Non-reductive homogeneous spaces, admitting affine connections

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The purpose of the work is a study of three-dimensional non-reductive homogeneous spaces, admitting invariant affine connections, description of the affine connections together with their curvature and torsion tensors, holonomy algebras.

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. We can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. An *isotropic* $\mathfrak{g}$*-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective.

We classify (up to isomorphism) faithful three-dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic.

Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [1].

It appears that the isotropy-faithfulness is not sufficient for the pair in order to have invariant connections. We say that a homogeneous space $\overline{G}/G$ is *reductive* if the Lie algebra $\bar{\mathfrak{g}}$ may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{g}$ and an ad(G)-invariant subspace $\mathfrak{m}$, that is, if $\bar{\mathfrak{g}} = \mathfrak{g}+\mathfrak{m}$, $\mathfrak{g} \cap \mathfrak{m} = 0$ and $\mathrm{ad}(G)\mathfrak{m} \subset \mathfrak{m}$. Last condition implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ and, conversely, if G is connected. If a homogeneous space is reductive, then the space always admits an invariant connection.

We describe all three-dimensional non-reductive homogeneous spaces, allowing invariant affine connections (the local classification of such spaces is equivalent to the description of the effective pairs of Lie algebras) and all invariant affine connections on the spaces together with their curvature and torsion tensors, holonomy algebras.

Studies are based on the use of properties of the Lie algebras, Lie groups and homogeneous spaces and they mainly have local character. The peculiarity of techniques presented in the work is the application of purely algebraic approach for description of homogeneous spaces and connections on them, as well as combination of methods of differential geometry, the theory of Lie groups and algebras and the theory of homogeneous spaces.

The results of the work can be used to study of the manifolds, and have applications in various fields of mathematics and physics, since many fundamental problems in these areas is associated with study of invariant objects on homogeneous space. The methods presented in the work can be used for the analysis of physical models, and algorithms can be computerized and used for the decision of similar problems in large dimensions.

References

[1] S. Kobayashi, K. Nomizu, Foundations of differential geometry. *John Wiley and Sons, New York.* **1** (1963); **2** (1969).