# *Exactly solvable problems for a 2-particle system with a homogeneous relative space*

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**Abstract:** The Schrödinger equation that admits the freedom to choose the transitivity group of the relative space of a 2-particle system is considered. With the help of dynamic symmetry considerations, potentials that provide exact solutions of this equation are found. The explicit form of these solutions for both continuous and discrete energy spectra is obtained and the physical aspects of these solutions are discussed.

## 1. Introduction

In the previous article [1], the dynamic equation for a closed $N-$particle $\left(N\geq 2\right)$ system whose relative space can be a homogeneous space of any of the three groups isomorphic to $E\left(3N-3\right)$, $SO\left(3N-2\right)$ or $SO\left(3N-3,1\right)$ was formulated. In the case of the two particles this equation takes the form

$$i\hbar\frac{\partial\Psi\left(\vec{R},\vec{r},t\right)}{\partial t}=\left(\frac{\hat{\vec{P}}^2}{2M}+\hat{H}_{rel}\right)\Psi\left(\vec{R},\vec{r},t\right). \tag{1}$$

Here $\hat{\vec{P}}=-i\hbar\partial/\partial\vec{R}$, $\vec{R}=\alpha_1\vec{x}_1+\alpha_2\vec{x}_2$, $\alpha_1+\alpha_2=1$, $\alpha_i>0$, $\vec{x}_i$ is a position vector of $i-$th particle $\left(i=1,2\right)$ in the physical space $\mathbb{R}^3_E$ (a space $\mathbb{R}^3$ with Euclidean metric),

$$\hat{H}_{rel}=\frac{\hat{\vec{p}}^2}{2\mu}+\frac{1}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r}-\frac{1}{r^2}\right)\left(\hat{\vec{r}}\times\hat{\vec{p}}\right)^2+U\left(\vec{r}\right), \tag{2}$$

where $\hat{\vec{p}}=-i\hbar\partial/\partial\vec{r}\equiv-i\hbar\nabla$, $\vec{r}=\vec{x}_1-\vec{x}_2$, $C$ is is the real constant defining the transitivity group of the relative space $\mathbb{R}^3_{rel}$ of a system (the set of points with radius-vectors $\vec{r}$ ), $m$ is the observable mass of a system, and $\mu=\alpha_1\alpha_2 m.$[1]

The stationary states of a system in the centre-of-mass reference frame at any value of the constant $C$ are given by solutions of equation (1) of the form $\Psi\left(\vec{R},\vec{r},t\right)\Big|_{\vec{P}=\vec{0}}=\exp\left(-iEt/\hbar\right)\psi\left(\vec{r}\right)$, where $E$ is the system's energy and the function $\psi\left(\vec{r}\right)$ satisfies the equation

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2+\frac{1}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r}-\frac{1}{r^2}\right)\hat{\vec{l}}^{\,2}+U\left(r\right)\right)\psi\left(\vec{r}\right)=E\psi\left(\vec{r}\right), \tag{3}$$

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[1] With this choice of the mass parameter $\mu$ , the masses of the system components will be equal to $\alpha_1 m$ and $\alpha_2 m$ .

where $\hat{\vec{l}} = \hat{\vec{r}} \times \hat{\vec{p}}$ is a proper angular momentum of the system.

The boundary conditions to which solutions of the equation (3) must satisfy depend substantially on the value of the constant $C$ entering into the term describing the interaction of the particles

$$W = \frac{1}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r} - \frac{1}{r^2}\right)\hat{\vec{l}}^2 + U(r). \tag{4}$$

Really, in the case of $C < 0$ the space $\mathbb{R}^3_{rel}$, which is a homogeneous space of the group isomorphic to $SO(3,1)$, is realized in all infinite space $\mathbb{R}^3$, and as $\hat{\vec{l}}^2$ does not depend from radial variable $r$ the first item in (4), defined for all values of $r$, vanishes at $r \to 0$ and $r \to \infty$. In this case both finite motion of particles of system and their infinite motion, free on large relative distances between them, are possible. We will assume, as well as in case of $C = 0$, the last is possible if for the given function $U(r)$ at $C < 0$ exists a solution of equation (3), regular at the origin of coordinates and with an asymptotic behavior at the relative distances $r >> \rho = 1/\sqrt{|C|}$ of the following form:

$$\psi_{\vec{k}}(\vec{r}) \simeq \frac{i}{2kr}\sum_{lm}(2l+1)\left((-1)^l e^{-ikr} - S_l(k)e^{ikr}\right)P_l\left(\frac{\vec{k}\vec{k}'}{k^2}\right), \tag{5}$$

where $k = \sqrt{2\mu E}/\hbar$, $\vec{k}' = k\vec{r}/r$ and $\hbar\vec{k}$ – relative momentum of the particles.

Then analytical properties of the coefficients $S_l(k)$ in a complex plane of the variable $k$ will give information both about scattering states and bound states of a system.

At $C > 0$ the first item in (4) is defined in $\mathbb{R}^3$ almost everywhere, excepting spheres with the radiuses $r_n = n\rho$, $n = 1,2,3,\ldots$, $\rho = \pi/\sqrt{C}$. This allows us to restrict the relative space of a system $\mathbb{R}^3_{rel}$ to any of the regions of the space $\mathbb{R}^3$ defined by the inequalities $0 \le r < \rho$ or $n\rho < r < (n+1)\rho$, $n = 1,2,3\ldots$, which for $C > 0$ are the transitivity regions of the group isomorphic to $SO(4)$. In this case the physically significant solutions of equation (3) must satisfy the boundary conditions $\psi(\vec{r})\big|_{r=r_n} = 0$ leading to the vanishing on the spheres $r = r_n$ of radial component of the density vector of the flow of probability $\vec{j} = \left(\left(Cr^2/\sin^2\sqrt{C}r\right)\left(1 - P_{\vec{r}}\right) + P_{\vec{r}}\right)\vec{j}_0$, where $P_{\vec{r}} = \vec{r}\cdot\vec{r}/r^2$ is the projector on the radial direction ( $\vec{r}\cdot\vec{r}$ – the dyad), and $\vec{j}_0 = (\hbar/2\mu i)\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$. In particular, if to choose as the relative space $\mathbb{R}^3_{rel}$ the domain

$0 \le r < \rho$, a solution of equation (3) should be finite at $r = 0$ and vanish for all values of $r \ge \rho$:

$$\psi(\vec{r})\Big|_{r=0} < \infty, \quad \psi(\vec{r})\Big|_{r\ge\rho} = 0. \tag{6}$$

It means that in this case only finite motion of the particles of a system in a neighbourhood of point $\vec{R}$ on the relative distances $r = |\vec{x}_1 - \vec{x}_2|$ not exceeding $\rho$ is possible, i.e. their confinement in the region of physical space $|\vec{x} - \vec{R}| < \max(\alpha_i \rho),\ i = 1,2.$

Thus, in the case $C < 0$ the constant $C$ defines the interaction radius of particles $\rho = 1/\sqrt{|C|}$, and in the case $C > 0$ the constant $C$ defines the upper bound of possible relative distances between particles $\rho = \pi/\sqrt{C}$ in conventional Euclidean sense.

The solution of the equation (3) by well known procedure is reduced to the solution of the radial equation

$$\frac{d^2 f_l(k,r)}{dr^2} - \left( \frac{Cl(l+1)}{\sin^2 \sqrt{C}r} + \frac{2\mu}{\hbar^2} U(r) - k^2 \right) f_l(k,r) = 0, \tag{7}$$

where $k^2 = 2\mu E/\hbar^2$.

Thus, the concrete choice of the transitivity group of the relative space of a system leads in the radial Schrödinger equation only to modification of centrifugal potential.

**2. Dynamic symmetry of operator $\hat{H}_{rel}$ and selection of function $U(r)$**

As the requirement of Galilean invariance restricts arbitrariness in selecting a function $U(r)$ only by condition of its rotational invariance, the further concrete definition of the form of this function may be based only on a dynamic symmetry of the operator $\hat{H}_{rel}$ defined by formula (2).

If $U(r) = 0$, then operator (2) commutes with Hermitian quadratic combinations $[\tau_{kj}, \tau_j]_+/2$ and $[\tau_k, \tau_l]_+/2$ of the representation generators of the the transitivity group of the relative space of a system $\mathbb{R}^3_{rel}$ [1]

$$\tau_k = -\left([a, \nabla_k]_+ + [br_k r_j, \nabla_j]_+\right)/2, \tag{8}$$

where $[\ ,\ ]_+$ is a sign of the anticommutator, $a(r) = \sqrt{C}r \cot \sqrt{C}r$, $a(r) + r^2 b(r) = 1$,

$$\tau_{ij} = -(r_i \nabla_j - r_j \nabla_i). \tag{9}$$

Latin indices here and further in this section take the values 1, 2, 3.

Now, going to the operator $\hat{H}$ with $U(r) \neq 0$, and meaning the condition of dynamic symmetry, we'll demand its commutation either with operators $A_k = -[\tau_{kj}, \tau_j]_+/2 + \varphi(r) r_k$ or with operators $A_{kl} = -[\tau_k, \tau_l]_+/2 + \chi(r) r_k r_l$, where $\varphi(r)$ and $\chi(r)$ are some real $SO(3)-$ invariant functions.

The requirement $[A_k, \hat{H}] = 0$ leads to a set of equations for the functions $\varphi(r)$ and $U(r)$

$$\left.\begin{aligned} &\frac{dU}{dr} + \frac{\hbar^2}{\mu}\frac{Cr}{\sin^2\sqrt{C}r}\varphi = 0 \\ &\frac{dU}{dr} - \frac{\hbar^2}{\mu}\left(\frac{d\varphi}{dr} - \left(\frac{Cr}{\sin^2\sqrt{C}r} - \frac{1}{r}\right)\varphi\right) = 0 \end{aligned}\right\}.$$

Their integration gives $\varphi(r) = \alpha/r$ and

$$U(r) = \left(\alpha\hbar^2\sqrt{C}/\mu\right)\cot\sqrt{C}r + \beta, \tag{10}$$

where $\alpha$ and $\beta$ are some real constants with the dimensionalities of inverse length and energy, respectively. It is seen that at $C \to 0$ the function $U(r)$ coincides with the Coulomb potential, and in the case $C < 0$ for $\beta = -\alpha\hbar^2\sqrt{|C|}/\mu = -\alpha\hbar^2/\mu\rho$, where $\rho = \sqrt{1/|C|}$, takes the form

$$U(r) = \frac{2\alpha\hbar^2}{\mu\rho}\frac{e^{-2r/\rho}}{1 - e^{-2r/\rho}}. \tag{11}$$

In the case of the operators $A_{kl}$ the same requirement leads to a set of equations for $\chi(r)$ and $U(r)$

$$\left.\begin{aligned} &\frac{dU}{dr} - \frac{\hbar^2}{\mu}r\left(\chi + \frac{r}{2}\frac{d\chi}{dr}\right) = 0 \\ &\frac{dU}{dr} - \frac{\hbar^2\sqrt{C}}{\mu}\frac{r^2\chi}{\sin\sqrt{C}r\cos\sqrt{C}r} = 0 \end{aligned}\right\},$$

the integration of which gives $\chi(r) = \gamma\tan^2\sqrt{C}r/Cr^2$ and

$$U(r) = \left(\gamma\hbar^2/2\mu C\right)\tan^2\sqrt{C}r + \delta, \tag{12}$$

where $\gamma$ and $\delta$ are some arbitrary real constants with the dimensionalities of inverse length to the fourth power and energy, respectively. If $C \to 0$, function (12) apparently coincides with a potential of the isotropic oscillator. Choosing $\delta = \gamma\hbar^2/2\mu C$, we get

$$U(r) = \frac{\gamma\hbar^2}{2\mu C}\frac{1}{\cos^2\sqrt{C}r}. \tag{13}$$

From (49) obviously follows that for $s$-states $\psi$-functions are the solutions of the conventional Schrödinger equation with a potential $U(\vec{r})$ for any significance of constant $C$. In particular, the function (11) coincides with the Hulthen potential, and function (13) – with one of two types of the Pöschl-Teller potential (depending on a sign of the constant $C$). Therefore further we'll use the same names for potentials (11) and (13) too. At the same time, at $C > 0$ and $\beta = 0$ the function (10), first used in [2] for the description of a hydrogen atom in space of a constant positive curvature, may be termed as the Schrödinger potential[1].

It is possible to show that the algebraic properties of sets of the operators $\{A_k, \tau_{mn}\}$ and $\{A_{kl}, \tau_{mn}\}$ are expressed by the following commutation relations:

$$\left.\begin{aligned} &[A_k, A_l] = \left(\left(2\mu/\hbar^2\right)\left(\hat{H}_{rel} - \beta\right) + C\tau_{mn}\tau_{mn}\right)\tau_{kl}, \\ &[A_k, \tau_{lm}] = \left(\delta_{km}A_l - \delta_{kl}A_m\right). \end{aligned}\right\}, \tag{14}$$

$$\left.\begin{aligned} &[A_{kl}, A_{mn}] = -(C/2)\left([A_{km}, \tau_{ln}]_+ + [A_{ln}, \tau_{km}]_+ + [A_{kn}, \tau_{lm}]_+ + [A_{lm}, \tau_{kn}]_+\right) + \\ &\qquad + \left(C^2/4 - \gamma\right)\left(\delta_{km}\tau_{ln} + \delta_{ln}\tau_{km} + \delta_{kn}\tau_{lm} + \delta_{lm}\tau_{kn}\right), \\ &[A_{kl}, \tau_{mn}] = \delta_{kn}A_{ml} + \delta_{ln}A_{km} - \delta_{km}A_{nl} - \delta_{lm}A_{kn}. \end{aligned}\right\} \tag{15}$$

Certainly, in both cases

$$[\tau_{kl}, \tau_{mn}] = \delta_{kn}\tau_{ml} + \delta_{ln}\tau_{km} - \delta_{km}\tau_{nl} - \delta_{lm}\tau_{kn}. \tag{16}$$

As well as in geometrical interpretation of homogeneous spaces of groups isomorphic to $SO(3,1)$, $E(3)$ and $SO(4)$, when three values of $\operatorname{sgn} C$ specify various types of geometries (of Lobachevsky, Euclid and Riemann), relations (14) – (16) may be considered as algebraic structures determining different types of a dynamic symmetry of equation (3) in accordance with the values of $\operatorname{sgn} C$[2].

**3. Scattering states and bound states in the case $C < 0$**

Now we'll discuss solutions of the equation (3) with potentials (11) and (13), satisfying at the relative distances $r >> \rho = 1/\sqrt{|C|}$ the asymptotic requirement (5) under condition that axis $Z$ is chosen in direction of vector $\vec{k}$.

[1] In our approach to construction of quantum dynamics there are no sense to treat in this case the function (10) as the Coulomb potential in space of a constant positive curvature.

[2] These relations one can use for determination of energy levels and correspondent set of eigenfunctions in discussed problems ( look in this connecion [3-6]).

For this purpose we use the Jost functions method giving the simplest way to find the partial scattering amplitudes

$$S_l(k) = (-1)^l f_l(k) / f_l(-k), \tag{17}$$

where the Jost functions $f_l(k)$ are defined by the formula

$$f_l(k) = \lim_{r\to 0}(2l+1) r^l f_l(k,r), \tag{18}$$

and functions $f_l(k,r)$ are solutions of the radial equation (7) with $C = -1/\rho^2$

$$\frac{d^2 f_l(k,r)}{dr^2} - \left( \frac{l(l+1)}{\rho^2 \sinh^2(r/\rho)} + \frac{2\mu}{\hbar^2} U(r) - k^2 \right) f_l(k,r) = 0 \tag{19}$$

satisfying the asymptotic condition

$$\lim_{r\to\infty} e^{ikr} f_l(k,r) = 1. \tag{20}$$

*7.1. Hulthen potential*

Let's consider equation (19) with potential (11). Introducing a new variable $z = e^{-2r/\rho}$ and substituting $f_l(k,r) = \sinh^\nu(r/\rho) F_\nu(z)$ into (19), we get for $F_\nu(z)$ the equation related to the hypergeometric equation [7]:

$$z^2(z-1)F_\nu'' + \left(\nu - 1 + (\nu+1)z\right) z F_\nu' + \left[ \left( \frac{\rho^2 k^2 + \nu^2}{4} + \tilde{\alpha} \right) z - \frac{\rho^2 k^2 + \nu^2}{4} \right] F_\nu = 0, \tag{21}$$

where $\tilde{\alpha} = \alpha\rho$ and the parameter $\nu$ takes values $l+1$ or $-l$. Here and further the strokes above a letter denote derivatives with respect to the variable $z$.

In our case the solution of the equation (21) it is convenient to take in the form $F_\nu(z) = c_\nu z^{a_2} {}_2F_1(a_1 + a_2, b_1 + a_2; a_2 - b_2 + 1; z)$ (look [7]), where $c_\nu$ is some constant, and

$$a_1 = \frac{\nu + i\rho\sqrt{k^2 + 4\tilde{\alpha}/\rho^2}}{2},\quad b_1 = \frac{\nu - i\rho\sqrt{k^2 + 4\tilde{\alpha}/\rho^2}}{2},\quad a_2 = \frac{\nu + i\rho k}{2},\quad b_2 = \frac{\nu - i\rho k}{2}.$$

The function $F_\nu(z)$ hasn't singularity at the point $r = 0$ for $\nu = -l$. Therefore, supposing $\nu = -l$ and choosing $c_\nu$ such that the asymptotic condition (20) is satisfied, we come to a solution of the radial equation (19) with the explicitly selected singularity at the point $r = 0$:

$$f_l(k,r) = \left( \frac{1 - e^{-2r/\rho}}{2} \right)^{-l} e^{-ikr} {}_2F_1\left( -l + \frac{i\rho}{2}(k - \varkappa), -l + \frac{i\rho}{2}(k + \varkappa); 1 + i\rho k; e^{-2r/\rho} \right), \tag{22}$$

where

$$\varkappa = \sqrt{k^2 + 4\tilde{\alpha}/\rho^2}. \tag{23}$$

In this form the function $f_l(k,r)$ is most convenient for calculations of the Jost functions $f_l(k)$. Using (18) and (17), we find

$$f_l(k)=\frac{\rho^l(2l+1)}{2^l}\frac{\Gamma(1+i\rho k)}{\Gamma\left(1+l+\frac{i\rho}{2}(k-\varkappa)\right)\Gamma\left(1+l+\frac{i\rho}{2}(k+\varkappa)\right)}, \tag{24}$$

$$S_l(k)=(-1)^l\frac{\Gamma(1+i\rho k)\Gamma\left(1+l-\frac{i\rho}{2}(k-\varkappa)\right)\Gamma\left(1+l-\frac{i\rho}{2}(k+\varkappa)\right)}{\Gamma(1-i\rho k)\Gamma\left(1+l+\frac{i\rho}{2}(k-\varkappa)\right)\Gamma\left(1+l+\frac{i\rho}{2}(k+\varkappa)\right)}. \tag{25}$$

The bound states correspond to zeros of $S_l(k)$ on the imaginary negative semiaxis in a plane of the complex variable $k$, i. e. at the points satisfying the equation[1]

$$1+l+i\rho(k-\varkappa)/2=-n_r, \qquad n_r=0,1,2,... \tag{26}$$

Then, introducing $n=n_r+l+1$ and taking into account (23), from (26) we get $k=\sqrt{2mE}/\hbar=i\left(n^2+\tilde{\alpha}\right)/n\rho$. Therefore, the bound states are possible if $n^2+\tilde{\alpha}<0$, i.e. for $\tilde{\alpha}=\alpha\rho<0$ and $|\alpha|\rho>1$. Thus, the number of bound states is restricted, and their energy spectrum is determined by the formula

$$E_n=-\frac{\hbar^2}{2\mu}\left(\frac{n}{\rho}+\frac{\alpha}{n}\right)^2, \quad n=1,2,...,\left[\sqrt{|\alpha|\rho}\right], \tag{27}$$

where the symbol $[\, x \,]$ means the integer part of number $x$. It is easy to see that at $\rho\to\infty$ the number of bound states becomes infinite and the energy spectrum coincides with the Coulomb spectrum. Besides, from (25) follows that $\lim_{\rho\to\infty}S_l(k)=\Gamma(1+l+i(\alpha/k))/\Gamma(1+l-i(\alpha/k))$ in full conformity with the known expression for partial scattering amplitudes in a Coulomb field. However it is necessary to underline that within the limits of dynamics presented here the limiting transition $\rho\to\infty$ is absolutely formal and isn't dictated by any physical reasons. It has only a test meaning.

Using (22), (23), and (24) with $k=i\left(n^2+\tilde{\alpha}\right)/n\rho$, and also formulas of the transformations of hypergeometric functions the everywhere finite radial functions of the bound states may be represented as

[1] Poles of $\Gamma(1-i\rho k)$ give superfluous zeros because they are poles of the Jost functions $f_l(-k)$.

$$f_l(n,r)=c_{nl}\left(1-e^{-\frac{2r}{\rho}}\right)^{l+1}e^{r\left(\frac{n}{\rho}+\frac{\alpha}{n}\right)}{}_2F_1\left(1+l-\frac{\alpha\rho}{n},1+l-n;1-n-\frac{\alpha\rho}{n};e^{-\frac{2r}{\rho}}\right),$$

where $c_{nl}$ is a normalizing factor, the explicit form of which is inessential for given work.

*7.2. Pöschl-Teller potential (the first type)*

Now we'll consider the equation (19) with potential (13) at $C<0$. Introducing a new variable $z=\cosh^{-2}(r/\rho)$, where $\rho=\sqrt{1/|C|}$, and substituting $f_l(k,r)=\sinh^{\nu}(r/\rho)F_\nu(z)$ into (7), we again come to an equation related to the hypergeometric equation:

$$z^2(z-1)F_\nu''+\left(\nu-1+\frac{3}{2}z\right)zF_\nu'+\left[-\frac{\tilde{\gamma}^2}{4}z-\left(\frac{\rho^2k^2}{4}+\frac{\nu^2}{4}\right)\right]F_\nu=0\ , \qquad (28)$$

where $\tilde{\gamma}=\gamma\rho^4$, and the parameter $\nu$ takes values $l+1\,\text{or}\,-l$. Repeating the actions of the previous section as a solution of radial equation in this case we select the function

$$f_l(k,r)=\frac{\tanh^{-l}(r/\rho)}{\left(1+e^{-2r/\rho}\right)^{ik\rho}}e^{-ikr}{}_2F_1\left(\frac{ik\rho+\Omega_l^+}{2},\frac{ik\rho+\Omega_l^-}{2};1+ik\rho;\cosh^{-2}\frac{r}{\rho}\right), \qquad (29)$$

$$\Omega_l^\pm=-l+\left(1\pm\sqrt{1+4\tilde{\gamma}}\right)/2\ . \qquad (30)$$

Then, using (29), (18), and (17) for partial scattering amplitudes, we get

$$S_l(k)=(-1)^l\frac{\Gamma(1+ik\rho)\Gamma\left(\frac{1}{2}\left(2-ik\rho-\Omega_l^+\right)\right)\Gamma\left(\frac{1}{2}\left(2-ik\rho-\Omega_l^-\right)\right)}{\Gamma(1-ik\rho)\Gamma\left(\frac{1}{2}\left(2+ik\rho-\Omega_l^+\right)\right)\Gamma\left(\frac{1}{2}\left(2+ik\rho-\Omega_l^-\right)\right)}. \qquad (31)$$

The existence of bound states, as well as in the considered above case, is dependent on the value of the interaction constant $\gamma=\tilde{\gamma}/\rho^4$. From expression (31) it follows that zeros of the partial amplitude $S_l(k)$ are at the points

$$k=-\frac{i}{\rho}\left(\frac{1}{2}\sqrt{1+4\tilde{\gamma}}-\left(2n_r+l+\frac{3}{2}\right)\right),\quad n_r=0,1,2,... \qquad (32)$$

As follows from (32), if $\tilde{\gamma}<0$ the bound states are impossible, and under the condition $\tilde{\gamma}>2$, there is a finite number of the bound states with the energy spectrum given by the formula

$$E_n=-\frac{\hbar^2}{2\mu\rho^2}\left(\frac{1}{2}\sqrt{1+4\gamma\rho^4}-\left(n+\frac{3}{2}\right)\right)^2,\ n=2n_r+l=0,1,2,...,\left[\frac{\sqrt{1+4\gamma\rho^4}-3}{2}\right].$$

Also as in the previous problem the number of the bound states increases with growth $\rho$, but the limiting transition $\rho\to\infty$ here is impossible[1].

The regular radial functions of bound states may be represented in the following form:

$$f_l(n,r)=c_{nl}\left(\tanh\frac{r}{\rho}\right)^{l+1}\frac{e^{\left(n+2-\left(1+\sqrt{1+4\gamma\rho^4}\right)/2\right)r/\rho}}{\left(1+e^{-2r/\rho}\right)^{-\left(n+2-\left(1+\sqrt{1+4\gamma\rho^4}\right)/2\right)}}\Phi_{nl}(r),$$

where $c_{nl}$ – normalizing factor and

$$\Phi_{nl}(r)={}_2F_1\left(\frac{1+\sqrt{1+4\gamma\rho^4}}{2}+\frac{l-n-1}{2},\frac{l-n}{2};\frac{1+\sqrt{1+4\gamma\rho^4}}{2}-n-1;\cosh^{-2}\frac{r}{\rho}\right).$$

**8. Systems of confined particles**.

Let's consider equation (3) with the potentials given by formulas (10) and (13) in the case $C>0$ with the boundary conditions $\psi(\vec{r})\big|_{r=0}<\infty,\quad \psi(\vec{r})\big|_{r=\rho}=0.$

*8.1. Schrödinger potential*

At first we'll find the energy spectrum and $\psi$ – functions of states corresponing to it in the case of the potential (10), assuming $\beta=0$ and $\sqrt{C}=\pi/\rho$. In this case equation (3) can be written in the following form:

$$\nabla^2\psi-\left(\left(\frac{\pi^2}{\rho^2\sin^2(\pi r/\rho)}-\frac{1}{r^2}\right)\frac{\vec{l}^{\,2}}{\hbar^2}+\frac{2\pi\alpha}{\rho}\cot\frac{\pi r}{\rho}-k^2\right)\psi=0\,, \tag{34}$$

where $k^2=2\mu E/\hbar^2$. The substitution $\psi(\vec{r})=\left(\sin^{\nu}(\pi r/\rho)/r\right)F_\nu(z)\mathrm{Y}_{lm}(\theta,\varphi)$, where $z=e^{-2i\pi r/\rho}$, again leads to the equation related to the hypergeometric equation

$$z^2(z-1)F_\nu''+\left(\nu-1+(\nu+1)z\right)zF_\nu'+\left[\left(\frac{\nu^2-\tilde{\rho}^2\varkappa^2}{4}\right)^{*}z-\left(\frac{\nu^2-\tilde{\rho}^2\varkappa^2}{4}\right)\right]F_\nu=0, \tag{35}$$

where $\varkappa=\sqrt{k^2-4i\tilde{\alpha}/\tilde{\rho}^2}$, $\tilde{\rho}=\rho/\pi$, $\tilde{\alpha}=\alpha\rho/2\pi$, and the parameter $\nu$ takes values $l+1$ or $-l$.

Following the way used in the previous sections, as the solution of equation (35) it is convenient to take the function

[1] If in (12) instead of $\delta=\gamma\hbar^2/2\mu C=-\gamma\hbar^2\rho^2/2\mu$ to take $\delta=0$ the energy spectrum will be given by formula $E_n=-\left(\hbar^2/2\mu\rho^2\right)\left(1/4-\sqrt{1+4\gamma\rho^4}\,(n+3/2)+(n+3/2)^2\right)$ and at $\rho\to\infty$ will coincide , as well as should be, with equidistant spectrum of the isotropic oscillator $E_n=\hbar\omega(n+3/2)$, where $\omega=\hbar\sqrt{\gamma/\mu}$.

$$F_\nu = c_\nu z^{a_2}{}_2F_1\left(a_1+a_2, b_1+a_2; a_2-b_2+1; z\right) , \tag{36}$$

where $a_1 = \left(\nu+\tilde{\rho}\varkappa^*\right)/2,\quad a_2 = \left(\nu+\tilde{\rho}\varkappa\right)/2,\quad b_1 = \left(\nu-\tilde{\rho}\varkappa^*\right)/2,\quad b_2 = \left(\nu-\tilde{\rho}\varkappa\right)/2$.

The requirement of convergence of the hypergeometric series in all unit disk, including the point $z=1$ (i.e. $r=0$), takes the form $\mathrm{Re}\left(a_1+a_2+b_1+b_2-1\right) = 2\nu-1<0$ and obviously is not fulfilled at $\nu = l+1$. But at $\nu = l+1$ the solutions of equation (35) are vanishing at the origin of coordinates. Thus, the both boundary conditions (6) can be satisfied simultaneously when the hypergeometric function in (36) is a polynomial, i.e. when $a_1+a_2 = l+1+\tilde{\rho}\,\mathrm{Re}\,\varkappa = -n_r$, $n_r = 0, 1, 2\ldots$. This results in the following formula for an energy spectrum of a system:

$$E_n = \frac{\hbar^2}{2\mu}\left(\frac{\pi^2 n^2}{\rho^2} - \frac{\alpha^2}{n^2}\right), \qquad n = n_r + l + 1, \quad n = 1, 2, 3, \ldots . \tag{37}$$

As seen from (37), $E_n \to \infty$ at unlimitedly growing of number $n$. It means that there is no finite quantity of the energy sufficient for separation of the considered system into its components in complete agreement with the fact established earlier that the function (4) describes at $C>0$ the particles interaction ensuring their confinement in the limited domain of the physical space.

Thus the required solutions of equation (34) have the following form:

$$\psi_{nlm}\left(\vec{r}\right) = \begin{cases} \dfrac{c_{nl}}{r}\sin^{l+1}\left(\dfrac{\pi r}{\rho}\right)\Phi_{nl}\left(r\right)\mathrm{Y}_{lm}\left(\theta,\varphi\right), & 0 \le r \le \rho, \\ 0, & r \ge \rho, \end{cases}$$

where $c_{nl}$ – normalizing factor, and

$$\Phi_{nl}\left(r\right) = e^{\frac{i\pi(n-l-1)r}{\rho}} e^{\frac{\alpha r}{n}}\,{}_2F_1\left(l+1-n, l+1+\frac{i\alpha\rho}{\pi n}; 1-n+\frac{i\alpha\rho}{\pi n}; e^{-\frac{2i\pi r}{\rho}}\right).$$

*8.2. Pöschl-Teller potential (the second type)*

Now we'll find solutions of equation (3) with potential (13) at $\sqrt{C} = \pi/\rho$ :

$$\nabla^2\psi - \left(\left(\frac{\pi^2}{\rho^2\sin^2\left(\pi r/\rho\right)} - \frac{1}{r^2}\right)\frac{\vec{l}^{\,2}}{\hbar^2} + \frac{\gamma\rho^2}{\pi^2}\frac{1}{\cos^2\left(\pi r/\rho\right)} - k^2\right)\psi = 0.$$

Again by the substitution $\psi\left(\vec{r}\right) = \left(\sin^\nu\left(\pi r/\rho\right)/r\right)F_\nu\left(z\right)\mathrm{Y}_{lm}\left(\theta,\varphi\right)$, where $z = \cos^2\pi r/\rho$, we get for $F_\nu\left(z\right)$ the equation related to the hypergeometric equation

$$z^2(z-1)F_\nu''+\left(-\frac{1}{2}+(\nu+1)z\right)zF_\nu'+\left[\left(\frac{\nu^2-\tilde{\rho}^2k^2}{4}\right)z+\frac{\tilde{\rho}^2\tilde{\gamma}^2}{4}\right]F_\nu=0,$$

where $\tilde{\rho}=\rho/\pi$, $\tilde{\gamma}=\gamma\tilde{\rho}$ and the parameter $\nu$ takes values $l+1$ or $-l$.

Repeating the reasoning of the previous section, we'll find the energy spectrum and the set of $\psi-$ functions of a system describing the confinement of its particles in this case:

$$E_n=\frac{\pi^2\hbar^2}{2\mu\rho^2}\left(n+\frac{3}{2}+\sqrt{\frac{1}{4}+\frac{\gamma\rho^4}{\pi^4}}\right)^2,\ n=2n_r+l=0,1,2....,\qquad(38)$$

$$\psi_{nlm}(\vec{r})=\begin{cases}\dfrac{c_{nl}}{r}\sin^{l+1}\left(\dfrac{\pi r}{\rho}\right)\Phi_{nl}(r)\mathrm{Y}_{lm}(\theta,\varphi), & 0\le r\le\rho,\\ 0, & r\ge\rho,\end{cases}$$

where $c_{nl}-$ normalizing factor, and

$$\Phi_{nl}(r)=\cos^{\sqrt{\frac{1}{4}+\frac{\lambda\rho^4}{\pi^4}}}(\pi r/\rho)\,{}_2F_1\left(\frac{n+l+3}{2}+\sqrt{\frac{1}{4}+\frac{\lambda\rho^4}{\pi^4}},\frac{l-n}{2};1+\sqrt{\frac{1}{4}+\frac{\lambda\rho^4}{\pi^4}};\cos^2(\pi r/\rho)\right).$$

**9. Concluding remarks**

So, if from the very beginning we use for formulation of dynamics of two-particle system instead of position vectors $\vec{x}_1$ and $\vec{x}_2$ of its particles in physical space, the variables $\vec{R}=\vec{R}(\vec{x}_1,\vec{x}_2)$ and $\vec{r}=\vec{r}(\vec{x}_1,\vec{x}_2)$, subordinated to special transformational laws under Galilean transformations of physical space [1], we get a freedom in choosing of the transitivity group of the relative space of this system. Then the Galilean-invariant dynamic equation, which allows such a freedom, can be presented in the following form

$$i\hbar\frac{\partial\Psi(\vec{R},\vec{r},t)}{\partial t}=\left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\vec{R}^2}-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial\vec{r}^2}-\frac{\hbar^2}{2\mu}\left(\frac{C}{\sin^2\sqrt{C}r}-\frac{1}{r^2}\right)\left(\vec{r}\times\frac{\partial}{\partial\vec{r}}\right)^2+U(\vec{r})\right)\Psi(\vec{R},\vec{r},t).\qquad(39)$$

Let's notice, that at $C\neq0$ in accordance with obtained above results this equation has exact solutions of the problems with Hulthen and Pöschl-Teller potentials for all values of the quantum number $l$ unlike the conventional theory ($C=0$) giving exact solutions of the Schrödinger equation with the same potentials only for $s$-states. Taking into account also that in these problems there is only a finite number of the bound states at $C<0$ and there are no scattering states at $C>0$ one can assume that in these cases equation (39) is more suitable than at $C=0$ for description of coupled systems with interpartial interactions by means of short-range forces and confining forces, accordingly. In particular, the results obtained here can be apply to construct of deuteron and quarkonium models. But for their more adequate description, as twofermion systems, it is necessary to formulate a dynamic equation, using suitable spinor representations of the $SO(3,1)$ and $SO(4)$ groups, respectively. We intend to discuss this problem in the next article.

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