Nonlinear Phenomena in Complex Systems, vol. **21**, no. 4 (2018), pp. 406 - 410

The Classical Energy–Momentum Problem and Fock Tensor in Relativistic Theory of Gravitation[§]

A. A. Leonovich*
University of Informatics and Radioelectronics,
68 P. Brovki Ave., 220013 Minsk, BELARUS

Yu. P. Vyblyi[†]
Institute of Physics, National Academy of Sciences of Belarus,
68 Nezavisimosti Ave., 220072 Minsk, BELARUS
(Received 15 September, 2018)

The problem of gravitational field energy-momentum as a classical high spin problem (spin 2) is discussed. In the frameworks of the relativistic theory of gravity, the spherical gravitational wave and Fock energy–momentum tensor is considered.

PACS numbers: 02.30.GP; 03.65.Ge; 04.30.Db
Keywords: gravitational field, spin 2, Fock energy–momentum tensor, gravitational wave

1. Introduction

Basic physical categories both in classical and quantum theories are that of the energy, momentum and angular momentum. General Relativity (GR) which is a classical theory of gravitation formulated by Einstein, is the only theory facing serious difficulties with the energy. Main roots of this difficulties lie in the Equivalence Principal which leads to the geometrization of a gravitational field. The detailed analysis of this situation is presented in [1].

GR is a gauge theory where a gauge field is presented with a metric tensor [2]. The Hilbert–Einstein action for the gravitational field can be formulated using the Riemann geometry. This action is the only physically consistent (gauge invariant) action when this theory describes selfacting massless field of spin 2.

A complicated problem of constructing physically correct (nonlinear, gauge invariant) field theory of selfacting spin 2 in Minkowski space–time is the well known classical energy-momentum problem of high spin theory [2], [3].

There is giant gap still between spin 1 and spin 2 description [2]. All known results refer to linearized theory, and without exaggeration we can say that until now nobody in any field-theory-like approaches has got positive value of energy density of strong gravitational waves, and this fact forms an essence of the problem of high spin theory.

Thus, there is a difference between physical description of spin 1 and spin 2. In the case of vector field the stress tensor F_{ik} and Hilbert energy-momentum tensor (EMT) does not contain the the second derivatives of the potentials A_i. The same is true for the Yang–Mills fields, which means that there is no energy problem for vector fields (spin 1).

Quiet different situation arises in the case of tensor fields (spin 2). Now the Christoffel symbols cannot be shorten in stress tensor and they are present in EMT, which contains now the second derivatives in respect to field components. In [2] is mentioned that there is no expression for symmetrical EMT with square dependence on fields and their first derivatives and the second derivatives necessarily present. This fact leads

[§]Delivered at 25th International Seminar "Nonlinear Phenomena in Complex Systems", 21–25 May, 2018, Minsk (Belarus)
*E-mail: kaffiz@bsuir.by
[†]E-mail: Vyblyi@gmail.com

to principal sign uncertainty in energy density of radiation and absence a square structure of energy expression – this is a high spin theory problem. There is the only one way to solve it within the frameworks of classical field theory – to find a formulation of a theory in which the second derivatives in EMT turn to zero under the fulfillment of equations of motions and additional conditions. This just the case of linear theory of spin 2 with transverse and traceless (TT) gauge. In the paper we will discuss any approaches to solving of this problem.

2. Maxwellization of Hilbert energy-momentum tensor for dynamical torsion in Riemann–Cartan geometry

It was shown that there is a fundamental relation between symmetrization of EMT and non-commutativity of the covariant derivatives operation and a simple algebraic method to obtaining the symmetric EMT in general case was proposed [4], [5].

Let us consider the curvature tensor in the Riemann–Cartan geometry

$$\tilde{R}^l_{ijk} = R^l_{ijk} + 2\nabla_{[i}T^l_{j]k} + 2T^l_{[i|m|}T^l_{j]k}, \quad (1)$$

where T_{ijk} is a contorsion tensor [6], ∇_i is Riemann covariant derivative and

$$\nabla_{[m}\tilde{R}_{ij]kn} - T^l_{[m|k|}\tilde{R}_{ij]ln} - T^l_{[m/n/}\tilde{R}_{ij]lk} \equiv 0. \quad (2)$$

Let us consider the gravitational Lagrangian $L = L(g_{ik}, \tilde{R}_{iklm})$. The Hilbert EMT is obtained by using the method developed earlier for Lagrange theories with the help of Lie derivatives taken with respect to Lagrangian

$$\nabla^i(\tilde{G}_{ij} - [L]^{kmn}S_{ij}T_{kmn}) \equiv -[L]^{kmn}\nabla_j T_{kmn}, \quad (3)$$

where

$$\tilde{G}_{ij} = X^{kmn}_i\tilde{R}_{jkmn} + X^{kmn}_j\tilde{R}_{ikmn} - g_{ij}L, \quad (4)$$

$$[L]_{jmn} = \nabla^i X_{ijmn} - T^{ks}_m X_{kjsn} + T^{ks}_n X_{kjsm} \quad (5)$$

is the Lagrange-Euler derivative and

$$X_{ijkl} = \frac{\partial L}{\partial \tilde{R}_{ijkl}}. \quad (6)$$

Now, we have the expansion $\tilde{G}_{ij} = G_{ij} + T_{ij}$, where G_{ij} is a generalized Einstein tensor, T_{ij} is a Hilbert EMT for dynamical tensor field T_{ijk}. The second derivatives $\nabla_i\nabla_j T_{klm}$ disappear under equations of motion $[L]_{ijk} = 0$. Hilbert EMT for dynamical contorsion assumes Maxwellian type under equations of motions. This is a solution of above mentioned problem of higher spin for contorsion field.

Analogously, Gilbert EMT for gauge tensor fields of Yang-Mills type in the Dirac–Kahler theory assumes Maxwellian type under equations of motions [7], [8], although contorsion tensor T_{ijk} in the Riemann–Cartan geometry and gauge tensor field A_{ijk} are differently included in corresponding covariant derivatives.

3. Spherical gravitational wave in relativistic theory of gravitation

As known, in General Relativity the energy of gravitational field is described with help the energy-momentum pseudotensor, but the expression for corresponding tensor is absent. It is one of reasons for which the gravitation interaction may regarded as tensor interaction in the Minkowski space–time. The most consistently such approach was realized in so-called relativistic theory of gravitation (RTG) [9], [10]. This theory can be regarded as a gauge theory of the group of Lie variations for dynamical variables. The related transformations are variations of the form of the function for generally covariant transformations. That the action be invariant for this group under the transformations of the dynamic variables alone requires replacing the "nondynamic" Minkowski metric γ^{ik} with expression g^{ik}: $\tilde{g}^{ik} = \sqrt{-g}g^{ik} = \sqrt{-\gamma}(\gamma^{ik} + k\psi^{ik})$, where $\gamma = \det\gamma_{ik}, g = \det g_{ik}$, k^2 is the

Einstein constant, and thus introducing the gauge gravitational potential ψ^{ik}. The expression g^{ik} is interpreted here as the metric of the effective space–time from which the connection, the Christoffel bracket, can be uniquely constructed. The RTG field equations in its massless variant are the Einstein ones for this effective metric, added the conditions, restricting the spin states of the field ψ^{ik}:

$$D_i \tilde{g}^{ik} = 0, \tag{7}$$

where D_i is the covariant derivative in the Minkowski space. This conditions play the significant role in RTG, removing the gauge arbitrariness of Einstein equations and they coincide with Fock harmonic conditions in Galilean coordinates [11].

Although RTG field equations coincide with General Relativity ones locally, its global solutions, generally speaking, will be different, since this solutions are defined on the various manifolds. RTG, founding on the simple space-time topology, allows to introduce the global Galilean coordinate system, that distinguish RTG from the bimetric theories, in which a flat space plays the auxiliary role and its topology don't definite the character of the physical processes. This distinction may take place at interpretation of the field solutions, since the coordinate system in RTG is defined by Minkowski metric, but it is fixed by noncovariant coordinate conditions in GR. Just this situations takes place for spherical-symmetric gravitational fields. In RG according the Birkhoff theorem [12] any spherical gravitational field in vacuum is static. The proof of this theorem is ground on the transformation of certain spherically-symmetric metric to the coordinates in which it has a static form. But in RTG such transformation is the transfer from the spherical coordinates in the Minkowski space with the metric $\gamma_{ik} = \mathrm{diag}(1, -1, -r^2, -r^2 \sin^2\theta)$ to some "nonstatic" coordinates. The Birkhoff theorem means that in the case of spherical symmetry the coordinate system in which the vacuum metric depends from one coordinate always exists, but it not means that the field was static in the starting coordinates.

Hence the task of the investigation of nonstatic spherical-symmetric solutions, which was in general view investigated in [13], arises. In this paper one of the possible nonstatic spherically-symmetric wave solution,containing one arbitrary function, is found in implicit form. The corresponding solution without this function was founded in [14].

To find the spherical wave solutions we use the Birkhoff theorem and present a nonstatic spherical vacuum solution in certain coordinate system (T, R, θ, ϕ) in the form of Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{R}\right) dT^2 - A^2(r)\left(1 - \frac{2m}{R}\right)^{-1} dR^2 - R^2 d\Omega^2 \ . \tag{8}$$

To find the solution in spherical coordinates of (t, r, θ, ϕ) we make the coordinate transformation

$$t = t(T, R), r = r(T, R) \ , \tag{9}$$

and transformation coefficients will be found from the condition (1). The corresponding equations connecting the variables (t, r) and (T, R) will have a form

$$\frac{R}{R-2m}\frac{\partial^2 t}{\partial T^2} - R^{-2}\partial_R\left[(R^2 - 2mR)\frac{\partial t}{\partial R}\right] = 0 \ , \tag{10}$$

$$\frac{R}{R-2m}\frac{\partial^2 r}{\partial T^2} - R^{-2}\partial_R\left[(R^2 - 2mR)\frac{\partial r}{\partial R}\right] + \frac{2r}{R^2} = 0 \ . \tag{11}$$

We will search for the partial solution of the equations (10, 11) in the next form

$$T = t, R = R(u), u = r_0 \ln\frac{r}{r_0} - t \ , \tag{12}$$

where r_0 is a certain parameter, u is the retarded argument, which is finite at any values of r; the light speed and gravitational constant are believed equal to unit. Finding with help of (8) the transformations coefficients and substituting its to the equations (10, 11), we find that the equation (10) is satisfied identically, and the equation (11) is reduced to ordinary differential equation for the function $R(u)$ (that is possible due to the choosing form of a solution):

$$\ddot{R} + \frac{R}{R-2m}\left(\frac{R}{R-2m} + \frac{2}{R^2}\right)\dot{R}^3 - \frac{2(R-m)}{R(R-2m)}\dot{R}^2 - \dot{R} = 0 \ , \qquad (13)$$

where the point denote the differentiation on the argument u.

The metric components g_{ik} in the spherical coordinates, are received with help of the corresponding transformation coefficients, have a form

$$g_{00} = 1 - \frac{2m}{R} - \frac{R\dot{R}^2}{A^2(R-2m)} \ , g_{01} = \frac{R\dot{R}^2}{A^2 r(R-2m)} \ ,$$

$$g_{11} = -\frac{R\dot{R}^2}{A^2 r^2 (R-2m)} \ , g_{22} = -R^2 \ ,$$

$$g_{33} = -R^2 \sin^2\theta \ . \qquad (14)$$

The finding of the metric may be used for investigation of a sign of gravitational radiate density t^{00}, which according to RTG has a form [9]

$$t^{00} = -\frac{1}{\sqrt{-\gamma}}\gamma^{ik} D_i D_k \tilde{g}^{00} \ . \qquad (15)$$

The problem concerning positive definite of gravitational energy density is not trivial, because in RTG the expression t^{00} don't possess square-law structure relative the field functions and its first derivatives and it contains second derivatives too. This situation is the consequence of above mentioned difference between spin one and spin two description.

The choice of function A is the additional gauge condition and it must be connect with the demand of positive definiteness of energy density (15). The choice of this additional conditions is not simple and is discussed in framework of General Relativity, for instance, in [15].

4. Fock EMT in relativistic theory of gravity

Let us consider Fock representation for Einstein tensor G^{ik}

$$2gG^{ik} = \partial_m \partial_n (\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}) + L^{ik}, \qquad (16)$$

where term L^{ik} contains the first derivatives only. Fock interpreted the expression

$$U^{ik} = \partial_m \partial_n (\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}) \qquad (17)$$

as weigh $+2$ density of energy-momentum pseudotensor of gravitational field, which equals to L^{ik} under field equations [11]. Thus, this pseudotensor may be expressed with help the first derivatives only. This pseudotensor is well-known Landau–Lifshitz one [16] and it was received its authors independently.

In relativistic theory of gravity we may to receive corresponding tensor density, replacing the partial derivatives to covariant ones in Minkowski space.

$$t_g^{ik} = \frac{1}{\sqrt{-\gamma}} D_m D_n (\tilde{g}^{ik}\tilde{g}^{mn} - \tilde{g}^{im}\tilde{g}^{kn}). \qquad (18)$$

This tensor may be received with help the procedure regarded in sec. 1 for Lagrangian

$$L = \tilde{R}(\tilde{g}^{ik}) + \frac{1}{\sqrt{-f}} R_{ijkl}(f_{mn})\tilde{g}^{ik}\tilde{g}^{jl}, \qquad (19)$$

where $\tilde{g}^{ik} = \tilde{f}^{ik} + \tilde{\psi}^{ik}$, f_{mn} is a background metric which is supposed equal to zero after variation. Analogous approach was used in [17] to obtain EMT for conform invariant scalar field. It is essentially that under field equations this tensor leads to the expression with the first derivatives only.

5. Conclusions

In the frameworks of the relativistic theory of gravitation the energy and momentum of the gravitational field may be correctly defined in a flat Minkowski space–time. The use of the flat background space–time allows to give a new interpretation of some solutions of the Einstein equations. In Relativistic theory of gravitation, spherically-symmetrical wave solutions have a physical sense as far as the temporal coordinate of the Minkowski space–time has it. Essentially that this wave possesses the positive-defined energy and momentum densities, and although the receiving solution has adequately formal character, it illustrates the possibility of the existence of spherical gravitational waves.

References

[1] V. I. Denisov and A. A. Logunov. *Modern problems of mathematics.* Itogi nauki i tehniki, vol. **21** (VINITI, Moscow, 1984). (in Russian).

[2] P. Penrouse and V. Rindler. *Spinors and space-time.* (Mir, Moscow, 1987). (in Russian).

[3] M.A. Vasiliev. *Higher-spin gauge theories.* Prepr. Lebedev Phys. Inst. (Lebedev Phys. Inst., Moscow, 1991).

[4] B.M. Barbashov, A.A. Leonovich, A.B. Pestov. Nucl. Phys. **38**, 261 (1983) (Russian).

[5] B. M. Barbashov and A. A. Leonovich. *Prepr. JINR P5-5-83-398.* (Dubna, 1983). (in Russian).

[6] J.A. Schouten and D.J. Struik. Introduction to New Methods of Differential Geometry.V.1. ONTI NKTP, M.-L.(1948), 176 p. (in Russian)

[7] A. A. Leonovich and A.B. Pestov. *Prepr. JINR P2-5-80-823.* (Dubna, 1980). (in Russian).

[8] A. A. Leonovich. TMF. **57**, 12 (1983). (in Russian).

[9] A.A. Logunov and M. A. Mestvirishvili. *The Relativistic Theory of Gravitation.* (Mir, Moscow, 1989).

[10] A.A. Logunov. *The Theory of Gravity.* (Nova Scifnce Publ., New York, 1998).

[11] V.A. Fock. *The Theory of Space, Time and Gravity.* (GIFML, Moscow, 1961). (in Russian).

[12] G.Birkhoff. *Relativity and Modern Physics.* (Harvard Un. Press, 1923).

[13] A.A. Vlasov. *Nongauge Approach to Relativistic Theory of Gravity.* (MSU Press, Moscow, 1992). (in Russian).

[14] A. Leonovich and Yu.P. Vyblyi. In: *Proc. of the IX Int. Conf. Bolyai-Gauss-Lobachevsky (BGL-9.* (Institute Physics Press, Minsk, 2015). P. 130.

[15] L.D. Faddeyev. Uspekhi Phys. Nauk. **136**(3), 435, (1982). (in Russian).

[16] L.D. Landau and E.M. Lifshitz. *Field theory.* 7-th edit. (Science, Moscow, 1979). (in Russian).

[17] N.A. Chernikov and E.A. Tagirov. Ann. Inst. H. Poincare. **A9**, 109 (1968).