admits a reduction of order and they are reduced to nonlinear nonautonomous equations of the second order, which are not included in the list in [3] of equations of P-type. Consequently, the following is true

Theorem 7 None of the systems (3.1), (3.3), (3.5) is of the Painlevé-type. At the same time, a component z of these systems has no movable critical singular points at all.

Each of the systems (3.2), (3.4), (3.14) admits a reduction of order. In this connection the following is true

Theorem 8 None of the systems (3.2), (3.4), (3.14) is of the Painlevé-type. At the same time, one of the components (z for (3.2), (3.4)) and y for (3.14) has no movable critical singular points at all.

Theorem 9 The system (3.6) is of the Painlevé-type. Its general solution has the form

$$x = \frac{\dot{y}}{z},\ z = c_3e^{c_1t + c_2e^{-t}},\ y = c_1 + c_2e^{-t},$$

where c_1, c_2, c_3 are arbitrary constants.

Indeed, the system (3.6) $\dot{x} = xy - x, \dot{y} = xz, \dot{z} = -yz$ has the first integral $xz + y = c$, which allows to reduce it to the linear equation $\dot{y} + y = c$ and find the explicit form of y.

For systems (3.7), (3.9), (3.10), (3.12) the condition of the second step of Painlevé test is not fulfilled: the resonance equations have two complex conjugate roots with positive real part.

Systems (3.8), (3.11), (3.13) do not pass the third step of the Painlevé test:

1. The system for determining constant solutions of the system (3.8) in formal Laurent expansions is not joint.

2. Laurent expansions of the solutions of the system (3.11) contain 2 (instead 3) arbitrary constants. In this connection the resonance equation has a positive root of multiplicity 2. In this case the system (3.11) has the first integral

$$\frac{x^2}{2} + \frac{\varepsilon z^2}{2} + y = K,$$

where K is an arbitrary constant.

3. Laurent expansions of the solutions of the system (3.13) contain 2 (instead 3) arbitrary constants. The resonance equation has the same as in the case of the system (3.11) positive root of multiplicity 2.

Theorem 10 None of the systems (3.8), (3.11), (3.13) is of the Painlevé-type.

Systems (4.1) – (4.5) are reduced to two-dimensional non-autonomous systems, each of which does not pass the Painlevé test. Thus, the following is true

Theorem 11 None of the systems (4.1) – (4.5) is of the Painlevé-type. At the same time, one of the components (z for (4.1), (4.2), (4.4), (4.5) and x for (4.3)) has no movable critical singular points at all.

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