***The Schrödinger equation for N-particle system with homogeneous relative space***

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**Abstract:** The Schrödinger equation for a closed system of $N \geq 2$ particles, the relative space of which can be a homogeneous space of any of groups isomorphic to $E(3N-3)$, $SO(3N-2)$ or $SO(3N-3,1)$ , is formulated. The formulation is given in termins of variables subordinated to special transformational laws under Galilean transformations of physical space by use of the group representations theory allowing to construct dynamics of particles directly at a quantum level.

## 1. Introduction

For the first time quantum dynamics of one particle in homogeneous spaces of groups isomorphic to $SO(4)$ and $SO(3,1)$, treated respectively as 3-dimensional spaces of constant positive and negative curvature, has been considered in [1] and [2], where the Kepler-Coulomb problem has been formulated and solved. The isotropic oscillator in the same spaces factually has been studied in [3]. Further these problems have been discussed from various points of view in [4-9].

In this paper the possibility of use of homogeneous spaces of groups isomorphic to $SO(3N-2)$ and $SO(3N-3,1)$ for $N \geq 2$ in completely other physical context is discussed. Unlike the listed articles, where the physical space possesses a non-Euclidean structure, in our approach to formulation of dynamical equation for $N-$particle system $(N \geq 2)$ the physical space is Euclidean, but above mentioned groups are the transitivity groups of the relative space of this system and are entered for the description of the interpartial interaction. For this purpose, from the very beginning, instead of the position vectors $\vec{x}_i$ $(1 \leq i \leq N)$ of each particle of a system in the physical space, new variables $\vec{R} = \vec{R}(\vec{x}_1,...,\vec{x}_N)$ and $\vec{r} = \vec{r}(\vec{x}_1,...,\vec{x}_N)$ are introduced, specfying respectively the position of a point in spaces $\mathbb{R}^3$ and $\mathbb{R}^{3N-3}$ and subordinated to special transformational laws under Galilean transformations of physical space (these variables as well as the relative space of a system will be defined below). In terms of these variables it is possible in full consent with requirement of Galilean invariance to formulate for considered system the dynamical equation, which admits freedom in a choice of the transitivity group of its relative space. Dynamics based on this equation enables one to describe the interaction between particles of a system not only with help of different interaction potentials, but also by means of a choice of such a group. Its formulation may be most consecutively and simply given by use of the group representations theory allowing to construct dynamics of particles directly at a quantum level, bypassing the construction and subsequent quantization of corresponding classical counterpart. With this purpose for all transitive groups viewed in this paper their unitary representations with an operator factor are used. Such representations are defined in some linear space $\mathfrak{L}$ of the functions $\Psi(x)$ specified on the set $M$ by the map $T(g)$ of the following form [10]:

$$T(g)\Psi(x)=A(x,g)\Psi\left(g^{-1}x\right), \tag{1}$$

where $A(x,g)-$ function of a point $x\in M$ and of an element $g$ from the group of transformations $G$ of the set $M$ satisfying the functional equation

$$A(x,g_1g_2)=A(x,g_1)A\left(g_1^{-1}x,g_2\right). \tag{2}$$

**2. Galilei group and relative space of a system**

It is well known that the nonrelativistic space of events (*the Galilean space*) is the set $\mathbb{R}_E^3\times\mathbb{R}$ of points $(\vec{x},t)$ with the Galilean group of motion $\Gamma$, where $\vec{x}$ is a position vector of a point of physical space $\mathbb{R}_E^3$ (a space $\mathbb{R}^3$ with Euclidean metric) and $t$ is a point of the time axis $\mathbb{R}$. The transformations $(\tau,\vec{a},\vec{V},\vec{c})\in\Gamma$ of this space are given by the formula

$$(\tau,\vec{a},\vec{V},\vec{c})(\vec{x},t)=\left(\mathfrak{R}(\vec{c})\vec{x}+\vec{V}t+\vec{a},t+\tau\right), \tag{3}$$

with the composition law [11]

$$(\tau_1,\vec{a}_1,\vec{V}_1,\vec{c}_1)(\tau_2,\vec{a}_2,\vec{V}_2,\vec{c}_2)=\left(\tau_1+\tau_2,\vec{a}_1+\mathfrak{R}(\vec{c}_1)\vec{a}_2+\tau_2\vec{V}_1,\vec{V}_1+\mathfrak{R}(\vec{c}_1)\vec{V}_2,\langle\vec{c}_1,\vec{c}_2\rangle\right), \tag{4}$$

where $\mathfrak{R}(\vec{c})$ is the rotation of space $\mathbb{R}_E^3$ by the angle $\varphi=2arctg|\vec{c}|$ about the vector $\vec{c}$ ($\vec{c}$ is a vector-parameter of the rotation group $SO(3)$), and the symbol $\langle\vec{c}_1,\vec{c}_2\rangle$ denotes the composition law of vector-parameters [12]):

$$\mathfrak{R}(\vec{c})\vec{x}=\frac{\left(1-\vec{c}^2\right)\vec{x}+2\vec{c}(\vec{c}\vec{x})+2\vec{c}\times\vec{x}}{1+\vec{c}^2},$$

$$\langle\vec{c}_1,\vec{c}_2\rangle=\frac{\vec{c}_1+\vec{c}_2+\vec{c}_1\times\vec{c}_2}{1-\vec{c}_1\vec{c}_2}.$$

Here $\vec{c}\vec{x}$ is the scalar product of vectors, $\vec{c}^2=\vec{c}\vec{c}$ and $\vec{c}\times\vec{x}$ is the vector product of vectors.

The unity element and the inverse element in such parameterization are $(0,\vec{0},\vec{0},\vec{0})$ and

$\left(-\tau, \Re\left(-\vec{c}\right)\left(\tau\vec{V}-\vec{a}\right), -\Re\left(-\vec{c}\right)\vec{V}, -\vec{c}\right)$ accordingly, where it is accepted into account, that $\Re^{-1}\left(\vec{c}\right)=\Re\left(-\vec{c}\right)$.

Let's put in correspondence to set of $N$ simultaneous events $\left(\vec{x}_1,t\right),\ldots,\left(\vec{x}_N,t\right)$ a point $\left(\vec{R},\vec{r},t\right)$ of the space $\mathbb{R}^3\times\mathbb{R}^{3N-3}\times\mathbb{R}$, where the symbol $\vec{r}$ denotes a $\left(3N-3\right)-$ component position vector in the space $\mathbb{R}^{3N-3}$ with coordinates $r_\nu\left(\vec{x}_1,\ldots,\vec{x}_N\right)$, $\nu=s+3\left(j-1\right)$, where $j$ is a number of $3-$ component vector $\vec{r}_j=\vec{r}_j\left(\vec{x}_1,\ldots,\vec{x}_N\right)$, and $s$ is a number of its the Cartesian projection in space $\mathbb{R}^3$ (i.e. $1\le j\le N-1$, $s=1,2,3$ and $1\le\nu\le 3N-3)$ ), and the vectors $\vec{R}=\vec{R}\left(\vec{x}_1,\ldots,\vec{x}_N\right)$ and $\vec{r}_j=\vec{r}_j\left(\vec{x}_1,\ldots,\vec{x}_N\right)$ satisfy the next functional equations:

$$\left.\begin{aligned}\vec{R}\left(\Re\left(\vec{c}\right)\vec{x}_1+\vec{b}\left(t\right),\ldots,\Re\left(\vec{c}\right)\vec{x}_N+\vec{b}\left(t\right)\right)=\Re\left(\vec{c}\right)\vec{R}\left(\vec{x}_1,\ldots,\vec{x}_N\right)+\vec{b}\left(t\right),\\ \vec{r}_j\left(\Re\left(\vec{c}\right)\vec{x}_1+\vec{b}\left(t\right),\ldots,\Re\left(\vec{c}\right)\vec{x}_N+\vec{b}\left(t\right)\right)=\Re\left(\vec{c}\right)\vec{r}_j\left(\vec{x}_1,\ldots,\vec{x}_N\right).\end{aligned}\right\} \tag{5}$$

Then at $\vec{b}\left(t\right)=\vec{V}t+\vec{a}$ from (5) follows, that transformations of Galilean space (3) specify the transformations $\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}$ of the space $\mathbb{R}^3\times\mathbb{R}^{3N-3}\times\mathbb{R}$, set by the formula

$$\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\left(\vec{R},\vec{r},t\right)=\left(\Re\left(\vec{c}\right)\vec{R}+\vec{V}t+\vec{a},\tilde{\Re}\left(\vec{c}\right)\vec{r},t+\tau\right), \tag{6}$$

where the vector $\vec{r}'=\tilde{\Re}\left(\vec{c}\right)\vec{r}$ has, according to the above definition, coordinates $r'_\nu$ corresponding to the set of vectors $\vec{r}'_j=\Re\left(\vec{c}\right)\vec{r}_j$, $1\le j\le N-1$.

If we attach to the space $\mathbb{R}^{3N-3}$ the group of rotations $SO\left(3N-3\right)$ that leaves invariant the quadratic form

$$r^2=\sum_{\nu=1}^{3N-3}r_\nu^2, \tag{7}$$

the symbol $\tilde{\Re}\left(\vec{c}\right)$ will represent an element of its subgroup, isomorphic to $SO\left(3\right)$, and it is clear that transformations $\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}$ form the group $\mathfrak{G}$, isomorphic to the Galilei group $\Gamma$.

Now the explicit form of a solution of equations (6) is inessential. Only the law of transformation (6) dictated by these equations is important. From (6) follows that under these transformations a position vector $\vec{R}$ will be transformed as well as a position vector $\vec{x}\in\mathbb{R}_E^3$. Consequently, from the group viewpoint, the space $\mathbb{R}^3$ described by a position vector $\vec{R}$ is a homogeneous space of the group $E\left(3\right)\subset\mathfrak{G}$ of Euclidean transformations $\left\{0,\vec{a},\vec{0},\vec{c}\right\}$ and at introduction in it of the Euclidean metric can be identified with the space $\mathbb{R}_E^3$.

On the contrary, the transitivity group of the space $\mathbb{R}^{3N-3}$ described by a position vector $\vec{r}$, is not predetermined by transformations (3). Really, from (6) follows, that transformations (3) induce in this space only 3-parameter rotations $\tilde{\Re}(\vec{c}) \in SO(3N-3)$, leaving it motionless at the translations $\{0,\vec{a},\vec{0},\vec{0}\}$ and at the pure Galilean transformations $\{0,\vec{0},\vec{V},\vec{0}\}$. Then, supplementing the rotation group $SO(3N-3)$ of this space with translations of various types, we can transform this space or its some domain into a homogeneous space of the $(3N-2)(3N-3)/2-$parameter group G isomorphic to any of the groups $E(3N-3)$, $SO(3N-2)$ or $SO(3N-3,1)$. Further, using such a space for construction of dynamics of $N-$particle system, we'll term it as *relative space of a system* and designate by symbol $\mathbb{R}^{3N-3}_{rel}$.

One of the simplest realization of the map $(\vec{x}_1,...,\vec{x}_N) \to (\vec{R},\vec{r}_1,...,\vec{r}_{N-1})$ satisfying the required conditions is the linear transformation

$$\vec{R} = \sum_{i=1}^{N} \alpha_i \vec{x}_i , \quad \vec{r}_j = \left( \alpha_{j+1} \left( \prod_{i=1}^{N} \alpha_i \right)^{-\frac{1}{N-1}} \mathrm{A}_j \mathrm{A}_{j+1}^{-1} \right)^{\frac{1}{2}} \left( \mathrm{A}_j^{-1} \sum_{i=1}^{j} \alpha_i \vec{x}_i - \vec{x}_{j+1} \right), \tag{8}$$

where $\vec{x}_i$ is the radius-vector of *i*-th particle in the physical space $\mathbb{R}^3_E$, $\alpha_i > 0$ for all *i*, $\mathrm{A}_j = \sum_{i=1}^{j} \alpha_i$, $1 \le j \le N$, and $\mathrm{A}_N = \sum_{i=1}^{N} \alpha_i = 1$. The module of the Jacobian of the transformation (8) is equal to unity.

It is easy to see that any simultaneous cyclic permutation of the coefficients $(\alpha_1,...,\alpha_N)$ and radius-vectors $(\vec{x}_1,...,\vec{x}_N)$ does not change the vector $\vec{R}$, but lead to a new set of vectors $(\vec{r}_1,...,\vec{r}_{N-1})$. However, for our purposes it is important that the quadratic form (7), having in variables $(\vec{x}_1,...,\vec{x}_N)$ the form

$$r^2 = \left( \prod_{i=1}^{N} \alpha_i \right)^{-1} \sum_{1 \le k < j \le N} \alpha_k \alpha_j \left( \vec{x}_k - \vec{x}_j \right)^2, \tag{9}$$

doesn't depend from such permutations. Clearly, that in the space $\mathbb{R}^{3N-3}$ these permutations will represent by rotations forming finite subgroup of $SO(3N-3)$ isomorphic to cyclical group of the permutations of $N$ objects[1].

Further we'll use the space $\mathbb{R}^3_E \times \mathbb{R}^{3N-3}_{rel} \times \mathbb{R}$ as a «material» for construction of the

[1]In [13], where 3-particle system are discussed, such transformations are termed the *kinematic rotations*.

unitary representations of the Galilei group $\mathfrak{G}$ and the transitivity group G of the space $\mathbb{R}_{rel}^{3N-3}$.

**3. Unitary representation of the Galilei group $\mathfrak{G}$**

From (6) obviously follows, that group $\mathfrak{G}$ contains subgroup $\mathfrak{G}_{\tau,\vec{a}}$ of Abelian translations $\left\{\tau,\vec{a},\vec{0},\vec{0}\right\}$, subgroup $\mathfrak{G}_{\vec{c}}$ of rotations $\left\{0,\vec{0},\vec{0},\vec{c}\right\}$ isomorphic to $SO(3)$ and subgroup $\mathfrak{G}_{\vec{V}}$ of the pure Galilei transformations $\left\{0,\vec{0},\vec{V},\vec{0}\right\}$.

Since the composition law for elements $\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\in\mathfrak{G}$ coincides with (4), each transformation of G can be represented in the following form:

$$\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}=\left\{\tau,\vec{a},\vec{0},\vec{0}\right\}\left\{0,\vec{0},\vec{V},\vec{0}\right\}\left\{0,\vec{0},\vec{0},\vec{c}\right\}. \tag{10}$$

Let's introduce now the operator $\mathfrak{T}(\mathfrak{g})$, $\mathfrak{g}=\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\in\mathfrak{G}$, acting in the linear space $\mathfrak{L}$ of the complex-valued functions $\Psi=\Psi\left(\vec{R},\vec{r},t\right)$ and forming on specified above subgroups $\mathfrak{G}_{\vec{c}}$, $\mathfrak{G}_{\tau,\vec{a}}$ and $\mathfrak{G}_{\vec{V}}$ their representations of a following kinds:

$$\left.\begin{aligned}&\mathfrak{T}\left(\left\{0,\vec{0},\vec{0},\vec{c}\right\}\right)\Psi\left(\vec{R},\vec{r},t\right)=\Psi\left(\Re\left(-\vec{c}\right)\vec{R},\tilde{\Re}\left(-\vec{c}\right)\vec{r},t\right),\\&\mathfrak{T}\left(\left\{\tau,\vec{a},\vec{0},\vec{0}\right\}\right)\Psi\left(\vec{R},\vec{r},t\right)=\Psi\left(\vec{R}-\vec{a},\vec{r},t-\tau\right),\\&\mathfrak{T}\left(\left\{0,\vec{0},\vec{V},\vec{0}\right\}\right)\Psi\left(\vec{R},\vec{r},t\right)=A\left(\vec{R},t,\vec{V}\right)\Psi\left(\vec{R}-\vec{V}t,\vec{r},t\right),\end{aligned}\right\} \tag{11}$$

where the factor $A\left(\vec{R},t,\vec{V}\right)$ satisfies to the functional equation

$$A\left(\vec{R},t,\vec{V}_1+\vec{V}_2\right)=A\left(\vec{R},t,\vec{V}_1\right)A\left(\vec{R}-\vec{V}_1t,t,\vec{V}_2\right). \tag{12}$$

The equation (12) is a special case of the equation (2) in which it is taken into account that for elements of subgroup $\mathfrak{G}_{\vec{V}}$ the product $\left\{0,\vec{0},\vec{V}_1,\vec{0}\right\}\left\{0,\vec{0},\vec{V}_2,\vec{0}\right\}=\left\{0,\vec{0},\vec{V}_1+\vec{V}_2,\vec{0}\right\}$ and the inverse element $\left\{0,\vec{0},\vec{V}_1,\vec{0}\right\}^{-1}=\left\{0,\vec{0},-\vec{V}_1,\vec{0}\right\}$.

As according to (11) and (12) $\mathfrak{T}\left(\left\{0,\vec{0},\vec{0},\vec{0}\right\}\right)$=E (E is the unit operator in $\mathfrak{L}$) we can present the function $\mathfrak{T}\left(\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\right)$ in the multiplicative form corresponding to product (10):

$$\mathfrak{T}\left(\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\right)=\mathfrak{T}\left(\left\{\tau,\vec{a},\vec{0},\vec{0}\right\}\right)\mathfrak{T}\left(\left\{0,\vec{0},\vec{V},\vec{0}\right\}\right)\mathfrak{T}\left(\left\{0,\vec{0},\vec{0},\vec{c}\right\}\right).$$

Then according to (11)

$$\mathfrak{T}\left(\left\{\tau,\vec{a},\vec{V},\vec{c}\right\}\right)\Psi\left(\vec{R},\vec{r},t\right)=A\left(\vec{R}-\vec{a},t-\tau,\vec{V}\right)\Psi\left(\Re\left(-\vec{c}\right)\left(\vec{R}-\vec{V}\left(t-\tau\right)-\vec{a}\right),\tilde{\Re}\left(-\vec{c}\right)\vec{r},t-\tau\right). \tag{13}$$

From (11) and (12) follows, that this map (13) forms on subgroup of the pure Galilean transformations the representation belonging to the class of representations (1), and gives on the subgroups of time and space translations, and also on the rotation subgroup their quasiregular representations. Further we'll suppose the unitarity of representation (13) with respect to the conventional scalar product

$$\left(\Psi_1,\Psi_2\right)=\int\limits_{\mathbb{R}^3_E\times\mathbb{R}^{3N-3}_{rel}}\Psi_1^*\left(\vec{R},\vec{r},t\right)\Psi_2\left(\vec{R},\vec{r},t\right)d^3R\,d^{3N-3}r, \tag{14}$$

where $d^3R$ and $d^{3N-3}r=\prod_{\nu=1}^{3N-3}dr_\nu=\prod_{j=1}^{N-1}d^3r_j$ are the Euclidean measures in spaces $\mathbb{R}^3_E$ and $\mathbb{R}^3_{rel}$ respectively.

For $t=0$ the equation (12) takes the form $A\left(\vec{R},0,\vec{V}_1+\vec{V}_2\right)=A\left(\vec{R},0,\vec{V}_1\right)A\left(\vec{R},0,\vec{V}_2\right)$ and hence one can assume $A\left(\vec{R},0,\vec{V}\right)=\exp\left(\lambda\vec{R}\vec{V}\right)$, where $\lambda$ – some scalar parameter. Transferring to the case $t\neq 0$, we note that a solution of equation (2) may be given in the general form as follows[1]:

$$A\left(x,g\right)=f\left(x\right)/f\left(g^{-1}x\right), \tag{15}$$

where $f\left(x\right)$ – some function not vanishing on the set $\mathrm{M}$.

Bearing this in mind, we represent the factor $A\left(\vec{R},t,\vec{V}\right)$ as $A\left(\vec{R},t,\vec{V}\right)=f\left(\vec{R},t\right)/f\left(\vec{R}-\vec{V}t,t\right)$ under condition that $\left(f\left(\vec{R},t\right)/f\left(\vec{R}-\vec{V}t,t\right)\right)\Big|_{t\to 0}=\exp\left(\lambda\vec{R}\vec{V}\right)$. This condition is easily satisfied assuming $f\left(\vec{R},t\right)=\exp\left(\lambda\vec{R}^2/\left(2t\right)\right)$. That leads to the following solution of equation (10):

$$A\left(\vec{R},t,\vec{V}\right)=\exp\left(\lambda\left(\vec{R}\vec{V}-\vec{V}^2t/2\right)\right). \tag{16}$$

Then, using the law of composition (4), formula (13) and expression (16), one can easily show that for any $\mathfrak{g}_1=\left\{\tau_1,\vec{a}_1,\vec{V}_1,\vec{c}_1\right\}$ and $\mathfrak{g}_2=\left\{\tau_2,\vec{a}_2,\vec{V}_2,\vec{c}_2\right\}$ the equality

$$\mathfrak{T}\left(\mathfrak{g}_1\mathfrak{g}_2\right)=\exp\left(-\lambda\left(\Re\left(\vec{c}_1\right)\vec{a}_2\vec{V}_1+\vec{V}_1^2\tau_2/2\right)\right)\mathfrak{T}\left(\mathfrak{g}_1\right)\mathfrak{T}\left(\mathfrak{g}_2\right) \tag{17}$$

is fulfilled. Thus the operators $\mathfrak{T}\left(\mathfrak{g}\right)$ with a purely imaginary parameter $\lambda$ form the well-known physically significant projective representation of the Galilei group [14].

From expression (17) we can derive the inverse operator of representation (13)

$$\mathfrak{T}^{-1}\left(\mathfrak{g}\right)=\exp\left(\lambda\left(\vec{a}\vec{V}-\vec{V}^2\tau/2\right)\right)\mathfrak{T}\left(\mathfrak{g}^{-1}\right). \tag{18}$$

Then, using (13), (16) and (18), we can show that homothetic transformations of the

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[1] It is proved by the following chain of obvious equalities:
$A\left(x,g_1g_2\right)=f\left(x\right)/f\left(\left(g_1g_2\right)^{-1}x\right)=\left[f\left(x\right)/f\left(\left(g_1\right)^{-1}x\right)\right]\left[f\left(\left(g_1\right)^{-1}x\right)/f\left(g_2^{-1}\left(g_1^{-1}x\right)\right)\right]=A\left(x,g_1\right)A\left(g_1^{-1}x,g_2\right).$

infinitesimal operators (generators) of time and space translations $d_t = -\partial/\partial t$ and $\vec{D} = -\partial/\partial \vec{R}$, carried out by the operator $\mathfrak{T}(\mathfrak{g})$, are given by the formulas

$$\mathfrak{T}(\mathfrak{g}) d_t \mathfrak{T}^{-1}(\mathfrak{g}) = \left(d_t + \vec{V}\vec{D} + \lambda \vec{V}^2/2\right), \tag{19}$$

$$\mathfrak{T}(\mathfrak{g}) \vec{D} \mathfrak{T}^{-1}(\mathfrak{g}) = \Re(-\vec{c})\left(\vec{D} + \lambda \vec{V}\right). \tag{20}$$

Now obviously, that the operator $\vec{D}^2$ will be conversed as follows:

$$\mathfrak{T}(\mathfrak{g}) \vec{D}^2 \mathfrak{T}^{-1}(\mathfrak{g}) = \left(\vec{D}^2 + 2\lambda \vec{V}\vec{D} + \lambda^2 \vec{V}^2\right). \tag{21}$$

Consequently, in compliance with (19) and (21), the invariant operator of the Galilei group $\mathfrak{G}$ one can represent in the following form

$$K = d_t - \vec{D}^2/2\lambda - K_{rel}, \tag{22}$$

where $K_{rel} = F(\vec{r}, \nabla)$ is some $SO(3N-3)$-invariant operator dependent only on $\vec{r}$ and $\nabla \equiv \partial/\partial \vec{r}$.

As the Galilei group contains a subgroup of the time translatios, the unitarity of representation (13) with respect to the scalar product (14) may be provided only in the subspace $\mathfrak{S} \subset \mathfrak{L}$ of the functions $\Psi = \Psi(\vec{R}, \vec{r}, t)$, for which this scalar product is independent of time. It is easy to show, that under the condition $\lambda^* = -\lambda$ and of anti-Hermiticity of the operator $K_{rel}$ with respect to the scalar product (14) this requirement is fulfilled for the functions, satisfying the Galilean-invariant equation

$$K\Psi(\vec{R}, \vec{r}, t) = 0. \tag{23}$$

**4. Generators of unitary representation of transitivity group of space $\mathbb{R}^{3N-3}_{rel}$**

To specify the form of the operator $K_{rel}$, included in (22), we'll suppose that in the subspace $\mathfrak{S} \subset \mathfrak{L}$ the unitary with respect to scalar product (14) representation

$$\mathrm{T}(g)\Psi(\vec{R}, \vec{r}, t) = \mathrm{A}(\vec{r}, g)\Psi(\vec{R}, g^{-1}\vec{r}, t) \tag{24}$$

of some transitive group $G$ of $\mathbb{R}^{3N-3}_{rel}$ is realized too. Here $\mathrm{A}(\vec{r}, g)$ is an operator factor satisfying equation (2) under condition $\mathrm{A}(\vec{r}, g) = 1$, if $g \in SO(3N-3) \subset G$. The last condition conforms representation (24) with definition (11) according to which contraction of the representation $\mathfrak{T}(\{\tau, \vec{a}, \vec{V}, \vec{c}\})$ on the rotation subgroup is quasiregular. These requirements can be satisfied, if to take, in accordance with (15), $\mathrm{A}(\vec{r}, g) = f(\vec{r})/f(g^{-1}\vec{r})$, where $f(\vec{r})$ is some

$SO(3N-3)-$ invariant function (i.e $f(\vec{r})=f(r)$, $r=\sqrt{\sum_{\nu=1}^{3N-3} r_\nu^2}$ ) not vanishing in $\mathbb{R}_{rel}^{3N-3}$. The explicit form of this function will be set using the anti-Hermiticity requirement of the generators

$$\tau_J \Psi(\vec{R},\vec{r},t) = \partial\left(\mathrm{T}(g_J(\chi))\Psi(\vec{R},\vec{r},t)\right)/\partial\chi\Big|_{\chi=0} \quad (25)$$

of representation (24) with respect to the scalar product of (14). Here $g_J(\chi)$ is an element of one-parameter subgroup $G_J \subset G$ $\left(1 \le J \le (3N-2)(3N-3)/2\right)$, $\chi$ is the transformation parameter, and $g_J(0)$ is the unity element of the group $G$.

Taking into account that $\mathrm{A}(\vec{r},g) = f(\vec{r})/f(g^{-1}\vec{r})$, substitution of (24) into (25) gives

$$\tau_J = f(\vec{r})\tilde{\tau}_J f^{-1}(\vec{r}), \quad (26)$$

where

$$\tilde{\tau}_J = \left(\partial\left(g_J(-\chi)\vec{r}\right)/\partial\chi\right)\nabla\Big|_{\chi=0} \quad (27)$$

is the corresponding generator of a quasiregular representation of the group $G$.

Let's take for definiteness the values $J=\nu$ $(1 \le \nu \le 3N-3)$ for the generators of translations $\tau_\nu$ and the remaining $(3N-3)(3N-4)/2$ values of for the generators of rotations. As representation (24) on the rotation subgroup is quasiregular, these generators are of the well-known form $\tau_{\mu\nu} = \tilde{\tau}_{\mu\nu} = -\left(r_\mu \nabla_\nu - r_\nu \nabla_\mu\right)$, where $1 \le \mu,\nu \le 3N-3$, $\nabla_\mu = \partial/\partial r_\mu$, with the permutable relations

$$\left[\tau_{\mu\nu}, \tau_{\kappa\lambda}\right] = \delta_{\mu\lambda}\tau_{\kappa\nu} + \delta_{\nu\lambda}\tau_{\mu\kappa} - \delta_{\mu\kappa}\tau_{\lambda\nu} - \delta_{\nu\kappa}\tau_{\mu\lambda}.$$

Then the values $J = 3N-2, 3N-1, \ldots, (3N-3)(3N-4)/2$ in (25) can, for example, renumber the operators $\tau_{\mu\nu}$ with $\mu < \nu$.

The commutators of operators $\tau_{\mu\nu}$ with translation generators $\tau_\kappa$ should have the following form

$$\left[\tau_\kappa, \tau_{\mu\nu}\right] = \delta_{\kappa\nu}\tau_\mu - \delta_{\kappa\mu}\tau_\nu. \quad (28)$$

But since the explicit form of the one-parameter transformations of translations of the space $\mathbb{R}_{rel}^{3N-3}$ is not given, the corresponding generators of representation (24) may be constructed as follows.

Taking into account that according to (27) all quasiregular representation generators of the group $G$ are homogeneous linear forms of the operators $\nabla_\kappa$, the most general form of the corresponding generators of translations, which commutators with generators of rotations $\tau_{\mu\nu} = \tilde{\tau}_{\mu\nu}$ look like (28), may be written as

$$\tilde{\tau}_{\kappa} = -\left(a(r)\delta_{\kappa\mu} + b(r) r_{\kappa} r_{\mu}\right)\nabla_{\mu}, \tag{29}$$

where $a(r)$ and $b(r)$ are some real functions of the radial variable $r$. Here and further the repeating index is used in accordance with the Einstein rule.

Using (29) and (26), and also the rotational invariance of the function $f(r)$, one can easily show that the commutators of the generators of translations $\tau_{\nu}$ and $\tau_{\kappa}$ are given by the formula

$$\left[\tau_{\nu}, \tau_{\kappa}\right] = \left(ab - \left(a'/r\right)\left(a + r^2 b\right)\right)\tau_{\nu\kappa}. \tag{30}$$

Here and further in this section of paper the stroke above a letter denotes a derivative with respect to the variable $r$.

As on construction $\tau_{\kappa}$ and $\tau_{\mu\nu}$ are the representation generators of the Lie algebra of the transitivity group of $\mathbb{R}^{3N-3}_{rel}$, the following equation must be fulfilled:

$$ab - \left(a'/r\right)\left(a + r^2 b\right) = C, \tag{31}$$

where, considering (27) and (29), $C$ is a real dimensionality constant of the inverse square of length. Functions $a(r)$ and $b(r)$ satisfying this equation lead to the Lie algebras of groups isomorphic to $E(3N-3)$, $SO(3N-2)$ or $SO(3N-3,1)$ at $C = 0$, $C > 0$, and $C < 0$ respectively. The connection of this constant with physical quantities having the dimensionality of length we'll discuss later[1].

Let's introduce now, supposing that everywhere in $\mathbb{R}^{3N-3}_{rel}$ the sum $a(r) + r^2 b(r)$ does not vanish, a new variable $\sigma = \sigma(r)$ with dimensionality of length, the derivative of which with respect to the variable $r$ is

$$\sigma' = \left(a + r^2 b\right)^{-1}. \tag{32}$$

Then (31) may be represented as an equation for the function $y = a\big/\left(r\sqrt{C}\right)$:

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[1] The approach, applied here to realization of space $\mathbb{R}^{3N-3}_{rel}$ as the homogeneous space of group does not demand interpretation of constant $C$ in a spirit of non-Euclidean geometries as curvature of space or as metric constant of geometry [15].

$$\frac{dy}{d\sigma} = -\sqrt{C}\left(1+y^2\right).$$

As according to (32) variable $\sigma$ is specified to within an additive constant the result of integration of this equation without generality loss may be written in the form of $\sigma = \left(1/\sqrt{C}\right)\cot^{-1} y$, and then

$$a(r) = \sqrt{C} r \cot\sqrt{C}\sigma(r). \tag{33}$$

The substitution of (29) into (26) after simple transformations gives

$$\tau_\kappa = -\left(\left[a, \nabla_\kappa\right]_+ + \left[b r_\kappa r_\mu, \nabla_\mu\right]_+\right)\Big/2 + \left(\left(f'/rf\right)\left(a + r^2 b\right) + a'/2r + rb'/2 + \left(3N-2\right)b/2\right) r_\kappa, \tag{34}$$

where $\left[\ ,\ \right]_+$ is a sign of the anticommutator. But as representation (24) is supposed to be unitary with respect to the scalar product (14), the operators $\tau_\kappa$ specified by expression (34) must be anti-Hermitian with respect to this product. Then assumption, that the function $f(r)$ is real, leads to the equation

$$\left(f'/rf\right)\left(a + r^2 b\right) + a'/2r + rb'/2 + \left(3N-2\right)b/2 = 0, \tag{35}$$

and generators of translations become

$$\tau_\kappa = -\left(\left[a, \nabla_\kappa\right]_+ + \left[b r_\kappa r_\mu, \nabla_\mu\right]_+\right)\Big/2. \tag{36}$$

Using now (32) and (33), and considering that according to (11) function $f(r)$ is defined to within a multiplicative constant, one can easily show that a solution of equation (35) is the function

$$f(r) = \left(\sin\sqrt{C}\sigma(r)\Big/\sqrt{C} r\right)^{\frac{3N-4}{2}} \sqrt{\sigma'(r)}. \tag{37}$$

As the function $f(r)$ is real, the choice of some monotonically increasing function $\sigma(r)$ with dimensionality of length enables us to find with help of (33) and (32) the explicit form of generators (36), giving together with generators of rotations $\tau_{\mu\nu} = -\left(r_\mu \nabla_\nu - r_\nu \nabla_\mu\right)$ specific realization of the Lie algebra of the unitary representation of the transitivity group $G$ of the relative space of a system in Cartesian coordinates[1].

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[1] If to endow the space $\mathbb{R}^{3N-3}_{rel}$ with $G-$ invariant metric, one can treat the function $\sigma(\vec{r})$ as a length of radius-vector $\vec{r} \in \mathbb{R}^{3N-3}_{rel}$ in the sense corresponding to this metric [16]. But physically such a treatment is absolutely useless because for introduction in the space $\mathbb{R}^{3N-3}_{rel}$ of $G-$ invariant metric there are no reasonable basis.

The simplest realization is defined by choosing $\sigma(r)=r$ for any value of $\operatorname{sgn} C$ , and this, according to definition (32), leads to the equality

$$a(r)+r^2b(r)=1. \tag{38}$$

Function $\sigma(\vec{r})=r$ is unique function with required properties, which contains no constant $C$ and in fundamental case $N=2$ has for any $C$ the value equal to Euclidean distance between particles. Therefor further we'll use namely this realization to formulate quantum dynamics of a system (though it is possible to develop the description in which arbitrariness in a choice of function $\sigma(r)$ is persisted till the very end [17]).

## 5. Schrödinger equation and dynamic variables

Let's note that at any conceptual level (classical or quantum) the dynamic variables describing nonrelativistic motion of a particle are operationally determined by the measuring procedures carried out with the help of the macroscopic devices, to which it is possible to apply transformations (3)[1]. Therefore the correlations between average values of the data obtained by means of measuring devices bound by such transformations should be of the same form both for classical dynamic variables and for their quantum counterparts. It is this circumstance that must be put in the basis of the definition of dynamic variables in quantum mechanics.

As it is known, the invariant operator of group $G$ is

$$K_G=\tau_\nu\tau_\nu+C\tau_{\mu\nu}\tau_{\mu\nu}/2.$$

With the help of (33), (36) and (38) operator $K_G$ can be presented in the form

$$K_G=\nabla^2-\left(C/\sin^2\sqrt{C}r-1/r^2\right)\left(q(q+1)-\tau_{\mu\nu}\tau_{\mu\nu}/2\right)+C(q+1)^2, \tag{40}$$

where $q=(3N-6)/2$.

Let's define now operator $K_{rel}$ in the expression (22) as

$$K_{rel}=\left(K_G-C(q+1)^2\right)/2\eta+i\Phi(\vec{r}), \tag{41}$$

where $\eta$ is some purely imaginary parameter, and $|\eta|$ has the same dimensionality as $|\lambda|$ in (22), i.e. $[|\eta|]=[|\lambda|]=TL^{-2}$, and $\Phi(\vec{r})$ is some real function invariant at least with respect to 3-parameter rotations $\tilde{\Re}(\vec{c})\in SO(3N-3)$. Now, as Planck's constant has the dimensionality

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[1] I.e. the turns and shifts of measuring devices in space, a giving to them of translational motion with a constant velocity and shift of zero of time at the installation of a clock.

$\left[\hbar\right] = ML^2T^{-1}$, we assume $\lambda = i\,m/\hbar$ and $\eta = i\,\mu/\hbar$, where $m$ and $\mu$ are real parameters with the dimensionalities of mass. Then we can represent Galilean-invariant equation (23) as the Schrödinger equation

$$i\hbar\frac{\partial\Psi\left(\vec{R},\vec{r},t\right)}{\partial t} = \hat{H}\Psi\left(\vec{R},\vec{r},t\right), \qquad (42)$$

where according to (40) and (41)

$$\hat{H} = \frac{1}{2m}\hat{\vec{P}}^2 + \frac{1}{2\mu}\hat{\vec{p}}^2 + \frac{\hbar^2}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r} - \frac{1}{r^2}\right)\left(q\left(q+1\right) + \frac{\hat{L}^2}{\hbar^2}\right) + U\left(\vec{r}\right), \qquad (43)$$

where $\hat{\vec{P}} = i\hbar\vec{D} = -i\hbar\partial/\partial\vec{R}$, $\hat{\vec{p}} = -i\hbar\nabla$, $\hat{L}^2 = \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}_{\mu\nu}$, $\hat{L}_{\mu\nu} = i\hbar\tau_{\mu\nu}$ and $U\left(\vec{r}\right) = \hbar\Phi\left(\vec{r}\right)$. Here and further the symbol « ^ » above a letter is used to denote Hermitian operators (with respect to the scalar product (14)). In particular, the operators $\hat{\vec{R}}$ and $\hat{\vec{r}}$ are specified in the usual way, as multiplication of the function $\Psi\left(\vec{R},\vec{r},t\right)$ by the variables $\vec{R}$ and $\vec{r}$. Similarly, any functions of these operators are defined. Therefor everywhere further the sign « ^ » above the variables $\vec{R}$ and $\vec{r}$ is omitted.

From (20) and (21) follows, that for operators $\hat{\vec{P}}$ and $\hat{H}$ the following homothetic transformations are fulfilled:

$$\hat{\vec{P}}' = \mathfrak{T}\left(\mathfrak{g}\right)\hat{\vec{P}}\mathfrak{T}^{-1}\left(\mathfrak{g}\right) = \Re\left(-\vec{c}\right)\left(\hat{\vec{P}} - m\vec{V}\right), \qquad (44)$$

$$\hat{H}' = \mathfrak{T}\left(\mathfrak{g}\right)\hat{H}\mathfrak{T}^{-1}\left(\mathfrak{g}\right) = \hat{H} - \vec{V}\hat{\vec{P}} + mV^2/2. \qquad (45)$$

We also note that, in accordance with (8) $\hat{\vec{P}} = \sum_{k=1}^{N}\hat{\vec{p}}_k$, where $\hat{\vec{p}}_k = -i\hbar\partial/\partial\vec{x}_k$.

Further one can show, using (8), (13) and (18), that for operators $\hat{\vec{x}}_k$ and $\hat{\vec{p}}_k$ such transformations give

$$\hat{\vec{x}}'_k = \mathfrak{T}\left(\mathfrak{g}\right)\hat{\vec{x}}_k\mathfrak{T}^{-1}\left(\mathfrak{g}\right) = \Re\left(-\vec{c}\right)\left(\hat{\vec{x}}_k - \vec{V}\left(t-\tau\right) - \vec{a}\right), \qquad (46)$$

$$\hat{\vec{p}}'_k = \mathfrak{T}\left(\mathfrak{g}\right)\hat{\vec{p}}_k\mathfrak{T}^{-1}\left(\mathfrak{g}\right) = \Re\left(-\vec{c}\right)\left(\hat{\vec{p}}_k - \alpha_k m\vec{V}\right). \qquad (47)$$

From (44) – (47) obviously follows, that under condition $\left(\Psi,\Psi\right) = 1$ the quantities $\left\langle q\right\rangle = \left(\Psi,\hat{q}\Psi\right)$ and $\left\langle q'\right\rangle = \left(\Psi,\hat{q}'\Psi\right) = \left(\Psi,\mathfrak{T}\left(\mathfrak{g}\right)\hat{q}\mathfrak{T}^{-1}\left(\mathfrak{g}\right)\Psi\right)$ for each of the introduced operators are related by the equations

$$\left\langle\vec{x}'_k\right\rangle = \Re\left(-\vec{c}\right)\left(\left\langle\vec{x}_k\right\rangle - \vec{V}\left(t-\tau\right) - \vec{a}\right), \qquad (48)$$

$$\langle \vec{p}'_k \rangle = \Re(-\vec{c})\left(\langle \vec{p}_k \rangle - \alpha_k m\vec{V}\right), \tag{49}$$

$$\langle \vec{P}' \rangle = \Re(-\vec{c})\left(\langle \vec{P} \rangle - m\vec{V}\right), \tag{50}$$

$$\langle H' \rangle = \langle H \rangle - \vec{V}\langle \vec{P} \rangle + mV^2/2. \tag{51}$$

Then from (48) – (51) follows, that the quantities $\langle \vec{x}_k \rangle$, $\langle \vec{p}_k \rangle, \langle \vec{P} \rangle, \langle H \rangle$ it is necessary to interpret respectively as average values of the results of measurements of dynamic variables of position and momentum of a particle with the mass $m_k = \alpha_k m$, total momentum and mechanical energy of a material system with the mass $m$ in the quantum state $\Psi(\vec{R},\vec{r},t)$, having put to them in correspondence Hermitian operators $\hat{\vec{x}}_k, \hat{\vec{p}}_k, \hat{\vec{P}}, \hat{H}$. Then operators $\hat{\vec{x}}_k, \hat{\vec{p}}_k, \hat{\vec{P}}, \hat{H}$ and $\hat{\vec{x}}'_k, \hat{\vec{p}}'_k, \hat{\vec{P}}', \hat{H}'$ will represent the measuring procedures carried out by means of the devices bound by transformations (3).

The physical meaning of the tensor operator $\hat{L}_{\mu\nu} = i\hbar\tau_{\mu\nu}$ given by the rotation generators depends on the number of particles of a system. In paper [13], where 3-particle system are discussed, this tensor termed the *grand angular momentum tensor*. For 2-particles system the non-diagjnal elements of this tensor form the vector of a proper angular momentum of a system.

Thus, all quantum mechanical dynamic variables and equation of motion of a system can be specified on the basis of the transformation laws (19 – 21) and (44 – 47) caused by the Galilei group representation (13), where the parameter $\lambda$ entering into the operator factor (16) is determined by the mass of a system: $\lambda = i\,m/\hbar$.

If now we choose in (43) the mass parameter $\mu$ in the form $\mu = m\left(\prod_{k=1}^{N}\alpha_k\right)^{1/(N-1)}$ then

$$\hat{\vec{P}}^2/(2m) + \hat{\vec{p}}^2/(2\mu) = \sum_{k=1}^{N}\hat{\vec{p}}_k^2/(2\alpha_k m),$$

and at $C \to 0$ operator (43) will coincide with Hamiltonian for a system of $N$ particles with masses $m_k = \alpha_k m$ and interaction potential $U(\vec{r})$. Thus we can consider equation (42) as the Schrödinger equation for a $N-$particle system with interaction presented in operator (43) by the item

$$W = \frac{\hbar^2}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r} - \frac{1}{r^2}\right)\left(q(q+1) + \frac{\hat{L}^2}{\hbar^2}\right) + U(r). \tag{52}$$

Further we'll retain for function $U(\vec{r})$ the term *potential* at any values of the constant $C$. A sign of the constant $C$, reflecting the specificity of non-Abelian translations of the relative space of a system, is the primary characteristic of the interaction described by function (52).

Really, in the case of $C < 0$ the space $\mathbb{R}^{3N-3}_{rel}$ coincides with all infinite space $\mathbb{R}^{3N-3}$, and as $\hat{L}^2$ does not depend from radial variable $r$ the first item in (52), defined for all values of $r$, vanishes at $r \to \infty$. In this case both finite motion of particles of system and their infinite motion, free on large relative distances between them, are possible. Then coefficients $\alpha_k$ in (8) one can define as $\alpha_k = m_k / m$, where $m_k$ is the mass of the $k-$th free particle, $m = \sum_{k=1}^{N} m_k$, $\mu = \left( m^{-1} \prod_{k=1}^{N} m_k \right)^{1/(N-1)}$, and $\vec{R}$ coincides with the radius-vector of the center of mass of a system in the classical mechanics (in this case variables $\{\vec{R}, \vec{r}_j\}, 1 \le j \le N-1,$ coincide with the rationalized Jacobi coordinates [13]). This sense can be retained for $\vec{R}$ and in the quantum mechanics.

At $C > 0$ the first item in (52) is defined in $\mathbb{R}^{3N-3}$ almost everywhere, excepting hypespheres with the radiuses $r_n = n\rho,\ n = 1,2,3,\ldots,\ \rho = \pi/\sqrt{C}$. This makes it possible to restrict a relative motion of the particles by any transitivity domain of group $G$ (isomorphic to $SO(3N-2)$ ) defined by the inequalities $0 \le r < \rho$ or $n\rho < r < (n+1)\rho,\ n = 1,2,3\ldots$ . In particular, if to choose as the relative space $\mathbb{R}^{3N-3}_{rel}$ the open ball $0 \le r < \rho$, a solution of equation (42) should be finite at $r = 0$ and vanish for all values of $r \ge \rho$. It means that in this case it is possible only finite motion of the particles of a system in a neighbourhood of point $\vec{R}$ of physical space on the relative distances $|\vec{x}_k - \vec{x}_j|$ satisfying in accordance with (9) the condition

$$\sum_{1 \le k < j \le N} \alpha_k \alpha_j |\vec{x}_k - \vec{x}_j|^2 \le \left( \prod_{i=1}^{N} \alpha_i \right) \rho^2,$$

i.e. their *confinement* in the limited region of physical space in a neighbourhood of it point $\vec{R}$. In this case the the considered system cannot be divided on the free components so that their masses are measurable separately, and therefor we can only define these masses by formula $m_k = \alpha_k m,$ where $\alpha_k$ are the coefficients of the transformation (8) and $m$ is the *observable* mass of a system. Bearing in mind that the first item in (52) vanishes at $r \to 0$ one can specify the mass parameter $\mu$ by the formula $\mu = m\left(\prod_{k=1}^{N}\alpha_k\right)^{1/(N-1)}$ only under condition $U(\vec{r}) \to 0$ at $r \to 0$, i.e. if the interpartial interaction allows asymptotic freedom of particles on a small relative distances.

The stationary states of a system in the centre-of-mass reference frame at any value of the constant $C$ are given by solutions of equation (42) of the form $\Psi(\vec{R},\vec{r},t)\big|_{\vec{P}=\vec{0}} = \exp(-iEt/\hbar)\psi(\vec{r})$, where $E$ is the system's energy and the function $\psi(\vec{r})$ satisfies the equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{\hbar^2}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r} - \frac{1}{r^2}\right)\left(q(q+1) + \frac{L^2}{\hbar^2}\right) + U(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r}). \tag{53}$$

## 6. Concluding remark

Since in our approach to the construction of quantum dynamics its classical counterpart is not used, the definition of the explicit form of the function $U(\vec{r})$ even in the case $C = 0$ can be based only on a dynamic symmetry of the equation (53). It is clear that fundamental to the establishment of the explicit form of the potential $U(\vec{r})$ is the 2-particle interaction. This issue will be discussed in detail in our next publication.

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