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Spin 1/2 Particle with two Mass States: Interaction with External Fields

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In the paper, a model for spin 1/2 particle with two mass states is developed on the base of the Gel'fand–Yaglom approach in the theory of relativistic wave equations with extended sets of irreducible representations of the Lorentz group. The main generalized equation is presented in spin-tensor form, and with the use of the Dirac matrices. We introduce two auxiliary bispinors, they determine initial 16-component wave function, and in the absence of an external field for these bispinors we derive two separate Dirac-like equations with different masses M_1 and M_2. It is shown that in the presence of external fields, electromagnetic field and gravitational non-Euclidean background with non-vanishing Ricci scalar curvature, the master wave equation is not split into separated equations, instead a quite definite mixing of two Dirac-like equations arises. This mixing also remains in the presence of only an electromagnetic field, as well it remains in the presence of only a gravitational field. It is shown that a generalized equation for Majorana particle with two mass parameters exists as well, such a generalized Majorana equation is not split into two separated equations if the Ricci scalar does not vanish.

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1. Spinor field with two mass parameters in Gel'fand–Yaglom approach

In the context of the existence of similar neutrinos of different masses, in the present parer we examine a possibility of existing, within the theory of relativistic wave equations, a spin 1/2 particle with two mass parameters. Existence of such more general wave equations in comparison

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with commonly used ones is well known – see references [1–51]

Model for a particle with single value of spin $s = 1/2$ and two mass states is constructed on the base of the following linking scheme for the Lorentz group representations (see notation in [2, 3]; the horizontal and vertical lines stands for linking of the respective representations)

$$
\begin{array}{ccc}
(\frac{1}{2}\ ,1) & - & (1,\ \frac{1}{2}) \\
| & & | \\
(0,\ \frac{1}{2}) & - & (\frac{1}{2},\ 0).
\end{array} \tag{1}
$$

As usual [2, 3] we numerate representations in the scheme (2) by indices

$$
(0,\frac{1}{2}) \sim 1, \quad (\frac{1}{2},0) \sim 2, \\
(\frac{1}{2},1) \sim 3, \quad (1,\frac{1}{2}) \sim 4. \tag{2}
$$

The first order wave equation has the general structure

$$
(\Gamma_a \partial_a - M)\Psi = 0.
$$

In the theory of relativistic wave equations, the most important role belongs to the matrix Γ_4, and all properties of $\Gamma_1, \Gamma_2, \Gamma_3$ are determined by the matrix Γ_4 (see [1–3]).

For spin blocks $C^{1/2}$ and $C^{3/2}$ (for more detail see [1–3]) of the matrix Γ_4 (symbols of direct sum $\oplus$ and direct product $\otimes$ of representations are used)

$$
\Gamma_4 = (C^{1/2} \otimes I_2) \oplus (C^{3/2} \otimes I_4)\,,
$$

related to the scheme (2), we have the following structure in the Gel'fand–Yaglom basis [49], [50] (lower indexes relate to numerated representations in (2))

$$
C^{1/2} = \left| \begin{array}{cccc} 0 & c^{1/2}_{12} & c^{1/2}_{13} & 0 \\ c^{1/2}_{21} & 0 & 0 & c^{1/2}_{24} \\ c^{1/2}_{31} & 0 & 0 & c^{1/2}_{34} \\ 0 & c^{1/2}_{42} & c^{1/2}_{43} & 0 \end{array} \right| , c^{3/2} = \left| \begin{array}{cc} 0 & c^{3/2}_{34} \\ c^{3/2}_{43} & 0 \end{array} \right| ,
$$

where the entries of the matrices are not yet fixed. Due to uniqueness of spin $S = 1/2$, we get the constraint $c^{3/2}_{34} = c^{3/2}_{43} = 0$; due to relativistic invariance of the wave equation we obtain the following restriction $c^{1/2}_{34} = c^{1/2}_{43} = 0$. Two last conditions mean the break of link between representations $(\frac{1}{2},1)$ and $(1,\frac{1}{2})$ in the scheme (2), so that it transforms into another one

$$
(\frac{1}{2},1) - (0,\frac{1}{2}) - (\frac{1}{2},0) - (1,\frac{1}{2})\,. \tag{3}
$$

From the invariance of the model under spatial reflection we obtain additional restrictions

$$
c^{1/2}_{21} = c^{1/2}_{12}, \quad c^{1/2}_{24} = c^{1/2}_{13}, \quad c^{1/2}_{42} = c^{1/2}_{31}\,. \tag{4}
$$

Finally, the existence of the Lagrangian formulation for the model provides us with the following restrictions

$$
c^{1/2}_{12} \ \text{is} \ \ \text{real}\,, \quad c^{1/2}_{12} = \frac{\eta^{1/2}_{34}}{\eta^{1/2}_{12}} (c^{1/2}_{34})^*, \tag{5}
$$

where $\eta^{1/2}_{..}$ designate the elements of the block $\eta^{1/2}$ in the matrix of the bilinear form. Having in mind all said and using the notations

$$
c^{1/2}_{12} = c_1,\ c^{1/2}_{13} = c_2,\ \frac{\eta^{1/2}_{34}}{\eta^{1/2}_{12}} = \delta\ ,\ \delta = \pm 1\,, \tag{6}
$$

we obtain the following representation for the spin block $C^{1/2}$:

$$
C^{1/2} = \left| \begin{array}{cccc} 0 & c_1 & c_2 & 0 \\ c_1 & 0 & 0 & c_2 \\ \delta c_2^* & 0 & 0 & 0 \\ 0 & \delta c_2^* & 0 & 0 \end{array} \right| . \tag{7}
$$

Characteristic equation of 4-th order for the matrix $C^{1/2}$ (its roots determine all possible values for the mass parameters of the object described by the wave equation under consideration – for more detail see in [48]–[50]) is a bi-quadratic one

$$
\Lambda^4 - \Lambda^2\ (\,c_1^2 + 2\delta |c_2|^2) + |c_2|^4 = 0\,. \tag{8}
$$

so we get the following roots:

$$
(\Lambda^2)_1 = \frac{(c_1^2 + 2\delta|c_2|^2) + c_1\sqrt{c_1^2 + 4\delta|c_2|^2}}{2}, \\
(\Lambda^2)_2 = \frac{(c_1^2 + 2\delta|c_2|^2) - c_1\sqrt{c_1^2 + 4\delta|c_2|^2}}{2}. \tag{9}
$$

These roots may be presented differently if one uses the quantities

$$\gamma_1 = \pm\frac{c_1 + \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} ,$$
$$\gamma_2 = \pm\frac{c_1 - \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} ; \qquad (10)$$

as easily verified, we have identities

$$\gamma_1^2 = \Lambda_1^2, \qquad \gamma_2^2 = \Lambda_2^2 . \qquad (11)$$

From the general theory of wave equations for particles with spectra of mass parameters (for more detail see in [48]–[50]), it is known that here we have models for a fermion with two positive and two negative masses:

$$M_1 = \frac{M}{\pm\sqrt{\Lambda_1^2}} = \frac{M}{\pm\sqrt{\gamma_1^2}} ,$$
$$M_2 = \frac{M}{\pm\sqrt{\Lambda_2^2}} = \frac{M}{\pm\sqrt{\gamma_2^2}} . \qquad (12)$$

We will ignore models with negative mass parameters by the following reason. Let us recall that for ordinary Dirac equation the variant with negative mass may be transformed to the variant with positive mass by means of a simple linear transformation over the wave function:

$$(i\gamma^a\partial_a - M)\Psi = 0 , \quad M > 0;$$
$$\Psi' = \gamma^5\Psi, \quad \gamma'^a = \gamma^5\gamma^a\gamma^5 = -\gamma^a ,$$
$$[i\gamma'^a\partial_a - (-M)]\Psi' = 0, \quad (-M) < 0 . \qquad (13)$$

As we see later, the variants $\delta = \pm 1$

$$\eta_{12}^{1/2} = \eta_{34}^{1/2} = +1 , \quad \eta_{12}^{1/2} = \eta_{34}^{1/2} = -1 . \qquad (14)$$

corresponds to nonequivalent models.

The freedom in parameters c_1, c_2 must be agreed with real-valuedness of both masses:

$$\Lambda_1^2 > 0, \quad \Lambda_2^2 > 0.$$

2. The model in the modified Gel'fand–Yaglom basis

The modified Gel'fand–Yaglom basis for particles with half-integer spins is based on the use of a special way to combine and enumerate basic vectors of the initial $G-Y$ basis. In the case under consideration we should use the following four groups of states:

in modified basis,

$$\left|\begin{array}{c} \psi^{(0,1/2)}_{1/2,1/2} \\ \psi^{(0,1/2)}_{1/2,-1/2} \\ \psi^{(1/2,0)}_{1/2,1/2} \\ \psi^{(1/2,0)}_{1/2,-1/2} \end{array}\right| , \quad \left|\begin{array}{c} \psi^{(1,1/2)}_{1/2,1/2} \\ \psi^{(1,1/2)}_{1/2,-1/2} \\ \psi^{(1/2,1)}_{1/2,1/2} \\ \psi^{(1/2,1)}_{1/2,-1/2} \end{array}\right| ,$$
$$\left|\begin{array}{c} \psi^{(1,1/2)}_{3/2,3/2} \\ \psi^{(1,1/2)}_{3/2,-3/2} \\ \psi^{(1/2,1)}_{3/2,3/2} \\ \psi^{(1/2,1)}_{3/2,-3/2} \end{array}\right| , \quad \left|\begin{array}{c} \psi^{(1,1/2)}_{3/2,1/2} \\ \psi^{(1,1/2)}_{3/2,-1/2} \\ \psi^{(1/2,1)}_{3/2,1/2} \\ \psi^{(1/2,1)}_{3/2,-1/2} \end{array}\right| ; \qquad (15)$$

where in $\psi^{(l,l')}_{s,s_3}$, the indices (l,l') determine irreducible representations of the Lorentz group; parameter s denotes the values of spins; s_3 is a third projection of the spin.

In canonical Gel'fand–Yaglom basis, the vectors are enumerated as follows

$$\left|\begin{array}{c} \psi^{(0,1/2)}_{0,1/2} \\ \psi^{(0,1/2)}_{0,-1/2} \\ \psi^{(1/2,0)}_{1/2,0} \\ \psi^{(1/2,0)}_{-1/2,0} \end{array}\right| , \quad \left|\begin{array}{c} \psi^{(1,1/2)}_{1,1/2} \\ \psi^{(1,1/2)}_{0,1/2} \\ \psi^{(1,1/2)}_{-1,1/2} \\ \psi^{(1,1/2)}_{1,-1/2} \\ \psi^{(1,1/2)}_{0,-1/2} \\ \psi^{(1,1/2)}_{-1,-1/2} \end{array}\right| , \quad \left|\begin{array}{c} \psi^{(1/2,1)}_{1/2,1} \\ \psi^{(1/2,1)}_{1/2,0} \\ \psi^{(1/2,1)}_{1/2,-1} \\ \psi^{(1/2,1)}_{-1/2,1} \\ \psi^{(1/2,1)}_{-1/2,0} \\ \psi^{(1/2,1)}_{-1/2,-1} \end{array}\right| ; \qquad (16)$$

in $\psi^{(l,l')}_{l_3,l'_3}$ we use the notations: $l_3 = -l, -l + 1, ..., +l$; $l'_3 = -l', -l' + 1, ..., +l'$.

In spinor basis, the vectors are designated as a long column with the following components which we represent in a line as

$$\Big\{\psi^{\dot{1}},\ \psi^{\dot{2}},\ \psi_1,\ \psi_1, \psi^{\dot{1}}_{(11)},\ \psi^{\dot{1}}_{(12)},\ \psi^{\dot{1}}_{(22)},\ \psi^{\dot{2}}_{(11)},$$

$$\psi^{\dot{2}}_{(12)},\ \psi^{\dot{2}}_{(22)},\ \psi^{\dot{1}\dot{1}}_{1},\ psi^{\dot{1}\dot{2}}_{1},\ \psi^{\dot{2}\dot{2}}_{1},\ \psi^{\dot{1}\dot{1}}_{2},\ \psi^{\dot{1}\dot{2}}_{2},\ \psi^{\dot{2}\dot{2}}_{2}\Big\}$$

Relationships between these tree bases are given by the formulas

$$\psi^{(l,l')}_{l_3,l'_3} = \sum_{s,m} (ll' l_3 l'_3 | sm) \psi^{\tau}_{s,m}, \tag{17}$$

l_3, l'_3 are fixed, $m = l_3 + l'_3$;

$$\psi^{\tau}_{s,m} = \sum_{l_3,l'_3} (ll' l_3 l'_3 \ sm) \psi^{(l,l')}_{(l_3,l'_3)}, \tag{18}$$

where s, m are fixed, and $l_3 + l'_3 = m$;

$$\psi^{(l,l')}_{(l_3,l'_3)} = \left[\frac{(2l)!}{(l+l_3)!(l-l_3)!} \right] \times \left[\frac{(2l')!}{(l'+l'_3)!(l'-l'_3)!} \right] \psi^{\dot{1}\ldots\dot{1}\ \dot{2}\ldots\dot{2})}_{(1\ldots1\ 2\ldots2)}, \tag{19}$$

where the number of indices of the type $\dot{1}$ equals $l' + l'_3$; the number of indices of the type $\dot{2}$ equals $l' - l'_3$; the number of indices of the type 1 equals $l + l_3$; and the number of indices of the type 2 equals $l - l_3$.

With respect to (2), four groups of vectors in the modified basis are presented as

$$[\ \epsilon^{1}_{1/2,1/2}; \epsilon^{1}_{1/2,-1/2}; \epsilon^{2}_{1/2,1/2}; \epsilon^{2}_{1/2,-1/2}\ ],$$

$$[\ \epsilon^{4}_{1/2,1/2}; \epsilon^{4}_{1/2,-1/2}; \epsilon^{3}_{1/2,1/2}; \epsilon^{3}_{1/2,-1/2}\ ],$$

$$[\ \epsilon^{4}_{3/2,3/2}; \epsilon^{4}_{3/2,-3/2}; \epsilon^{3}_{3/2,3/2}; \epsilon^{3}_{3/2,-3/2}\ ],$$

$$[\ \epsilon^{4}_{3/2,1/2}; \epsilon^{4}_{3/2,-1/2}; \epsilon^{3}_{3/2,1/2}; \epsilon^{3}_{3/2,-1/2}\ ].$$

In this basis, the spin block $\Gamma^{(1/2)}_4$ has the structure

$$\Gamma^{(1/2)}_4 = \left| \begin{array}{cccccccc} 0 & 0 & c^{(1/2)}_{12} & 0 & 0 & 0 & c^{(1/2)}_{13} & 0 \\ 0 & 0 & 0 & c^{(1/2)}_{12} & 0 & 0 & 0 & c^{(1/2)}_{13} \\ c^{(1/2)}_{21} & 0 & 0 & 0 & c^{(1/2)}_{24} & 0 & 0 & 0 \\ 0 & c^{(1/2)}_{21} 0 & 0 & 0 & 0 & c^{(1/2)}_{24} & 0 & 0 \\ 0 & 0 & c^{(1/2)}_{42} & 00 & 0 & 0 & c^{(1/2)}_{24} & 0 \\ 0 & 0 & 0 & c^{(1/2)}_{42} & 0 & 0 & 0 & c^{(1/2)}_{43} \\ c^{(1/2)}_{31} & 0 & 0 & 0 & c^{(1/2)}_{34} & 0 & 0 & 0 \\ 0 & c^{(1/2)}_{31} & 0 & 0 & 0 & c^{(1/2)}_{34} & 0 & 0 \end{array} \right|;$$

from P-invariance it follows

$$c^{(1/2)}_{21} = c^{(1/2)}_{12}, \quad c^{(1/2)}_{24} = c^{(1/2)}_{13}, \quad c^{(1/2)}_{42} = c^{(1/2)}_{31}, \quad c^{(1/2)}_{34} = c^{(1/2)}_{43};$$

so the above spin block becomes simpler

$$\Gamma^{(1/2)}_4 = \left| \begin{array}{cc} c^{(1/2)}_{12} & c^{(1/2)}_{13} \\ c^{(1/2)}_{42} & c^{(1/2)}_{43} \end{array} \right| \otimes \gamma_4 \,. \tag{20}$$

Requirement of the existence of the Lagrangian form for the model gives an additional constraint

$$c^{(1/2)}_{42} = \frac{\eta^{(1/2)}_{34}}{\eta^{(1/2)}_{12}} \left(c^{(1/2)}_{13} \right)^{*} \,.$$

Further, with the shortening notations, we obtain

$$c_{12}^{(1/2)} = c_1, \quad c_{13}^{(1/2)} = c_2, \quad \frac{\eta_{34}^{(1/2)}}{\eta_{12}^{(1/2)}} = \delta, \quad \delta = \pm 1, \quad \Gamma_4^{(1/2)} = \begin{vmatrix} c_1 & c_2 \\ \delta c_2^* & c_{43}^{(1/2)} \end{vmatrix} \otimes \gamma_4 \, . \tag{21}$$

Let us consider the spin block $\Gamma_4^{(3/2)}$:

$$\Gamma_4^{(3/2)} = \begin{vmatrix} 0 & 0 & c_{43}^{(3/2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{43}^{(3/2)} & 0 & 0 & 0 & 0 \\ c_{34}^{(3/2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{34}^{(3/2)}0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{43}^{(3/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{43}^{(3/2)} \\ 0 & 0 & 0 & 0 & c_{34}^{(3/2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{34}^{(3/2)} & 0 & 0 \end{vmatrix} .$$

From P-invariance it follows

$$c_{34}^{(3/2)} = c_{43}^{(3/2)}, \qquad \Gamma_4^{(3/2)} = \begin{vmatrix} c_{43}^{(3/2)} & 0 \\ 0 & c_{43}^{(3/2)} \end{vmatrix} \otimes \gamma_4 \, .$$

Due to identity $c_{43}^{(3/2)} = 2c_{43}^{(1/2)}$, from the uniqueness of the spin value $S = 1/2$ we derive the constraints $c_{43}^{(3/2)} = 0$, $c_{43}^{(1/2)} = 0$. Let us simplify the notation as follows

$$c_{1/2}^{(1/2)} = c_1, \qquad c_{13}^{(1/2)} = i\sqrt{3}c_{13} = c_2 \, , \quad \Gamma_4^{(1/2)} = \begin{vmatrix} c_1 & c_2 \\ \delta c_2^* & 0 \end{vmatrix} \otimes \gamma_4 = C^{(1/2)} \otimes \gamma_4. \tag{22}$$

Characteristic equation for $C^{(1/2)}$ is

$$\det \begin{vmatrix} \lambda - c_1 & -c_2 \\ -\delta c_2^* & \lambda \end{vmatrix} = \lambda(\lambda - c_1) - \delta |c_2|^2 = 0 \, ,$$

for the roots we get (compare them with (10)–(11))

$$\lambda_1 = \frac{c_1 + \sqrt{c_1^2 + 4\delta|c_2|^2}}{2}, \quad \lambda_2 = \frac{c_1 - \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} \; ; \tag{23}$$

note two identities

$$\lambda_1 \lambda_2 = -\delta |c_2|^2, \qquad \lambda_1 + \lambda_2 = c_1, \quad \lambda_1 - \lambda_2 = \sqrt{c_1^2 + 4\delta|c_2|^2} \, , \tag{24}$$

where c_1, c_2 are parameters of the model. The minimal equation for $C^{(1/2)}$ has the form

$$\left(C^{(1/2)} - \lambda_1\right)\left(C^{(1/2)} - \lambda_2\right) = 0 \; ; \tag{25}$$

whence with (24) in mind we get another representation for the minimal polynomial equation

$$\left(C^{(1/2)}\right)^2 - c_1 C^{(1/2)} - \delta |c_2|^2 = 0 \, . \tag{26}$$

The minimal polynomial equation for the matrix $\Gamma_4^{(1/2)}$ reads

$$\left[\left(\Gamma_4^{(1/2)}\right) - \lambda_1^2\right]\left[\left(\Gamma_4^{(1/2)}\right) - \lambda_2^2\right] = 0\ . \tag{27}$$

Correspondingly, the minimal polynomial equation for the matrix Γ_4 looks as follows

$$\Gamma_4(\Gamma_4^2 - \lambda_1^2)\ (\Gamma_4^2 - \lambda_2^2) = 0\ . \tag{28}$$

3. Relationships between three bases

We start with the explicit form of the matrix Γ_4 in the modified basis and represent it as the following 4x4 block matrixes

$$\Gamma_4^{G-Y} = \begin{vmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

with nontrivial components given by

$$G_{11} = \begin{vmatrix} 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & c_1 \\ c_1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \end{vmatrix},\ G_{12} = \begin{vmatrix} 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_2 \\ c_2 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{vmatrix},\ G_{21} = \begin{vmatrix} 0 & 0 & \delta c_2^* & 0 \\ 0 & 0 & 0 & \delta c_2^* \\ \delta c_2^* & 0 & 0 & 0 \\ 0 & \delta c_2^* & 0 & 0 \end{vmatrix}$$

Using the formula (19), we can examine the transition to the canonical basis

$$\Psi_{canon} = B^+ \Psi_{G-Y}$$

where again the matrix B^+ is represented in the block form as

$$B^+ = \begin{vmatrix} B_{11} & 0 & 0 & 0 \\ 0 & B_{22} & B_{23} & B_{24} \\ 0 & B_{32} & B_{33} & B_{34} \\ 0 & B_{42} & B_{43} & B_{44} \end{vmatrix}$$

where B_{11} is 4x4 identity matrix and rest nontrivial components can be written as

$$B_{22} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 & 0 & 0 \end{vmatrix},\ B_{23} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},\ B_{24} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{vmatrix},$$

$$B_{32} = \begin{vmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{vmatrix},\ B_{33} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},\ B_{34} = \begin{vmatrix} 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 \end{vmatrix},$$

$$B_{42}=\begin{vmatrix}0&0&0&\sqrt{\frac{2}{3}}\\0&0&-\sqrt{\frac{2}{3}}&0\\0&0&0&-\frac{1}{\sqrt{3}}\\0&0&0&0\end{vmatrix},\quad B_{43}=\begin{vmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{vmatrix},\ B_{44}=\begin{vmatrix}0&0&0&\frac{1}{\sqrt{3}}\\0&0&\frac{1}{\sqrt{3}}&0\\0&0&0&\sqrt{\frac{2}{3}}\\0&0&0&0\end{vmatrix},$$

We are to find the matrix Γ_4 in the canonical basis and represent it in the block matrix form as

$$\Gamma_4^{canon}=B^{+}\Gamma_4^{G-Y}B==\begin{vmatrix}F_{11}&F_{12}&F_{13}&F_{14}\\F_{21}&0&0&0\\F_{31}&0&0&0\\F_{41}&0&0&0\end{vmatrix}$$

where nontrivial components F_{ij} are the following 4x4 matrices

$$F_{11}=\begin{vmatrix}0&0&c_1&0\\0&0&0&c_1\\c_1&0&0&0\\0&c_1&0&0\end{vmatrix},\ F_{12}=\begin{vmatrix}0&0&0&0\\0&0&0&0\\0&\frac{c_2}{\sqrt{3}}&0&c_2\sqrt{\frac{2}{3}}\\0&0&c_2\sqrt{\frac{2}{3}}&0\end{vmatrix},\ F_{13}=\begin{vmatrix}0&0&0&\frac{c_2}{\sqrt{3}}\\0&0&0&0\\0&0&0&0\\\frac{c_2}{\sqrt{3}}&0&0&0\end{vmatrix},\ F_{14}=\begin{vmatrix}0&-c_2\sqrt{\frac{2}{3}}&0&0\\c_2\sqrt{\frac{2}{3}}&0&-\frac{c_2}{\sqrt{3}}&0\\0&0&0&0\\0&0&0&0\end{vmatrix},$$

$$F_{21}=\begin{vmatrix}0&0&0&0\\0&0&-\frac{\delta c_2^*}{\sqrt{3}}&0\\0&0&0&-\delta\sqrt{\frac{2}{3}}c_2^*\\0&0&\delta\sqrt{\frac{2}{3}}c_2^*&0\end{vmatrix},\ F_{31}=\begin{vmatrix}0&0&0&\frac{\delta c_2^*}{\sqrt{3}}\\0&0&0&0\\0&0&0&0\\\frac{\delta c_2^*}{\sqrt{3}}&0&0&0\end{vmatrix},\ F_{41}=\begin{vmatrix}0&\delta c_2^*\sqrt{\frac{2}{3}}&0&0\\-\delta c_2^*\sqrt{\frac{2}{3}}&0&0&0\\0&-\frac{\delta c_2^*}{\sqrt{3}}&0&0\\0&0&0&0\end{vmatrix},$$

Now we relate canonical and spinor bases:

$$\Psi_{canon}=A\psi_{spin},$$

with matrix A represented in the block diagonal form as $A=\mathbf{diag}(A_{11},A_{22},A_{33},A_{44})$, where A_{11} is 4x4 identity matrix and rest three matrixes are diagonal and the following $A_{22}=\mathbf{diag}(1,\sqrt{2},1,1)$, $A_{33}=\mathbf{diag}(\sqrt{2},1,1,\sqrt{2})$,$A_{44}=\mathbf{diag}(1,1,\sqrt{2},1)$.

For Γ_4 in spinor basis we get

$$\Gamma_4^{spinorial}=A^{-1}\Gamma_4^{canonical}A=\begin{vmatrix}f_{11}&f_{12}&f_{13}&f_{14}\\f_{21}&0&0&0\\f_{31}&0&0&0\\f_{41}&0&0&0\end{vmatrix}$$

in the block matrix form with nontrivial 4x4 matrix components given by

$$f_{11}=\begin{vmatrix}0&0&c_1&0\\0&0&0&c_1\\c_1&0&0&0\\0&c_1&0&0\end{vmatrix},\ f_{12}=\begin{vmatrix}0&0&0&0\\0&0&0&0\\0&-c_2\sqrt{\frac{2}{3}}&0&c_2\sqrt{\frac{2}{3}}\\0&0&-c_2\sqrt{\frac{2}{3}}&0\end{vmatrix},\ f_{13}=\begin{vmatrix}0&0&0&c_2\sqrt{\frac{2}{3}}\\0&0&0&0\\0&0&0&0\\c_2\sqrt{\frac{2}{3}}&0&0&0\end{vmatrix},\ f_{31}=\begin{vmatrix}0&0&0&\frac{\delta c_2^*}{\sqrt{6}}\\0&0&0&0\\0&0&0&0\\\frac{\delta c_2^*}{\sqrt{6}}&0&0&0\end{vmatrix},$$

$$f_{14} = \begin{vmatrix} 0 & -c_2\sqrt{\frac{2}{3}} & 0 & 0 \\ c_2\sqrt{\frac{2}{3}} & 0 & -c_2\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad f_{21} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\delta c_2^*}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & -\delta\sqrt{\frac{2}{3}}c_2^* \\ 0 & 0 & \delta\sqrt{\frac{2}{3}}c_2^* & 0 \end{vmatrix}, \quad f_{41} = \begin{vmatrix} 0 & \delta c_2^*\sqrt{\frac{2}{3}} & 0 & 0 \\ -\delta c_2^*\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & -\frac{\delta c_2^*}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Thus, in spinor form the wave equation has the structure

$$c_1\, \partial^{\dot{a}b}\, \Psi_b + \sqrt{\frac{2}{3}} c_2 \partial^c_{\dot{b}}\, \psi_c^{(\dot{a}\dot{b})} + M\, \Psi^{\dot{a}} = 0\,, \quad c_1\, \partial_{a\dot{b}}\, \psi^{\dot{b}} + \sqrt{\frac{2}{3}} c_2 \partial^b{}_{\dot{c}}\, \psi^{\dot{c}}_{(ab)} + M\, \psi_a = 0\,,$$
$$-\frac{\delta}{\sqrt{6}} c_2^* (\partial_a^{\dot{b}}\, \psi^{\dot{c}} + \partial_a^{\dot{c}}\, \psi^{\dot{b}}) + M\, \psi_a^{\dot{b}\dot{c}} = 0\,, \quad -\frac{\delta}{\sqrt{6}} c_2^* (\partial_b^{\dot{a}}\, \psi_c + \partial_c^{\dot{a}}\, \psi_b) + M\, \psi^{\dot{a}}_{(bc)} = 0\,, \tag{29}$$

where the derivative operator in spinor form is determined by the formula $\partial_{\dot{a}b} = -i\partial_\mu(\sigma^\mu)_{\dot{a}b}$, σ^j stands for the Pauli matrices, $\sigma^4 = iI_2$.

4. The wave equation in spin-tensor form

It is convenient to present eq. (29) differently

$$c_1\, \partial^{\dot{a}b}\, \Psi_b + \beta_2\, \partial^c_{\dot{b}}\, \psi_c^{(\dot{a}\dot{b})} + M\, \Psi^{\dot{a}} = 0\,,$$
$$c_1\, \partial_{a\dot{b}}\, \psi^{\dot{b}} + \beta_2 \partial^b{}_{\dot{c}}\, \psi^{\dot{c}}_{(ab)} + M\, \psi_a = 0\,,$$
$$\frac{\beta_3}{2}(\partial_a^{\dot{b}}\, \psi^{\dot{c}} + \partial_a^{\dot{c}}\, \psi^{\dot{b}}) + M\, \psi_a^{\dot{b}\dot{c}} = 0\,,$$
$$\frac{\beta_3}{2}(\partial_b^{\dot{a}}\, \psi_c + \partial_c^{\dot{a}}\, \psi_b) + M\, \psi^{\dot{a}}_{(bc)} = 0\,, \tag{30}$$

where

$$\beta_2 = \sqrt{2/3}\, c_2, \quad \beta_3 = -f\sqrt{2/3}\, c_2^*. \tag{31}$$

To translate equations in (30) to spin-vector form, we are to use the known formulas

$$\Psi_a^{(\dot{b}\dot{c})} = \frac{1}{2}(\sigma^{\mu\dot{b}}{}_a \Psi^{\dot{c}}_\mu + \sigma^{\mu\dot{c}}{}_a \Psi^{\dot{b}}_\mu),$$
$$\Psi^{\dot{a}}_{(bc)} = \frac{1}{2}(\sigma^{\mu\dot{a}}{}_b \Psi_{\mu c} + \sigma^{\mu\dot{a}}{}_c \Psi_{\mu b})$$
$$\Psi^{\dot{a}} = \sigma^{\mu\dot{a}b}\Psi_{\mu b}, \quad \Psi_a = \sigma^\mu_{a\dot{b}}\Psi^{\dot{b}}_\mu\,. \tag{32}$$

Instead of the first equation in (30), we obtain

$$c_1 \partial^{\dot{a}b} \sigma^\mu_{b\dot{c}} \Psi^{\dot{c}}_\mu + \frac{1}{2}\beta_2 \partial^c_{\dot{b}} (\sigma^{\mu\dot{a}}{}_c \Psi^{\dot{b}}_\mu + \sigma^{\mu\dot{b}}{}_c \Psi^{\dot{a}}_\mu) + M\, \sigma^{\mu\dot{a}b}\Psi_{\mu b} = 0\,,$$
$$c_1 \partial_{a\dot{b}} \sigma^{\mu\dot{b}c}\Psi_{\mu c} + \frac{1}{2}\beta_2 \partial^b{}_{\dot{c}}\, (\sigma^{\mu\dot{c}}{}_a \Psi_{\mu b} + \sigma^{\mu\dot{c}}{}_b \Psi_{\mu a}) + M\, \sigma^\mu_{a\dot{b}}\Psi^{\dot{b}}_\mu = 0\,.$$

Whence it follows

$$c_1 \partial^{\dot{a}b} \sigma^\mu_{b\dot{c}} \Psi^{\dot{c}}_\mu + \frac{\beta_2}{2}(-\sigma^{\mu\dot{a}c}\partial_{c\dot{b}}\Psi^{\dot{b}}_\mu + \frac{2}{i}\partial_\mu \Psi^{\dot{a}}_\mu) + M\, \sigma^{\mu\dot{a}b}\Psi_{\mu b} = 0\,,$$
$$c_1 \partial_{a\dot{b}} \sigma^{\mu\dot{b}c}\Psi_{\mu c} + \frac{\beta_2}{2}(-\sigma^\mu_{a\dot{c}}\partial^{\dot{c}b}\Psi_{\mu b} + \frac{2}{i}\partial_\mu \Psi_{\mu a}) + M\, \sigma^\mu_{a\dot{b}}\Psi^{\dot{b}}_\mu = 0\,.$$

In new representation, the 16-component wave function makes up the vector-bispinor $\Psi_\mu = \{\Psi^{\dot{a}}_\mu,\ \Psi_{\mu b}\}$, and the last two equations are written with the use of the Dirac matrices as follows

$$c_1 \partial_\nu \gamma_\nu \gamma_\mu \Psi_\mu + \frac{\beta_2}{2}(-\gamma_\mu \partial_\nu \gamma_\nu \Psi_\mu - 2\partial_\mu \psi_\mu) + M\gamma_\mu \Psi_\mu = 0 \ .$$

Introducing the notation $\hat{\partial} = \gamma_\mu \Psi_\mu$ and taking into account the identity $\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = 2\delta_{\nu\mu}$, we rewrite the previous equation as

$$(c_1 + \frac{\beta_2}{2})\hat{\partial}(\gamma_\mu \Psi_\mu) - 2\beta_2(\partial_\mu \psi_\mu) + iM(\gamma_\mu \Psi_\mu) = 0 \ . \qquad (33)$$

Now, we consider third and fourth equations in (30): having in mind the formulas (32), we get

$$\frac{\beta_3}{2}(\partial^{\dot{a}}_b \sigma^{\mu\dot{c}d}\Psi_{\mu d} + \partial^{\dot{c}}_b \sigma^{\mu\dot{a}d}\Psi_{\mu d}) + \frac{M}{2}(\sigma^{\mu\dot{a}}{}_b \Psi^{\dot{c}}_\mu + \sigma^{\mu\dot{c}}{}_b \Psi^{\dot{a}}_\mu) = 0 \ ,$$
$$\frac{\beta_3}{2}(\partial^{\dot{b}}_a \sigma^\mu_{c\dot{d}}\Psi^{\dot{d}}_\mu + \partial^{\dot{b}}_c \sigma^\mu_{a\dot{d}}\Psi^{\dot{d}}_\mu + \frac{M}{2}(\sigma^{\mu\dot{b}}_a \Psi_{\mu c} + \sigma^{\mu\dot{b}}_c \Psi_{\mu a}) = 0 \ .$$

We multiply the first equation by$\sigma^{\lambda b}{}_{\dot{a}}$, and the second one – by $\sigma^{\lambda a}{}_{\dot{b}}$, so obtaining

$$\frac{\beta_3}{2}\sigma^{\lambda b}{}_{\dot{a}}(\partial^{\dot{a}}_b \sigma^{\mu\dot{c}d}\Psi_{\mu d} + \partial^{\dot{c}}_b \sigma^{\mu\dot{a}d}\Psi_{\mu d}) + \frac{M}{2}\sigma^{\lambda b}{}_{\dot{a}}(\sigma^{\mu\dot{a}}{}_b \Psi^{\dot{c}}_\mu + \sigma^{\mu\dot{c}}{}_b \Psi^{\dot{a}}_\mu) = 0 \ ,$$
$$\frac{\beta_3}{2}\sigma^{\lambda a}{}_{\dot{b}}(\partial^{\dot{b}}_a \sigma^\mu_{c\dot{d}}\Psi^{\dot{d}}_\mu + \partial^{\dot{b}}_c \sigma^\mu_{a\dot{d}}\Psi^{\dot{d}}_\mu + \frac{M}{2}\sigma^{\lambda a}{}_{\dot{b}}(\sigma^{\mu\dot{b}}_a \Psi_{\mu c} + \sigma^{\mu\dot{b}}_c \Psi_{\mu a}) = 0 \ .$$

These equations can be rewritten as

$$\frac{\beta_3}{2}[-(\sigma^{\lambda\dot{a}b}\partial_{b\dot{a}})\sigma^{\mu\dot{c}d}\Psi_{\mu d} - \partial^{\dot{c}b}\sigma^\lambda_{b\dot{a}}\sigma^{\mu\dot{a}d}\Psi_{\mu d}] + \frac{M}{2}[-\sigma^{\lambda\dot{a}b}\sigma^\mu_{b\dot{a}}\Psi^{\dot{c}}_\mu - \sigma^{\mu\dot{c}b}\sigma^\lambda_{b\dot{a}}\Psi^{\dot{a}}_\mu = 0 \ ,$$
$$\frac{\beta_3}{2}[-(\sigma^\lambda_{a\dot{b}}\partial^{\dot{b}a})\sigma^\mu_{c\dot{d}}\Psi^{\dot{d}}_\mu - \partial_{c\dot{b}}\sigma^{\lambda\dot{b}a}\sigma^\mu_{a\dot{d}}\Psi^{\dot{d}}_\mu] + \frac{M}{2}[-(\sigma^\lambda_{a\dot{b}}\sigma^{\mu\dot{b}a})\Psi_{\mu c} - \sigma^\mu_{c\dot{b}}\sigma^{\lambda\dot{b}a}\Psi_{\mu a}] = 0 \ ,$$

which after simple transformation may be presented in vector-bispinor form as

$$\beta_3\left[\partial_\lambda(\gamma_\mu \Psi_\mu) - \frac{1}{4}\gamma_\lambda \hat{\partial}(\gamma_\mu \Psi_\mu)\right] + M\left[\Psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu \Psi_\mu)\right] = 0 \ . \qquad (34)$$

Thus, we have arrived to the following set of equations, describing particle with one value of spin and two mass states (the first equation is multiplied by $\frac{1}{4}\gamma_\lambda$)

$$\frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\gamma_\lambda \hat{\partial}(\gamma_\mu \Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu \psi_\mu) + \frac{M}{4}\gamma_\lambda(\gamma_\mu \Psi_\mu) = 0 \ , \qquad (35)$$
$$\beta_3\left[\partial_\lambda(\gamma_\mu \Psi_\mu) - \frac{1}{4}\gamma_\lambda \hat{\partial}(\gamma_\mu \Psi_\mu)\right] + M\left[\Psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu \Psi_\mu)\right] = 0 \ . \qquad (36)$$

Summing equations in (35)–(36), we get

$$M\Psi_\lambda + \beta_3 \partial_\lambda(\gamma_\mu \Psi_\mu) + \frac{1}{4}\left(c_1 + \frac{\beta_2}{2} - \beta_3\right)\gamma_\lambda \hat{\partial}(\gamma_\mu \Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu \Psi_\mu) = 0 \ . \qquad (37)$$

We notice that multiplying this equation by $\frac{1}{4}\gamma_\lambda$, we obtain

$$\frac{1}{4}M\gamma_\mu\Psi_\mu + \frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}(\partial_\mu\Psi_\mu) = 0\ ,$$

which is equivalent to the above equation (35):

$$\frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu\psi_\mu) + \frac{M}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu) = 0\ .$$

This means that in fact we have only one independent equation (37), whereas eq. (35) is just its consequence. Let us rewrite this main equation (37) in a different form

$$\beta_3\gamma_\mu\,\partial_\rho\Psi_\mu + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\nu\,\partial_\mu\Psi_\nu - \frac{\beta_2}{2}\gamma_\rho\,\partial_\mu\Psi_\mu + M\Psi_\rho = 0\ , \tag{38}$$

where the shortening notation β_i for coefficients is used

$$\beta_1 = c_1 + \frac{\beta_2}{2} - \beta_3, \quad \beta_2 = \sqrt{\frac{2}{3}}\,c_2, \quad \beta_3 = -\delta\sqrt{\frac{2}{3}}\,c_2^*\ . \tag{39}$$

Equation (38) may be presented symbolically as $(\Gamma_\mu\partial_\mu + m)\Psi = 0$, its more detailed form is

$$\left\{\left[\beta_3\gamma_\nu\otimes(e^{\mu,\nu})_{\rho,\sigma} + \frac{\beta_1}{4}\gamma_\lambda\gamma_\mu\gamma_\nu\otimes(e^{\lambda,\nu})_{\rho,\sigma} - \frac{\beta_2}{2}\gamma_\nu\otimes(e^{\nu,\mu})_{\rho,\sigma}\right]\frac{\partial}{\partial x^\mu} + M\delta_{\rho,\sigma}\right\}\Psi_\sigma = 0\ . \tag{40}$$

5. Transforming the wave equation to the Petras structure

The structure of the matrices Γ_μ in eq. (40)

$$\beta_3\gamma_\nu\otimes e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\lambda\otimes e^{\rho,\lambda} - \frac{\beta_2}{2}\gamma_\nu\otimes e^{\nu,\mu} \tag{41}$$

is rather complicated: it includes triple products of the Dirac matrices. There exists special transformation which reduces the wave equation to a form without such triple products. New basis and respective wave equation should be determined as follows:

$$\Psi' = R\Psi, \quad \Gamma'_\mu = R\Gamma_\mu R^{-1},$$

$$R = I\otimes I + a\gamma_\rho\gamma_\sigma\otimes e^{\rho,\sigma}, \quad R^{-1} = I\otimes I + b\gamma_\rho\gamma_\sigma\otimes e^{\rho,\sigma}, \quad b = -\frac{1}{1+4a}\ . \tag{42}$$

Let us find

$$R\Gamma_\mu = [\beta_3\gamma_\nu\otimes e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\lambda\otimes e^{\rho,\lambda} - \frac{\beta_2}{2}\gamma_\nu\otimes e^{\nu,\mu}]$$
$$+a[\beta_3\gamma_\rho\gamma_\sigma\gamma_\nu\otimes e^{\rho,\sigma}e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\sigma\gamma_\eta\gamma_\mu\gamma_\lambda\otimes e^{\rho,\sigma}e^{\eta,\lambda} - \frac{\beta_2}{2}\gamma_\rho\gamma_\sigma\gamma_\nu\otimes e^{\rho,\sigma}e^{\nu,\mu}].$$

Taking into account the identities $e^{\rho,\sigma}e^{\mu,\nu} = \delta_{\sigma\mu}e^{\rho,\nu}$, $\gamma_\rho\gamma_\rho = 4$ we get

$$R\Gamma_\mu = \beta_3\gamma_\nu\otimes e^{\mu,\nu} - \beta_2(\frac{1}{2} + 2a)\gamma_\rho\otimes e^{\rho,\mu} + (\frac{\beta_1}{4} + a\beta_3 + a\beta_1)\gamma_\rho\gamma_\mu\gamma_\sigma\otimes e^{\rho,\sigma}\ . \tag{43}$$

Similarly, we derive the structure of the matrices Γ'_μ:

$$R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu}$$
$$+\left\{\frac{\beta_1}{4} + a(\beta_1+\beta_3) + b(\beta_1 - \frac{\beta_2}{2}) + 4ab(\beta_1 - \frac{\beta_2}{2} + \beta_3)\right\}\gamma_\rho\gamma_\mu\gamma_\sigma \otimes e^{\rho,\sigma} \ . \tag{44}$$

Due to (42) we have $4ab = -a-b$, so the relation (44) reduces to the form

$$R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} + \left\{\frac{\beta_1}{4} + a\frac{\beta_2}{2} - b\beta_3\right\}\gamma_\rho\gamma_\mu\gamma_\sigma \otimes e^{\rho,\sigma} \ . \tag{45}$$

Now, we demand that the coefficient at triple product of Dirac matrices vanishes, this results in

$$\Gamma'_\mu = R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho}$$
$$-\beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} \ . \tag{46}$$

and parameters a and b must obey two constraints:

$$\frac{\beta_1}{4} + a\frac{\beta_2}{2} - b\beta_3 = 0 \quad \Longrightarrow$$

$$b = \frac{\beta_1 + 2a\beta_2}{4\beta_3}, \quad a + b + 4ab = 0 \ .$$

Excluding b, we get a quadratic equation for a:

$$a^2 + 2a\frac{\beta_2 + 2(\beta_1+\beta_3)}{8\beta_2} + \frac{\beta_1}{8\beta_2} = 0 \ ;$$

its roots are

$$a = \frac{-C \pm \sqrt{C^2 - 8\beta_1\beta_2}}{8\beta_2}, \tag{47}$$

with designation $C = \beta_2 + 2\beta_1 + 2\beta_3$. Respective expressions for b are:

$$b = \frac{1}{4\beta_3}(\beta_1 + 2\beta_2 a)$$
$$= \frac{-(\beta_2 + 2\beta_3 - 2\beta_1) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{16\beta_3} \tag{48}$$

Now, turning to the formula (46),

$$\Gamma'_\mu = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} \ ,$$

we find

$$\beta_3(1+4b) = \frac{-(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{4} \equiv (+A+B) \ ,$$
$$\beta_2(\frac{1}{2}+2a) = \frac{+(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{4} \equiv (-A+B) \ ; \tag{49}$$

notice that for B we have two different in sign expressions (for definiteness, below we use the variant with the upper sign) Thus, in Petras basis, the matrices Γ'_μ may be written in a rather symmetric form

$$\Gamma'_\mu = (A+B)\gamma_\rho \otimes e^{\mu,\rho} + (A-B)\gamma_\rho \otimes e^{\rho,\mu} + M\Psi = 0 \ ; \tag{50}$$

correspondingly, the wave equation reads

$$(A+B)\ \gamma_\rho\partial_\mu\Psi_\rho + (A-B)\ \gamma_\mu\ \partial_\rho\Psi_\rho + M\ \Psi_\mu = 0 \ . \tag{51}$$

6. On parametrization of possible mass values

Recall main notations for parameters:

$$M_1 = \frac{M}{\lambda_1}\,, \quad M_2 = \frac{M}{\lambda_2}\,, \quad M > 0\,,$$

$$\lambda_1 = \frac{c_1 + \sqrt{c_1^2 + 4\delta|c_2|^2}}{2}, \quad \lambda_2 = \frac{c_1 - \sqrt{c_1^2 + 4\delta|c_2|^2}}{2}\,, \tag{52}$$

$$\lambda_1\lambda_2 = -\delta|c_2|^2, \quad \lambda_1 + \lambda_2 = c_1, \quad \lambda_1 - \lambda_2 = \sqrt{c_1^2 + 4\delta|c_2|^2}\,; \tag{53}$$

also

$$\beta_2 = \sqrt{\frac{2}{3}}\,c_2, \quad \beta_3 = -\delta\sqrt{\frac{2}{3}}\,c_2^*, \quad \beta_1 = (c_1 + \frac{\beta_2}{2} - \beta_3)\,; \tag{54}$$

and parameters A and B

$$\frac{1}{4}\left[-(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2}\,\right] \equiv +A + B\,,$$

$$\frac{1}{4}\left[+(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2}\,\right] \equiv -A + B\,. \tag{55}$$

First, we consider the model when $\delta = -1,\ c_1 = \rho,\ c_2 = c_2^* = \sigma$, in this case we have

$$\beta_2 = \sqrt{\frac{2}{3}}\sigma,\ \beta_3 = \sqrt{\frac{2}{3}}\sigma\,,\ \beta_1 = \rho - \frac{1}{2}\sqrt{\frac{2}{3}}\sigma\,,$$

$$\lambda_1 = \frac{\rho + \sqrt{\rho^2 - 4\sigma^2}}{2} > 0\,, \quad \lambda_2 = \frac{\rho - \sqrt{\rho^2 - 4\sigma^2}}{2} > 0\,,$$

$$\lambda_1\lambda_2 = \sigma^2,\ \lambda_1 + \lambda_2 = \rho,\ \lambda_1 - \lambda_2 = \sqrt{\rho^2 - 4\sigma^2}\,,$$

and

$$(\beta_2 - 2\beta_1 - 2\beta_3) = -2\rho\,, \quad (\beta_2 + 2\beta_1 + 2\beta_3) = 2(\rho + \sqrt{\frac{2}{3}}\sigma)\,,$$

$$\sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2} = 2\sqrt{\rho^2 + \frac{4}{3}\sigma^2}\,. \tag{56}$$

So, the expression for A, B in terms of λ_1, λ_2 read

$$A = \frac{\rho}{2} = \frac{\lambda_1 + \lambda_2}{2},\ \ B = \frac{\sqrt{\rho^2 + (4/3)\sigma^2}}{2} = \frac{\sqrt{(\lambda_1 + \lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2}\,. \tag{57}$$

The ratio of two masses is

$$M_1 = \frac{M}{\lambda_1}, \quad M_2 = \frac{M}{\lambda_2}, \quad \frac{M_2}{M_1} = \frac{\lambda_1}{\lambda_2} = \frac{1 + \sqrt{1 - 4\sigma^2/\rho^2}}{1 - \sqrt{1 - 4\sigma^2/\rho^2}}\,. \tag{58}$$

It is convenient to introduce an angular parametrization

$$\sin^2\gamma = \frac{4\sigma^2}{\rho^2}, \quad 4\sigma^2 \le \rho^2\,, \quad \gamma \in (0, \frac{\pi}{2})\,; \tag{59}$$

then the ratio of the masses is given by

$$\frac{M_2}{M_1} = \frac{1+\cos\gamma}{1-\cos\gamma} = \frac{1}{\tan^2(\gamma/2)} \in (1,\infty) \; . \tag{60}$$

Now, let us examine the second model, when $\delta = +1, \; c_1 = \rho, \; c_2 = c_2^* = \sigma$, for this case we have

$$\begin{gathered} \beta_2 = \sqrt{\frac{2}{3}}\sigma, \quad \beta_3 = -\sqrt{\frac{2}{3}}\sigma \; , \quad \beta_1 = \rho + \frac{3}{2}\sqrt{\frac{2}{3}}\sigma, \\ \lambda_1 = \frac{\rho + \sqrt{\rho^2+4\sigma^2}}{2}, \quad \lambda_2 = \frac{\rho - \sqrt{\rho^2+4\sigma^2}}{2} < 0, \\ \lambda_1\lambda_2 = -\sigma^2, \quad \lambda_1+\lambda_2 = \rho, \quad \lambda_1 - \lambda_2 = \sqrt{\rho^2+4\sigma^2} \; ; \end{gathered} \tag{61}$$

and

$$\begin{gathered} (\beta_2 - 2\beta_1 - 2\beta_3) = -2\rho \; , \qquad (\beta_2 + 2\beta_1 + 2\beta_3) = 2(\rho + \sqrt{\frac{2}{3}}\sigma) \; , \\ \sqrt{(\beta_2+2\beta_1+2\beta_3)^2 - 8\beta_1\beta_2} = 2\sqrt{\rho^2 - \frac{4}{3}\sigma^2} \; . \end{gathered} \tag{62}$$

So, the expressions for A, B in terms of λ_1, λ_2 are

$$A = \frac{\rho}{2} = \frac{\lambda_1+\lambda_2}{2}, \quad B = \frac{\sqrt{\rho^2-(4/3)\sigma^2}}{2} = \frac{\sqrt{(\lambda_1+\lambda_2)^2+(4/3)\lambda_1\lambda_2}}{2} \; . \tag{63}$$

The ratio of two mass is

$$M_1 = \frac{M}{\lambda_1} > 0, \quad M_2 = \frac{M}{\lambda_2} < 0, \quad \frac{M_2}{M_1} = \frac{\lambda_1}{\lambda_2} = \frac{\rho+\sqrt{\rho^2+4\sigma^2}}{\rho-\sqrt{\rho^2+4\sigma^2}} = \frac{1+\sqrt{1+4\sigma^2/\rho^2}}{1-\sqrt{1+4\sigma^2/\rho^2}} < 0 \; . \tag{64}$$

Let us introduce the following parametrization

$$\sinh^2\Gamma = 4\sigma^2/\rho^2 \; , \quad \Gamma \in (0,\infty) \; , \quad \frac{M_2}{M_1} = \frac{1+\cosh\Gamma}{1-\cosh\Gamma} = -\frac{1}{\tanh^2(\Gamma/2)} \in (-\infty,-1) \; . \tag{65}$$

In the paper we follow only the case of positive masses.

7. Independent components of the wave function

From this point, having in mind further extension of the model to a generally covariant case, we use the metrical tensor in Minkowski space with the signature $(+,-,-,-)$. Correspondingly, the wave equation (51) is written as

$$(A+B)\,\gamma^\rho\partial_\mu\Psi_\rho \quad + (A-B)\,\gamma_\mu\,\partial^\rho\Psi_\rho + iM\;\Psi_\mu = 0 \tag{66}$$

where we use Dirac matrices taken in the spinor basis, and the wave function is the vector-bispinor

$$\Psi_\rho(x) = \left| \begin{array}{cccc} \xi_0(x) & \xi_1(x) & \xi_2(x) & \xi_3(x) \\ \eta_0(x) & \eta_1(x) & \eta_2(x) & \eta_3(x) \end{array} \right|, \quad \gamma^\rho = \left| \begin{array}{cc} 0 & \bar{\sigma}^\rho \\ \sigma^\rho & 0 \end{array} \right| .$$

First, we convolute eq. (66) with γ^ν:

$$4(A-B)(\partial^\rho\Psi_\rho) + [(A+B)\hat{\partial} + iM](\gamma^\rho\Psi_\rho) = 0, \hat{\partial} = \gamma^\nu\partial_\nu \; , \tag{67}$$

and second, we act on eq. (66) by the operator ∂^ν:

$$(A+B)\Box(\gamma^\rho\Psi_\rho) + [\;(A-B)\hat{\partial} + iM\;](\gamma^\rho\Psi_\rho) = 0 \;, \Box = \partial^\nu\partial_\nu \; . \tag{68}$$

In order to exclude the term with the operator $\Box$, we should make additional calculation. Let us act on eq. (67) by the operator $\hat{\partial}$; taking into account the following identity

$$\hat{\partial}\hat{\partial} = \gamma^\alpha\partial_\alpha\gamma^\beta\partial_\beta = \partial_\alpha\partial_\beta\left(\frac{\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha}{2} + \frac{\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha}{2}\right) = \Box$$

we produce

$$(A+B)\Box(\gamma_\rho\Psi_\rho) = -4(A-B)\hat{\partial}(\gamma_\rho\Psi_\rho) - iM\hat{\partial}(\gamma_\rho\Psi_\rho) \; . \tag{69}$$

Now, we can substitute the relation into (68):

$$[\;3(A-B)\hat{\partial} - iM\;]\;(\partial^\rho\Psi_\rho) + iM\hat{\partial}(\gamma^\rho\Psi_\rho) = 0 \; , \tag{70}$$

additionally let us write down eq. (67) as

$$[\;(A+B)\hat{\partial} + iM\;]\;(\gamma^\rho\Psi_\rho) + 4(A-B)(\partial^\rho\Psi_\rho) = 0 \; . \tag{71}$$

It is convenient to introduce special notation for two bispinor functions:

$$(\gamma^\rho\Psi_\rho) = \Phi_1, \qquad (\partial^\rho\Psi_\rho) = \Phi_2 \; , \tag{72}$$

then the system (70)–(71) is written as follows

$$[\;3(A-B)\hat{\partial} - iM\;]\Phi_2 + iM\hat{\partial}\Phi_1 = 0 \; , \tag{73}$$

$$[\;(A+B)\hat{\partial} + iM\;]\;\Phi_1 + 4(A-B)\Phi_2 = 0 \; . \tag{74}$$

Let us make some transformations over these two equations (73)–(74); multiply the first one by $4(A-b)$, and the second one – by iM, and sum up the results (the fist equation (73) of the system remains the same); in this way we arrive at a new system

$$(5A-3B)\hat{\partial}\;\Phi_1 + \frac{12}{iM}(A-B)^2\hat{\partial}\;\Phi_2 + iM\;\Phi_1 = 0 \; , \tag{75}$$

$$-iM\hat{\partial}\;\Phi_1 - 3(A-B)\hat{\partial}\;\Phi_2 + iM\;\Phi_2 = 0 \; . \tag{76}$$

The last system may be presented in the matrix form

$$\hat{\partial}\begin{vmatrix} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{vmatrix}\begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} + iM\begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} = 0 \; . \tag{77}$$

The matrix W in the equation

$$W = \begin{vmatrix} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{vmatrix}, \qquad \hat{\partial}\;W\Phi + iM\Phi = 0$$

is to be reduced to a diagonal form with the help of a linear transformation in 2-dimensional space:

$$\left| \begin{array}{c} \Phi_1' \\ \Phi_2 \end{array} \right| = \left| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right| \left| \begin{array}{c} \Phi_1' \\ \Phi_2 \end{array} \right| , \qquad \Phi' = S\Phi, \quad \hat{\partial}(SWS^{-1})\Phi' + iM\Phi' = 0 \; ;$$

further we get $SW = W'S$, $\quad W' = \mathbf{diag}(W_1, W_2)$, or

$$\left| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right| \left| \begin{array}{cc} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{array} \right| = \left| \begin{array}{cc} W_1 & 0 \\ 0 & W_2 \end{array} \right| \left| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right| .$$

This is equivalent to linear sub-systems:

$$[(5A-3B) - R_1]s_{11} - iMs_{12} = 0 \; , \;\; \frac{12}{iM}(A-B)^2 s_{11} + [-3(A-B) - R_1]s_{12} = 0 \; ;$$

$$[(5A-3B) - R_2]s_{21} - iMs_{22} = 0 \; , \;\; \frac{12}{iM}(A-B)^2 s_{21} + [-3(A-B) - R_2]s_{22} = 0 \; .$$

From vanishing of the determinant of the matrix

$$\det \left| \begin{array}{cc} (5A-3B) - W_1 & (12/iM)(A-B)^2 \\ -iM & -3(A-B) - W_2 \end{array} \right| = 0$$

we obtain two eigenvalues W_1 and W_2:

$$W^2 - 2AW - 3(A^2 - B^2) = 0, \;\; W_1 = A - \sqrt{4A^2 - 3B^2} \; , \; W_2 = A + \sqrt{4A^2 - 3B^2}.$$

To fix elements of the matrix S, it suffices to use only one equation from each subsystem:

$$[(5A-3B) - R_1]s_{11} - iMs_{12} = 0 \; , \quad [(5A-3B) - R_2]s_{21} - iMs_{22} = 0 \; ;$$

its solution (one from possible) reads

$$s_{11} = 1, \quad s_{12} = \frac{5A - 3B - W_1}{iM} = \frac{4A - 3B + \sqrt{4A^2 - 3B^2}}{iM} \; ;$$

$$s_{21} = 1, \qquad s_{22} = \frac{5A - 3B - W_2}{iM} = \frac{4A - 3B - \sqrt{4A^2 - 3B^2}}{iM} \; .$$

Thus, we have arrived to separate Dirac-like equations

$$(\gamma^\nu \partial_\nu + i\frac{M}{W_1})\Phi_1' = 0 \; ,$$
$$(\gamma^\nu \partial_\nu + i\frac{M}{W_2})\Phi_2' = 0 \; ; \qquad (78)$$

these are what we need, because two identities are readily verified $W_1 \equiv \lambda_1, \; W_2 = \lambda_2$. New (primed) bispinors relate to initial ones by the formulas

$$\Phi_1' = \Phi_1 + \frac{5A - 3B - \lambda_1}{iM}\Phi_2,$$
$$\Phi_2' = \Phi_1 + \frac{5A - 3B - \lambda_2}{iM}\Phi_2 \; , \qquad (79)$$

where

$$\Phi_1 = \gamma^\rho \Psi_\rho, \quad \Phi_2 = \partial^\rho \Psi_\rho.$$

Initial wave equation (66), being written in the form

$$(A+B)\partial_\nu \Phi_1 + (A-B)\gamma_\nu \Phi_2 + iM\Psi_\nu = 0\ , \tag{80}$$

provides us with possibility to determine the complete 16-component vector-bispinor Ψ through the known bispinors Φ_1 and Φ_2.

8. Interaction with external fields

We start with equation in Minkowski space

$$(A+B)\ \gamma^\rho \partial_\nu \Psi_\rho + (A-B)\ \gamma_\nu\ \partial^\rho \Psi_\rho + iM\ \Psi_\nu = 0\ . \tag{81}$$

Extension to a generally covariant case (we are to use the tetrad formalism [51]) and to presence of an external electromagnetic field can be performed as follows

$$(A+B)\ D_\nu \gamma^\rho(x)\Psi_\rho(x) + (A-B)\ \gamma_\nu(x) D^\rho \Psi_\rho + iM\ \Psi_\nu(x) = 0\ ; \tag{82}$$

we use notations [51]

$$D_\alpha = \nabla_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x)\ , \qquad \hat{D} = \gamma^\alpha(x) D_\alpha, \quad D^\alpha D_\alpha = \Box\ . \tag{83}$$

Note two commutation rules [51]

$$\gamma^\rho(x) D_\nu = D_\nu \gamma^\rho(x)\ , \qquad D_\sigma g_{\alpha\beta}(x) = g_{\alpha\beta}(x) D_\sigma\ .$$

First, we convolute eq. (82) with γ^ν:

$$4(A-B)(D^\rho \Psi_\rho) + [\ (A+B)\hat{D} + iM\ ]\ (\gamma^\rho \Psi_\rho) = 0\ . \tag{84}$$

Act on eq. (82) by the operator D^ν, this results in

$$(A+B)\Box(\gamma^\rho \Psi_\rho) + [\ (A-B)\hat{D} + iM\ ](D^\rho \Psi_\rho) = 0\ . \tag{85}$$

In order to exclude the term with the operator $\Box$, we should make additional calculation. Let us act on eq. (84) by the operator $\hat{D}$; taking into account the following identity

$$(A+B)\hat{D}\hat{D}(\gamma^\rho \Psi_\rho) = -4(A-B)\hat{D}(D^\rho \Psi_\rho) - iM\hat{D}(\gamma^\rho \Psi_\rho)\ . \tag{86}$$

Taking into account the identity; (see [51]; $F_{\alpha\beta}(x)$ stands for the tensor of an external electromagnetic field $R(x)$ is the Ricci scalar)

$$\hat{D}\hat{D} = D_\alpha D_\beta \left[\frac{\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha}{2} + \frac{\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha}{2} \right] = \Box - \Sigma(x)\ , \quad \Sigma(x) = \left\{ -ieF_{\alpha\beta}\sigma^{\alpha\beta}(x) + \frac{R}{4} \right\}, \tag{87}$$

we derive the formula

$$(A+B)\Box(\gamma^\rho \Psi_\rho) = -4(A-B)\hat{D}(D^\rho \Psi_\rho) - iM\hat{D}(\gamma^\rho \Psi_\rho) + (A+B)\Sigma(x)(\gamma^\rho \Psi_\rho)\ . \tag{88}$$

Using the last relation, we may exclude form (85) the term with the operator $\Box$.

In this way we obtain two equations

$$(A+B)\hat{D}\Phi_1 + iM\Phi_1 + 4(A-B)\Phi_2 = 0\,, \tag{89}$$

$$3(A-B)\hat{D}\Phi_2 - iM\Phi_2 + iM\hat{D}\Phi_1 + (A+B)\Sigma(x)\Phi_1 = 0\,; \tag{90}$$

where the notations are used

$$\gamma^{\rho}(x)\Psi_{\rho}(x) = \Phi_1(x), \qquad D^{\rho}(x)\Psi_{\rho}(x) = \Phi_2(x) \tag{91}$$

In (89)-(90), let us multiply the second equation by $4(A-B)$, multiply by iM the first equation, and sum up the results; the second equation remains the same. In this way, we derive the system

$$\begin{gathered}\hat{D}\,\left\{(5A-3B)\Phi_1 + \frac{12}{iM}(A-B)^2\Phi_2\right\} + iM\Phi_1 + \frac{4(A^2-B^2)}{iM}\Sigma(x)\Phi_1 = 0,\\ \hat{D}\,\{-iM\Phi_1 - 3(A-B)\Phi_2\} + iM\Phi_2 - (A+B)\Sigma(x)\Phi_1 = 0.\end{gathered} \tag{92}$$

In matrix form, the system reads

$$\hat{D}W\begin{vmatrix}\Phi_1\\ \Phi_2\end{vmatrix} + iM\times I_2\begin{vmatrix}\Phi_1\\ \Phi_2\end{vmatrix} + (A+B)\Sigma(x)\begin{vmatrix}V_1 & 0\\ V_2 & 0\end{vmatrix}\begin{vmatrix}\Phi_1\\ \Phi_2\end{vmatrix} = 0\,, \tag{93}$$

where

$$W = \begin{vmatrix}(5A-3B) & 12(A-B)^2/iM\\ -iM & -3(A-B)\end{vmatrix}, \quad V = \begin{vmatrix}\beta & 0\\ -1 & 0\end{vmatrix} = \begin{vmatrix}\frac{4(A-B)}{iM} & 0\\ -1 & 0\end{vmatrix}. \tag{94}$$

Linear transformation over function Φ_1, Φ_2, which reduces the system to a diagonal form, is known (see in previous section). Symbolically, the problem under consideration is presented as

$$\hat{D}W\Phi + iM\ I_2\ \Phi + (A+B)\Sigma(x)\ V\Phi = 0, \quad \Phi' = S\Phi\,,$$

$$\hat{D}W'\Phi' + iM\ I_2\ \Phi' + (A+B)\Sigma(x)\ V'\Phi' = 0\,,$$

$$S = \begin{vmatrix}1 & a\\ 1 & b\end{vmatrix}, \quad S^{-1} = \frac{iM}{\lambda_1-\lambda_2}\begin{vmatrix}b & -a\\ -1 & 1\end{vmatrix}, \quad a = \frac{5A-3B-\lambda_1}{iM},\ b = \frac{5A-3B-\lambda_2}{iM}\,,$$

$$W' = SWS^{-1} = \begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{vmatrix}, \quad V' = SVS^{1} = \frac{iM}{\lambda_1-\lambda_2}\begin{vmatrix}b(\beta-a) & -a(\beta-a)\\ b(\beta-b) & -a(\beta-b)\end{vmatrix}.$$

In this way, we arrive at a simplified system

$$\begin{gathered}(\lambda_1\hat{D} + iM)\Phi_1' + \Sigma(x)(A+B)\frac{iM}{\lambda_1-\lambda_2}(\beta-a)(b\Phi_1' - a\Phi_2') = 0\,,\\ (\lambda_2\hat{D} + iM)\Phi_2' + \Sigma(x)(A+B)\frac{iM}{\lambda_1-\lambda_2}(\beta-b)(b\Phi_1' - a\Phi_2') = 0\,.\end{gathered} \tag{95}$$

Recall that

$$\Sigma(x) = -ieF_{\alpha\beta}(x)\sigma^{\alpha\beta}(x) + \frac{R(x)}{4}\,, \quad \beta = \frac{4(A-B)}{iM}\,.$$

Also notice that equation (82) written in the form

$$i(A+B)D_\nu\Phi_1 + i(A-B)\gamma_\nu(x)\Phi_2 - M\Psi_\nu = 0\ , \tag{96}$$

gives possibility to determine 16-component vector-bispinor Ψ_ν through bispinors Φ_1 and Φ_2.

Taking in mind the identities

$$iM(\beta - a) = iM[\frac{4(A-B)}{iM} - \frac{5A-3B-\lambda_1}{iM}] = \lambda_1 - A - B\ ,$$

$$iM(\beta - b) = iM[\frac{4(A-B)}{iM} - \frac{5A-3B-\lambda_2}{iM}] = \lambda_2 - A - B\ ,$$

we may rewrite equations (95) in a different way

$$\begin{aligned}(\lambda_1\hat{D} + iM)\Phi_1' + \Sigma(x)(A+B)\frac{\lambda_1 - A - B}{\lambda_1-\lambda_2}(b\Phi_1' - a\Phi_2') = 0\ ,\\ (\lambda_2\hat{D} + iM)\Phi_2' + \Sigma(x)(A+B)\frac{\lambda_2 - A - B}{\lambda_1-\lambda_2})(b\Phi_1' - a\Phi_2') = 0\ .\end{aligned} \tag{97}$$

Let us introduce shortening notations

$$(A+B)\frac{\lambda_1 - A - B}{\lambda_1(\lambda_1-\lambda_2)} = \Lambda_1, \qquad (A+B)\frac{\lambda_2 - A - B}{\lambda_2(\lambda_2-\lambda_1)} = \Lambda_2\ ; \tag{98}$$

then the above equations read

$$\begin{aligned}(i\hat{D} - M_1)\Phi_1' + \Sigma(x)\Lambda_1\ (b'\ \Phi_1' - a'\ \Phi_2') = 0\ ,\\ (i\hat{D} - M_2)\Phi_2' + \Sigma(x)\Lambda_2\ (b'\ \Phi_1' - a'\ \Phi_2') = 0\ ,\end{aligned} \tag{99}$$

where

$$a' = \frac{5A-3B-\lambda_1}{M}, \quad b' = \frac{5A-3B-\lambda_2}{M}\ ,$$

In the end, let us note that equations (99) allow for restrictions to Majorana case. Indeed, in any Majorana basis for Dirac matrices, $(i\gamma^a)^* = +(i\gamma^a)$, and real (imaginary) bispinors are determined by the formulas

$$(\Phi_1')^* = \pm\ (\Phi_1'), \quad (\Phi_2')^* = \pm\ (\Phi_2')\ . \tag{100}$$

Because such fields correspond to neutral particles, the term with $F_{\alpha\beta}$ vanishes and we have the identity $\Sigma^*(x) = +\Sigma(x)$. So we conclude that equations (99) for particles with two masses preserve their Lorentz-invariant form for neutral Majorana particles as well.

9. Conclusion

In the paper, starting from the general Gel'fand-Yaglom approach a new 16-component wave equation for spin 1/2 fermion, which is characterized by two mass parameters, is derived. On the base of 16-component wave function, two auxiliary bispinors are introduced, they determine initial 16-component wave function, and in the absence of an external field for these bispinors we derive two separate Dirac-like equations with different masses M_1 and M_2.

It is shown that in the presence of external fields, electromagnetic field and gravitational non-Euclidean background with non-vanishing Ricci scalar curvature, the main equation is not split into separated equations, instead a quite definite mixing of two Dirac-like equations arises.

It is shown that a generalized equation for Majorana particle with two mass parameters exists as well, such a generalized Majorana equation is not split into two separated equations if Ricci scalar of space-time model does not vanish.

It is desirable to get explicit solutions of such generalized wave equations in the presence of some external fields: magnetic, electric, or gravitational ones.

Also, it is desirable to elaborate a model for a spin $1/2$ particle with three mass parameters, as more interesting physically in the context of three type of neutrinos with different masses.

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