Spin 1/2 particle with two masses in magnetic field

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Abstract. In the present paper, the generalized equation for spin 1/2 particle with two mass states is investigated in presence of an external uniform magnetic field. After the separation of variables in cylindric coordinates, the problem reduces to a system of 8 first order differential equations, from where follows the system of related four second order differential equations. After the diagonalization of the mixing term, the separate equations for four functions are derived, in which the spectral parameters coincide with the roots of the 4-th order polynomial. The solutions are constructed in terms of confluent hypergeometric functions, and the analytical formulas for the two series of energy spectrum are found in explicit form as solutions of 4-th order algebraic equations; however these prove to be cumbersome and useless for our purposes. The numerical study of the energy levels is performed depending the parameter γ, determining the mass values, on the magnitude of magnetic field and magnetic and main quantum numbers: $E = E_{1,2}(\gamma, B; m, n)$. In particular, it is shown that the physical energy spectrum for a two mass fermion differs significantly from the energy spectrum of an ordinary Dirac particle.

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1 Introduction

In [1], a model for spin 1/2 particle with two mass states is developed on the base of Gel'fand–Yaglom approach [2–4] in the theory of relativistic wave equations with extended sets of irreducible representations of the Lorentz group. In the end, the main generalized equation is presented in spin-tensor basis, and with the use of the Dirac matrices. Besides the 16-component wave function, we introduce two auxiliary bispinors, which determine initial 16-component wave functions, and in absence of external field for these bispinors we derive two separate Dirac-like equations with masses M_1 and M_2. It is shown that in presence of external fields, electromagnetic and

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gravitational non-Euclidean background, (with non-vanishing Ricci scalar curvature), the main wave equations do not split into separated wave equations; instead, a quite definite mixing of two Dirac-like equations with additional Pauli interactions terms arises. This mixing also holds in presence of only the electromagnetic field, and, as well, it holds in presence of the gravitational field.

We start with the wave equation with respect to the bispinor functions $\Psi_1(x)$ and $\Psi_2(x)$ – it has the structure[1]:

$$[\, i\hat{D} - M_1 + b\Lambda_1\Sigma(x)\,]\; \Psi_1(x) - a\Lambda_1\Sigma(x)\; \Psi_2(x) = 0\,,$$

$$[\, i\hat{D} - M_2 - a\Lambda_2\Sigma(x)\,]\; \Psi_2(x) + b\Lambda_2\Sigma(x)\; \Psi_1(x) = 0\,; \tag{1.1}$$

where the involved operators are defined by the formulas[2]

$$\hat{D}(x) = \gamma^{\alpha}(x) + \Gamma_{\alpha}(x) + ieA_{\alpha}\,, \quad \gamma^{\alpha}(x) - e^{\alpha}_{(b)}\gamma^{b}, \quad e/\hbar c \Longrightarrow e\,,$$

$$\Sigma(x) = -ieF_{\alpha\beta}\sigma^{\alpha\beta}(x)\,,\; \sigma^{\alpha\beta}(x) = \frac{\gamma^{\alpha}(x)\gamma^{\beta}(x) - \gamma^{\beta}(x)\gamma^{\alpha}(x)}{4}\,.$$

We use the following parameters (the quantities ρ, σ, which appeared while determining (1.1)):

$$\sin^2\gamma = \frac{4\sigma^2}{\rho^2}\,, \quad \gamma \in (0, \pi/2)\,,$$

$$\lambda_1 = \frac{\rho + \sqrt{\rho^2 - 4\sigma^2}}{2} = \frac{\rho}{2}(1 + \cos\gamma)\,, \quad \lambda_2 = \frac{\rho - \sqrt{\rho^2 - 4\sigma^2}}{2} = \frac{\rho}{2}(1 - \cos\gamma)\,,$$

$$M_1 = \frac{M}{\lambda_1} = \frac{M/\rho}{(1+\cos\gamma)/2}\,, \qquad M_2 = \frac{M}{\lambda_2} = \frac{M/\rho}{(1-\cos\gamma)/2}\,,$$

$$A' = \frac{\lambda_1 + \lambda_2}{2}\,, \qquad B' = \frac{\sqrt{(\lambda_1+\lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2}\,,$$

$$a = \frac{5A' - 3B' - \lambda_1}{M}\,, \qquad b = \frac{5A' - 3B' - \lambda_2}{M}\,,$$

$$\Lambda_1 = (A' + B')\frac{\lambda_1 - A' - B'}{\lambda_1(\lambda_1 - \lambda_2)}\,, \quad \Lambda_2 = (A' + B')\frac{\lambda_2 - A' - B'}{\lambda_2(\lambda_2 - \lambda_1)}\,. \tag{1.2}$$

2 The separation of variables

We use the known representation of the uniform magnetic field in cylindric coordinates

$$A_t = 0\,, \quad A_r = 0\,, \quad A_z = 0\,, \quad A_{\phi} = -\frac{Br^2}{2}\,. \tag{2.1}$$

We further express the diagonal tetrad with respect to the coordinates $x^{\alpha} = (t, r, \phi, z)$,

$$dS^2 = dt^2 - dr^2 - r^2\, d\phi^2 - dz^2\,, \quad e^{\beta}_{(a)}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[1]When describing spinor fields, we use the known tetrad based approach [5].
[2]We employ here the tetrad formalism.

The non-vanishing Christoffel symbols and the Ricci rotation coefficients are

$$\Gamma^{r}{}_{jk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Gamma^{\phi}{}_{jk} = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Gamma^{z}{}_{jk} = 0 \,, \; \gamma_{122} = \frac{1}{r} \,.$$

It is convenient to apply the following shortening notations[3]

$$\frac{eB}{\hbar c} \Longrightarrow B, \quad [B] = L^{-2}, \qquad \frac{mc}{\hbar} \Longrightarrow M, \quad [M] = L^{-1} \,.$$

The operator $\hat{D}(x)$ from (1.1) takes the form

$$\hat{D} = i\gamma^0 \partial_t + i\gamma^1 (\partial_r + \frac{1}{2r}) + \gamma^2 (\frac{i\partial_\phi}{r} + \frac{Br}{2}) + i\gamma^3 \partial_z \,,$$

and it may be simplified by the substitution $\Psi = \psi/\sqrt{r}$:

$$\hat{D} = i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 (\frac{i\partial_\phi}{r} + \frac{Br}{2}) + i\gamma^3 \frac{\partial}{\partial z} \,. \tag{2.2}$$

In the system (1.1)

$$[\, i\hat{D} - M_1 - ieb\Lambda_1 \sigma^{\alpha\beta}(x) F_{\alpha\beta} \,]\, \psi_1(x) + iea\Lambda_1 \; \sigma^{\alpha\beta}(x) F_{\alpha\beta} \; \psi_2(x) = 0 \,,$$

$$[\, i\hat{D} - M_2 + iea\Lambda_2 e\sigma^{\alpha\beta}(x) F_{\alpha\beta} \,]\, \psi_2(x) - ieb\Lambda_2 \; \sigma^{\alpha\beta}(x) F_{\alpha\beta} \; \psi_1(x) = 0 \,, \tag{2.3}$$

one may see two blocks which enter the usual Dirac equation with additional Pauli interaction terms.

For the case of a uniform field[4], we have

$$e\sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) = 2\sigma^{\phi r} e F_{\phi r} = iB\gamma^2\gamma^1 = iB\Sigma_3 \,,$$

and therefore the system is written as

$$[\, i\hat{D} - M_1 + 2b\Lambda_1 B\Sigma_3 \,]\, \psi_1(x) - a\Lambda_1 B\Sigma_3 \; \psi_2(x) = 0 \,, \tag{2.4}$$

$$[\, i\hat{D} - M_2 - 2a\Lambda_2 B\Sigma_3 \,]\, \psi_2(x) + b\Lambda_2 B\Sigma_3 \; \psi_1(x) = 0 \,. \tag{2.5}$$

We shall further use the following notations (take notice of the signs)

$$2b\Lambda_1 B = \Gamma_1 \,, \; a\Lambda_1 B = R_1 \,, \quad 2a\Lambda_2 B = -\Gamma_2 \,, \; b\Lambda_2 B = -R_2 \,, \tag{2.6}$$

$$\frac{m}{r} - \frac{Br}{2} \Longrightarrow \mu(r) \,, \qquad \frac{\epsilon}{\hbar c} \Longrightarrow \epsilon \,, \quad [\epsilon] = L^{-1} \,.$$

Thus, the system of equations reads

$$(i\hat{D} - M_1 + \Gamma_1 \Sigma_3)\psi_1 - R_1 \Sigma_3 \; \psi_2 = 0 \,, \tag{2.7}$$

$$(i\hat{D} - M_2 + \Gamma_2 \Sigma_3)\psi_2 - R_2 \Sigma_3 \; \psi_1 = 0 \,, \tag{2.8}$$

[3] The physical dimensions are indicated as well.

[4] Here the quantity e is absorbed by the symbol B.

and the parameters $\Gamma_{1,2}$ and $R_{1,2}$ have the physical dimension of length.

We apply the following substitution for the wave function $\psi = \{ \psi_1(x) \otimes \psi_2(x) \}$:

$$\psi_1 = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{pmatrix} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{pmatrix}, \quad \psi_2 = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{pmatrix} g_1(r) \\ g_2(r) \\ g_3(r) \\ g_4(r) \end{pmatrix}. \tag{2.9}$$

Considering the Dirac matrix in spinor basis, we derive 8 differential equations in the variable r:

$$\begin{aligned}
&-i(\frac{d}{dr} + \mu) f_4 + (\epsilon + k) f_3 + (\Gamma_1 - M_1) f_1 - R_1 g_1 = 0\,, \\
&-i(\frac{d}{dr} - \mu) f_3 + (\epsilon - k) f_4 - (\Gamma_1 + M_1) f_2 + R_1 g_2 = 0\,, \\
&+i(\frac{d}{dr} + \mu) f_2 + (\epsilon - k) f_1 + (\Gamma_1 - M_1) f_3 - R_1 g_3 = 0\,, \\
&+i(\frac{d}{dr} - \mu) f_1 + (\epsilon + k) f_2 - (\Gamma_1 + M_1) f_4 + R_1 g_4 = 0\,;
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&-i(\frac{d}{dr} + \mu) g_4 + (\epsilon + k) g_3 + (\Gamma_2 - M_2) g_1 - R_2 f_1 = 0\,, \\
&-i(\frac{d}{dr} - \mu) g_3 + (\epsilon - k) g_4 - (\Gamma_2 + M_2) f_2 + R_2 f_2 = 0\,, \\
&+i(\frac{d}{dr} + \mu) g_2 + (\epsilon - k) g_1 + (\Gamma_2 - M_2) g_3 - R_2 f_3 = 0\,, \\
&+i(\frac{d}{dr} - \mu) g_1 + (\epsilon + k) g_2 - (\Gamma_2 + M_2) g_4 + R_2 f_4 = 0\,.
\end{aligned} \tag{2.11}$$

3 The analysis of the radial equations

We introduce the notations

$$i(\frac{d}{dr} + \mu) = D_+\,, \quad i(\frac{d}{dr} - \mu) = D_-\,,$$

and re-group the equations (2.10)–(2.11) as follows

$$\begin{aligned}
(\epsilon + k) f_3 + (\Gamma_1 - M_1) f_1 - R_1 g_1 &= +D_+ f_4\,, \\
(\epsilon - k) f_1 + (\Gamma_1 - M_1) f_3 - R_1 g_3 &= -D_+ f_2\,, \\
(\epsilon + k) g_3 + (\Gamma_2 - M_2) g_1 - R_2 f_1 &= +D_+ g_4\,, \\
(\epsilon - k) g_1 + (\Gamma_2 - M_2) g_3 - R_2 f_3 &= -D_+ g_2\,;
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
(\epsilon - k) f_4 - (\Gamma_1 + M_1) f_2 + R_1 g_2 &= +D_- f_3\,, \\
(\epsilon + k) f_2 - (\Gamma_1 + M_1) f_4 + R_1 g_4 &= -D_- f_1\,, \\
(\epsilon - k) g_4 - (\Gamma_2 + M_2) g_2 + R_2 f_2 &= +D_- g_3\,, \\
(\epsilon + k) g_2 - (\Gamma_2 + M_2) g_4 + R_2 f_4 &= -D_- g_1\,.
\end{aligned} \tag{3.2}$$

These are two linear homogeneous systems with respect to f_1, f_3, g_1, g_3 and f_2, f_4, g_2, g_4.

With the notation

$$A'' = \Big\{ \; R_1^2 {R_2}^2 + 2[-(\Gamma_1 - M_1)(\Gamma_2 - M_2) + k^2 - \epsilon^2] R_2 R_1$$
$$+[(\Gamma_1 - M_1)^2 - \epsilon^2 + k^2][(\Gamma_2 - M_2)^2 - \epsilon^2 + k^2] \Big\}^{-1} \; ,$$

we can write the solution of the system (3.1) as

$$f_1 = A'' \Big\{ (\epsilon + k) \Big[R_1 \, R_2 + (\Gamma_2 - M_2)^2 - \epsilon^2 + k^2 \Big] \, D_+ f_2$$
$$+ \Big[(\Gamma_1 - M_1) \Big((\Gamma_2 - M_2)^2 - \epsilon^2 + k^2 \Big) - R_1 \, R_2 \; (\Gamma_2 - M_2) \Big] \, D_+ f_4$$
$$+ R_1 \; (\Gamma_2 - M_2 - M_1 + \Gamma_1) \, (\epsilon + k) \; D_+ g_2$$
$$+ \left[-{R_1}^2 R_2 + \left((\Gamma_2 - M_2) \, (\Gamma_1 - M_1) + \epsilon^2 - k^2 \right) R_1 \right] D_+ g_4 \Big\} \; ,$$

$$f_3 = A'' \; \Big\{ \Big[R_2 \; (\Gamma_2 - M_2) \, R_1 + (M_1 - \Gamma_1) \Big((\Gamma_2 - M_2)^2 - \epsilon^2 + k^2 \Big) \Big] \, D_+ f_2$$
$$+ (-\epsilon + k) \Big[R_1 \, R_2 + (\Gamma_2 - M_2)^2 - \epsilon^2 + k^2 \Big] \, D_+ f_4$$
$$+ \left[{R_1}^2 R_2 + R_1 \; \left(-(\Gamma_2 - M_2) \, (\Gamma_1 - M_1) + k^2 - \epsilon^2 \right) \right] D_+ g_2$$
$$+ R_1 \; (\Gamma_2 - M_2 + \Gamma_1 - M_1) \, (-\epsilon + k) \; D_+ g_4 \} \; ,$$

$$g_1 = A'' \; \{ R_2 \; (\Gamma_2 - M_2 + \Gamma_1 - M_1) \, (\epsilon + k) \; D_+ f_2$$
$$+ \left[-R_1 \, {R_2}^2 - R_2 \; \left(-(\Gamma_2 - M_2) \, (\Gamma_1 - M_1) + k^2 - \epsilon^2 \right) \right] D_+ f_4$$
$$+ \left[(\epsilon + k) \, R_1 \, R_2 + (\epsilon + k) \left((\Gamma_1 - M_1)^2 - \epsilon^2 + k^2 \right) \right] D_+ g_2$$
$$+ \Big[-R_1 \, R_2 \; (\Gamma_1 - M_1) + (\Gamma_2 - M_2) \Big((\Gamma_1 - M_1)^2 - \epsilon^2 + k^2 \Big) \Big] \, D_+ g_4 \Big\} \; ,$$

$$g_3 = A'' \; \left\{ \left[R_1 \, {R_2}^2 + R_2 \; \left(-(\Gamma_2 - M_2) \, (\Gamma_1 - M_1) + k^2 - \epsilon^2 \right) \right] D_+ f_2 \right.$$
$$+ R_2 \; (\Gamma_2 - M_2 + \Gamma_1 - M_1) \, (-\epsilon + k) \; D_+ f_4$$
$$+ \Big[R_1 \, R_2 \; (\Gamma_1 - M_1) + (-\Gamma_2 + M_2) \Big((\Gamma_1 - M_1)^2 - \epsilon^2 + k^2 \Big) \Big] \, D_+ g_2$$
$$+ (-\epsilon + k) \Big[R_1 \, R_2 + (\Gamma_1 - M_1)^2 - \epsilon^2 + k^2 \Big] \, D_+ g_4 \Big\} \; ;$$

Hence this solution has the structure

$$\begin{pmatrix} f_1 \\ f_3 \\ g_1 \\ g_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} D_+ f_2 \\ D_+ f_4 \\ D_+ g_2 \\ D_+ g_4 \end{pmatrix} . \tag{3.3}$$

Similarly, using the above notation, we infer

$$B'' = \Big\{ \; R_1^2 {R_2}^2 + 2\,[-(\Gamma_1 + M_1)(\Gamma_2 + M_2) - \epsilon^2 + k^2] R_1\, R_2$$

$$+ [-\epsilon^2 + k^2 + (\Gamma_1 + M_1)^2][(\Gamma_2 + M_2)^2 - \epsilon^2 + k^2] \; \Big\}^{-1} ,$$

and the solution of the system (3.2) is

$$f_2 = B'' \; \Big\{ (\epsilon - k) \Big[R_1\, R_2 + (\Gamma_2 + M_2)^2 - \epsilon^2 + k^2 \Big] \, D_- f_1$$

$$+ \Big[R_1\, R_2 \; (\Gamma_2 + M_2) - \Big((\Gamma_2 + M_2)^2 - \epsilon^2 + k^2 \Big) \, (\Gamma_1 + M_1) \Big] \, D_- f_3$$

$$+ \, (\Gamma_2 + M_2 + \Gamma_1 + M_1) \, (\epsilon - k) \, R_1 \, D_- g_1$$

$$+ \big[{R_1}^2 R_2 + \big(- (\Gamma_2 + M_2) \, (\Gamma_1 + M_1) - \epsilon^2 + k^2 \big) \, R_1 \big] \, D_- g_3 \Big\} \; ,$$

$$f_4 = B'' \; \Big\{ \Big[-R_1\, R_2 \; (\Gamma_2 + M_2) + \Big((\Gamma_2 + M_2)^2 - \epsilon^2 + k^2 \Big) \, (\Gamma_1 + M_1) \Big] \, D_- f_1$$

$$+ \Big[- (\epsilon + k) \, \Big(R_1\, R_2 + (\Gamma_2 + M_2)^2 - \epsilon^2 + k^2 \Big) \Big] \, D_- f_3$$

$$+ \big[-{R_1}^2 R_2 + \big((\Gamma_2 + M_2) \, (\Gamma_1 + M_1) + \epsilon^2 - k^2 \big) \, R_1 \big] \, D_- g_1$$

$$- \, (\Gamma_2 + M_2 + \Gamma_1 + M_1) \, (\epsilon + k) \, R_1 \, D_- g_3 \} \; ,$$

$$g_2 = B'' \; \{ - \, (\Gamma_2 + M_2 + \Gamma_1 + M_1) \, (-\epsilon + k) \, R_2 \, D_- f_1$$

$$+ \big[R_1 \, {R_2}^2 + \big(- \, (\Gamma_2 + M_2) \, (\Gamma_1 + M_1) - \epsilon^2 + k^2 \big) \, R_2 \big] \, D_- f_3$$

$$+ \, (\epsilon - k) \, \Big[R_1\, R_2 + (\Gamma_1 + M_1)^2 - \epsilon^2 + k^2 \Big] \, D_- g_1$$

$$+ \Big[R_1\, R_2 \; (\Gamma_1 + M_1) - \Big((\Gamma_1 + M_1)^2 - \epsilon^2 + k^2 \Big) \, (\Gamma_2 + M_2) \Big] \, D_- g_3 \Big\} \; ,$$

$$g_4 = B'' \; \big\{ \big[-R_1 \, {R_2}^2 + \big((\Gamma_2 + M_2) \, (\Gamma_1 + M_1) + \epsilon^2 - k^2 \big) \, R_2 \big] \, D_- f_1$$

$$- \, (\Gamma_2 + M_2 + \Gamma_1 + M_1) \, (\epsilon + k) \, R_2 \, D_- f_3$$

$$+ \Big[-R_1\, R_2 \; (\Gamma_1 + M_1) + \Big((\Gamma_1 + M_1)^2 - \epsilon^2 + k^2 \Big) \, (\Gamma_2 + M_2) \Big] \, D_- g_1$$

$$- \, (\epsilon + k) \, \Big[R_2\, R_1 + (\Gamma_1 + M_1)^2 - \epsilon^2 + k^2 \Big] \, D_- g_3 \Big\} \; ;$$

so that

$$\begin{pmatrix} f_2 \\ f_4 \\ g_2 \\ g_4 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{pmatrix} D_- f_1 \\ D_- f_3 \\ D_- g_1 \\ D_- g_3 \end{pmatrix} . \tag{3.4}$$

Combining the above results, we can derive the two systems of second order:

$$\begin{pmatrix} f_1 \\ f_3 \\ g_1 \\ g_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{pmatrix} D_+D_-f_1 \\ D_+D_-f_3 \\ D_+D_-g_1 \\ D_+D_-g_3 \end{pmatrix},$$

$$\begin{pmatrix} f_2 \\ f_4 \\ g_2 \\ g_4 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & A_{22} & A_{23} & A_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} D_-D_+f_2 \\ D_-D_+f_4 \\ D_-D_+g_2 \\ D_-D_+g_4 \end{pmatrix}.$$

We may follow only one case, e.g., the one used for solving the system for f_1, f_3, g_1, g_3. Its structure may be rewritten as

$$(3.5)\quad \begin{pmatrix} f_1 \\ f_3 \\ g_1 \\ g_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix} \begin{pmatrix} D_+D_-f_1 \\ D_+D_-f_3 \\ D_+D_-g_1 \\ D_+D_-g_3 \end{pmatrix}, \quad C = AB\,,$$

or, in matrix form,

$$(3.6)\qquad F = C\,(D_+D_-)\,F \quad \Longrightarrow \quad (D_+D_-)\,F = \Lambda\,F\,, \quad \Lambda = C^{-1}.$$

To compute the matrix $C = AB$ turns out to be rather complicated. However, finding its inverse is feasible[5], and we get:

$$\Lambda = C^{-1} = \begin{pmatrix} {\Gamma_1}^2 - {M_1}^2 + R_2R_1 + E^2 & 2(\epsilon+k)\Gamma_1 \\ -2(-\epsilon+k)\Gamma_1 & {\Gamma_1}^2 - {M_1}^2 + R_2R_1 + E^2 \\ -R_2(\Gamma_1+\Gamma_2-M_1+M_2) & -2R_2(\epsilon+k) \\ 2(-\epsilon+k)R_2 & -R_2(\Gamma_1+\Gamma_2-M_1+M_2) \end{pmatrix.$$

$$\begin{matrix} -R_1(\Gamma_1+\Gamma_2+M_1-M_2) & -2R_1(\epsilon+k) \\ 2R_1(-\epsilon+k) & -R_1(\Gamma_1+\Gamma_2+M_1-M_2) \\ {\Gamma_2}^2 - {M_2}^2 + R_2R_1 + E^2 & 2(\epsilon+k)\Gamma_2 \\ -2(-\epsilon+k)\Gamma_2 & {\Gamma_2}^2 - {M_2}^2 + R_2R_1 + E^2 \end{matrix}\Bigg)$$

$$(3.7)\qquad = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}.$$

We further need to diagonalize the matrix Λ:

$$D_+D_-F = \Lambda F, \quad F = TF', \quad T^{-1}D_+D_-TF' = T^{-1}\Lambda TF',$$

$$(3.8)\quad D_+D_-F' = (T^{-1}\Lambda T)F', \quad T^{-1}\Lambda T = \Lambda' = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

[5] We assume $\epsilon^2 - k^2 = E^2$, and write down the matrix by columns.

For the matrix T, we have the following equation

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

so we obtain four systems with respect to the columns 1,2,3,4 of the matrix T:

$$(s_{11} - \lambda_1)t_{11} + s_{12}t_{21} + s_{13}t_{31} + s_{14}t_{41} = 0 \,,$$

$$s_{21}t_{11} + (s_{22} - \lambda_1)t_{21} + s_{23}t_{31} + s_{24}t_{41} = 0 \,,$$
$$s_{31}t_{11} + s_{32}t_{21} + (s_{33} - \lambda_1)t_{31} + s_{34}t_{41} = 0 \,,$$
$$s_{41}t_{11} + s_{42}t_{21} + s_{43}t_{31} + (s_{44} - \lambda_1))t_{41} = 0 \,;$$

$$(s_{11} - \lambda_2)t_{12} + s_{12}t_{22} + s_{13}t_{32} + s_{14}t_{42} = 0 \,,$$
$$s_{21}t_{12} + (s_{22} - \lambda_2)t_{22} + s_{23}t_{32} + s_{24}t_{42} = 0 \,,$$
$$s_{31}t_{12} + s_{32}t_{22} + (s_{33} - \lambda_2))t_{32} + s_{34}t_{42} = 0 \,,$$
$$s_{41}t_{12} + s_{42}t_{22} + s_{43}t_{32} + (s_{44} - \lambda_2))t_{42} = 0 \,;$$

$$(s_{11} - \lambda_3)t_{13} + s_{12}t_{23} + s_{13}t_{33} + s_{14}t_{43} = 0 \,,$$
$$s_{21}t_{13} + (s_{22} - \lambda_3)t_{23} + s_{23}t_{33} + s_{24}t_{43} = 0 \,,$$
$$s_{31}t_{13} + s_{32}t_{23} + (s_{33} - \lambda_3)t_{33} + s_{34}t_{43} = 0 \,,$$
$$s_{41}t_{13} + s_{42}t_{23} + s_{43}t_{33} + (s_{44} - \lambda_3)t_{43} = 0 \,;$$

$$(s_{11} - \lambda_4)t_{14} + s_{12}t_{24} + s_{13}t_{34} + s_{14}t_{44} = 0 \,,$$
$$s_{21}t_{14} + (s_{22} - \lambda_4)t_{24} + s_{23}t_{34} + s_{24}t_{44} = 0 \,,$$
$$s_{31}t_{14} + s_{32}t_{24} + (s_{33} - \lambda_4)t_{34} + s_{34}t_{44} = 0 \,,$$
$$s_{41}t_{14} + s_{42}t_{24} + s_{43}t_{34} + (s_{44} - \lambda_4)t_{44} = 0 \,.$$

In fact, the systems have the same structure

$$\begin{aligned} &(s_{11} - \lambda_{(i)})x_1 + s_{12}x_2 + s_{13}x_3 + s_{14}x_4 = 0 \,, \\ &s_{21}x_1 + (s_{22} - \lambda_{(i)})x_2 + s_{23}x_3 + s_{24}x_4 = 0 \,, \\ &s_{31}x_1 + s_{32}x_2 + (s_{33} - \lambda_{(i)})x_3 + s_{34}x_4 = 0 \,, \\ &s_{41}x_1 + s_{42}x_2 + s_{43}x_3 + (s_{44} - \lambda_{(i)})x_4 = 0 \,. \end{aligned} \tag{3.9}$$

From the vanishing of the determinant, we yield:

$$\det \begin{pmatrix} (s_{11} - \lambda) & s_{12} & s_{13} & s_{14} \\ s_{21} & (s_{22} - \lambda) & s_{23} & s_{24} \\ s_{31} & s_{32} & (s_{33} - \lambda) & s_{34} \\ s_{41} & s_{42} & s_{43} & (s_{44} - \lambda) \end{pmatrix} = 0 \,,$$

and we find 4 diagonal elements

$$\Lambda' = \left(\begin{array}{cccc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{array} \right) ,$$

$$\lambda_1 = \epsilon^2 - k^2 + R_1 R_2 - \frac{1}{2}({M_1}^2 + {M_2}^2) + \frac{1}{2}({\Gamma_1}^2 + {\Gamma_2}^2) - \sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}$$
$$-\frac{1}{2}\Big[4 R_1 R_2 \Big((\Gamma_1 + \Gamma_2)^2 - (M_1 - M_2)^2 + 4\,(\epsilon^2 - k^2)\Big)$$
$$+4\,(\epsilon^2 - k^2)(\Gamma_1 - \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)^2$$
$$-4\,\frac{\Big(4 R_1 R_2\,(\Gamma_1 + \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2\big)\big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)\Big)(\epsilon^2 - k^2)}{\sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}}\Bigg]^{\frac{1}{2}} ,$$

$$\lambda_2 = \epsilon^2 - k^2 + R_1 R_2 - \frac{1}{2}({M_1}^2 + {M_2}^2) + \frac{1}{2}({\Gamma_1}^2 + {\Gamma_2}^2) - \sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}$$
$$+\frac{1}{2}\Big[4 R_1 R_2 \Big((\Gamma_1 + \Gamma_2)^2 - (M_1 - M_2)^2 + 4\,(\epsilon^2 - k^2)\Big)$$
$$+4\,(\epsilon^2 - k^2)(\Gamma_1 - \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)^2$$
$$-4\,\frac{\Big(4 R_1 R_2\,(\Gamma_1 + \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2\big)\big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)\Big)(\epsilon^2 - k^2)}{\sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}}\Bigg]^{\frac{1}{2}} ,$$

$$\lambda_3 = \epsilon^2 - k^2 + R_1 R_2 - \frac{1}{2}({M_1}^2 + {M_2}^2) + \frac{1}{2}({\Gamma_1}^2 + {\Gamma_2}^2) + \sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}$$
$$-\frac{1}{2}\Big[4 R_1 R_2 \Big((\Gamma_1 + \Gamma_2)^2 - (M_1 - M_2)^2 + 4\,(\epsilon^2 - k^2)\Big)$$
$$+4\,(\epsilon^2 - k^2)(\Gamma_1 - \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)^2$$
$$+4\,\frac{\Big(4 R_1 R_2\,(\Gamma_1 + \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2\big)\big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)\Big)(\epsilon^2 - k^2)}{\sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}}\Bigg]^{\frac{1}{2}} ,$$

$$\lambda_4 = \epsilon^2 - k^2 + R_1 R_2 - \frac{1}{2}({M_1}^2 + {M_2}^2) + \frac{1}{2}({\Gamma_1}^2 + {\Gamma_2}^2) + \sqrt{(\epsilon^2 - k^2)(\Gamma_1 + \Gamma_2)^2}$$
$$+\frac{1}{2}\Big[4 R_1 R_2 \Big((\Gamma_1 + \Gamma_2)^2 - (M_1 - M_2)^2 + 4\,(\epsilon^2 - k^2)\Big)$$
$$+4\,(\epsilon^2 - k^2)(\Gamma_1 - \Gamma_2)^2 + \big({\Gamma_1}^2 - {\Gamma_2}^2 - {M_1}^2 + {M_2}^2\big)^2$$

$$+4\,\frac{\Big(4\,R_1\,R_2\;(\Gamma_1+\Gamma_2)^2+\big({\Gamma_1}^2-{\Gamma_2}^2\big)\big({\Gamma_1}^2-{\Gamma_2}^2-{M_1}^2+{M_2}^2\big)\Big)\,(\epsilon^2-k^2)}{\sqrt{(\epsilon^2-k^2)\,(\Gamma_1+\Gamma_2)^2}}\Bigg]^{\frac{1}{2}}\;.$$

(3.10)

To find the columns of the matrix T, we turn to the system (for $i=1,2,3,4$):

$$(s_{11}-\lambda^{(i)})x_1^{(i)}+s_{12}x_2^{(i)}+s_{13}x_3^{(i)}+s_{14}x_4^{(i)}=0\;,$$
$$s_{21}x_1^{(i)}+(s_{22}-\lambda^{(i)})x_2^{(i)}+s_{23}x_3^{(i)}+s_{24}x_4^{(i)}=0\;,$$
$$s_{31}x_1^{(i)}+s_{32}x_2^{(i)}+(s_{33}-\lambda^{(i)})x_3^{(i)}+s_{34}x_4^{(i)}=0\;,$$
$$s_{41}x_1^{(i)}+s_{42}^{(i)}x_2+s_{43}x_3^{(i)}+(s_{44}-\lambda^{(i)})x_4^{(i)}=0\;.$$

Because the rank of the matrix equals to 3, we may ignore the fourth equation and for definiteness we set $x_4^{(i)}=1$; in this way we obtain the linear non-homogeneous system

$$(s_{11}-\lambda_{(i)})x_1^{(i)}+s_{12}x_2^{(i)}+s_{13}x_3^{(i)}=-s_{14}\;,$$
$$s_{21}x_1^{(i)}+(s_{22}-\lambda_{(i)})x_2^{(i)}+s_{23}x_3^{(i)}=-s_{24}\;,$$
$$s_{31}x_1^{(i)}+s_{32}x_2^{(i)}+(s_{33}-\lambda_{(i)})x_3^{(i)}=-s_{34}\;. \tag{3.11}$$

Applying the Cramer rule,

$$x_1^{(i)}=\frac{\Delta_1^{(i)}}{\Delta^{(i)}}\,,\quad x_2^{(i)}=\frac{\Delta_1^{(i)}}{\Delta^{(i)}}\,,\quad x_2^{(i)}=\frac{\Delta_1^{(i)}}{\Delta^{(i)}}\,,$$

we get the four columns (remembering that $x_4^{(i)}=1$):

$$T=\begin{pmatrix} x_1^{(1)} & x_1^{(2)} & x_1^{(3)} & x_1^{(4)}\\ x_2^{(1)} & x_2^{(2)} & x_2^{(3)} & x_2^{(4)}\\ x_3^{(1)} & x_3^{(2)} & x_2^{(3)} & x_3^{(4)}\\ 1 & 1 & 1 & 1\end{pmatrix}=\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14}\\ t_{21} & t_{22} & t_{23} & t_{24}\\ t_{31} & t_{32} & t_{33} & t_{34}\\ 1 & 1 & 1 & 1\end{pmatrix}, \tag{3.12}$$

where (note that $i=1,2,3,4$)

$$t_{1i}=\Delta_{(i)}^{-1}[-s_{14}\,\lambda_{(i)}^2+(s_{14}\,s_{22}-s_{34}\,s_{13}-s_{24}\,s_{12}+s_{33}\,s_{14})\,\lambda_{(i)}$$
$$+(s_{34}\,s_{13}-s_{33}\,s_{14})\,s_{22}+s_{24}\,s_{12}\,s_{33}-s_{23}\,s_{34}\,s_{12}-s_{32}\;(s_{24}\,s_{13}-s_{23}\,s_{14})],$$
$$t_{2i}=\Delta_{(i)}^{-1}[-\lambda_{(i)}^2s_{24}+(-s_{34}\,s_{23}+s_{24}\,s_{33}+s_{11}\,s_{24}-s_{21}\,s_{14})\,\lambda_{(i)}$$
$$+(-s_{24}\,s_{33}+s_{34}\,s_{23})\,s_{11}+s_{21}\,s_{14}\,s_{33}-s_{21}\,s_{34}\,s_{13}+s_{31}\;(s_{24}\,s_{13}-s_{23}\,s_{14})],$$
$$t_{3i}=\Delta_{(i)}^{-1}[-s_{34}\,\lambda_{(i)}^2+(s_{34}\,s_{22}+s_{34}\,s_{11}-s_{14}\,s_{31}-s_{24}\,s_{32})\,\lambda_i$$
$$+(s_{24}\,s_{32}-s_{34}\,s_{22})\,s_{11}+s_{14}\,s_{31}\,s_{22}-s_{31}\,s_{24}\,s_{12}+s_{21}\;(s_{12}\,s_{34}-s_{32}\,s_{14})]\;,$$

(3.13)

and

$$\Delta_{(i)} = -{\lambda_1}^3 + (s_{11} + s_{22} + s_{33})\,{\lambda_1}^2$$
$$+ (-s_{11}s_{22} - s_{33}s_{22} + s_{21}\,s_{12} + s_{31}\,s_{13} + s_{32}\,s_{23} - s_{11}\,s_{33})\,\lambda_1$$
$$+ (s_{11}s_{22}\,s_{33} + s_{12}\,s_{31}\,s_{23} + s_{21}\,s_{13}s_{32} - s_{31}\,s_{13}s_{22} - s_{21}\,s_{12}\,s_{33} - s_{11}\,s_{23}s_{32})\,. \tag{3.14}$$

Thus, we have the following four differential equations

$$D_+D_-F'_{(i)} = \lambda_{(i)}F'_{(i)}\,, \quad i = 1,2,3,4\,.$$

They have the same structure; for shortness[6] we bring the factor 1/2 inside symbol B

$$\left[\frac{d^2}{dr^2} + \lambda + \frac{m}{r^2} + B - (\frac{m}{r} - Br)^2\right]\Phi(r) = 0\,,$$

or,

$$\left[\frac{d^2}{dr^2} + \lambda + B(1+2m) - \frac{m^2-m}{r^2} - B^2r^2\right]\Phi(r) = 0\,. \tag{3.15}$$

By changing the variable $x = Br^2$ (for definiteness, let $B > 0$), we yield:

$$x\,\frac{d^2\Phi}{dx^2} + \frac{1}{2}\,\frac{d\Phi}{dx} + (-\frac{1}{4}\,x + \frac{1}{4}\,\frac{\lambda + 2\,Bm + B}{B} - \frac{1}{4}\,\frac{m(m-1)}{x}\,)\,\Phi = 0\,, \tag{3.16}$$

and using the substitution $\Phi = x^\alpha\,e^{\beta x}\,\varphi(x)$, we get

$$x\frac{d^2\varphi}{dx^2} + (2\,\alpha + \frac{1}{2} + 2\,\beta\,x)\frac{d\varphi}{dx} + \left[\frac{1}{4}\,\left(4\,\beta^2 - 1\right)x\right.$$
$$\left.+\frac{1}{4}\,\frac{2\,Bm + B + \lambda + 8\,\alpha\,\beta\,B + 2\,\beta\,B}{B} + \frac{1}{4}\,\frac{4\,\alpha^2 - 2\,\alpha + m - m^2}{x}\right]\varphi = 0\,.$$

We note that if $\beta = -\frac{1}{2}$, $x \in (0, +\infty)$, then this equation becomes simpler:

$$x\frac{d^2\varphi}{dx^2} + (2\,\alpha + \frac{1}{2} - x)\frac{d\varphi}{dx} + (\frac{1}{4}\,\frac{2\,Bm - 4\,B\alpha + \lambda}{B} + \frac{1}{4}\,\frac{4\,\alpha^2 - 2\,\alpha + m - m^2}{x})\varphi = 0\,.$$

Assume that

$$4\,\alpha^2 - 2\,\alpha + m - m^2 = 0 \quad\Longrightarrow\quad \alpha = -\frac{1}{2}\,m + \frac{1}{2}\,,\ \frac{1}{2}\,m\,.$$

In order to describe the bond states we may use only positive values for α:

$$\alpha_1 = \frac{1-m}{2} > 0 \quad\Longrightarrow\quad m = -\frac{1}{2}, -\frac{3}{2}, \dots\,;$$
$$\alpha_2 = \frac{m}{2} \quad\Longrightarrow\quad m = +\frac{1}{2}, +\frac{3}{2}, \dots\,. \tag{3.17}$$

[6] We perform the change in notation: $eB/2\hbar c \rightsquigarrow B$.

So we arrive to the confluent hypergeometric equation for $F(a,c;x)$ with the parameters

$$a = -\frac{1}{4}\,\frac{2\,Bm - 4\,B\alpha + \lambda}{B}\,, \qquad c = 2\,\alpha + \frac{1}{2}\,.$$

The ordinary condition for getting polynomials as solutions ($a = -n$) gives the quantization rule:

$$\frac{1}{2}\,m - \alpha + \frac{\lambda}{4\,B} = n\,. \tag{3.18}$$

According to two expressions for α in (3.17), we have two formulas for spectra (by physical reasons we assume that $\lambda > 0$:

$$m = -\frac{1}{2}, -\frac{3}{2}, ... \qquad \lambda = 4B(n + \frac{1}{2} - m)\,, \quad n = 1, 2, ...\,; \tag{3.19}$$

$$m = +\frac{1}{2}, +\frac{3}{2}, ... \qquad \lambda = 4B\,n, \quad n = 0, 1, 2, ...\,. \tag{3.20}$$

These formulas may be unified into a single one, namely

$$\lambda = 4b\,N, \qquad N \in \{0, 1, 2...\}\,. \tag{3.21}$$

We introduce the following notations

$$\epsilon^2 - k^2 = E^2\,, \quad M_1 + M_2 = \mu\,, \quad M_1 - M_2 = \nu\,,$$

$$R_1 R_2 = R\,, \qquad \Gamma_1 + \Gamma_2 = x, \qquad \Gamma_1 - \Gamma_2 = y\,,$$

$$M_1^2 + M_2^2 = \frac{1}{2}(\mu^2 + \nu^2), \quad \Gamma_1^2 + \Gamma_2^2 = \frac{1}{2}(x^2 + y^2)\,.$$

Then, the roots λ_i may be written as[7]:

$$\pm\, 2\sqrt{(16R + 4y^2)\,E^2 - 4\,[4Rx + y(yx - \nu\mu)]E + 4\,R(x^2 - \nu^2) + (yx - \nu\mu)^2}$$
$$= 4E^2 - 4Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_{1,2}\,, \tag{3.22}$$

$$\pm\, 2\sqrt{(16R + 4y^2)E^2 + 4\,[4Rx + y(yx - \nu\mu) + 4\,R(x^2 - \nu^2) + (yx - \nu\mu)^2]E}$$
$$= 4E^2 + 4\,Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_{3,4}\,. \tag{3.23}$$

In fact, the last equation differs only by sign at E, and therefore it suffices to follow only one of them, (let it be Eq. (3.22)); also we should take in mind that due to physical reasons, only the positive values for $E > 0$ are acceptable.

We have the following restrictions on the energy parameters E related to the roots λ_1 and λ_2:

$$4E^2 - 4Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_1 > 0\,,$$

$$4E^2 - 4Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_2 < 0\,,$$

[7] We look for them in pairs λ_1, λ_2 and λ_3, λ_4.

from where it follows

$$(E - \frac{x}{2})^2 > \frac{4\lambda_1 + \mu^2 + \nu^2 - y^2}{4}\,, \tag{3.24}$$

$$(E - \frac{x}{2})^2 < \frac{4\lambda_2 + \mu^2 + \nu^2 - y^2}{4}\,. \tag{3.25}$$

We shall further show that $y = 0$. We conclude from (3.24)-(3.25) that a spectrum of positive and not-bounded values of E is possible only for the case (3.24). A spectrum of negative values of E may occur as well; however it should be related to the variant $\lambda_3 : E \to -E$.

The algebraic equation with respect to E takes the form[8]

$$16\,E^4 - 32x\,E^3 + \big(-8\,\mu^2 - 8\,\nu^2 + 24\,x^2 - 8\,y^2 - 32\,R - 128\,BN\big)\,E^2$$
$$+ \big[-8\,x^3 + \big(128\,BN + 8\,\nu^2 + 32\,R + 8\,y^2 + 8\,\mu^2\big)\,x - 16\,y\nu\,\mu\big]\,E$$
$$+x^4 + \big(-2\,\nu^2 - 8\,R - 2\,y^2 - 32\,BN - 2\,\mu^2\big)\,x^2$$
$$+8\,yx\nu\,\mu + \mu^4 + \big(-8\,R - 2\,\nu^2 + 32\,BN - 2\,y^2\big)\,\mu^2$$
$$+\nu^4 + (8\,R - 2\,y^2 + 32\,BN)\nu^2 + 256\,(-\frac{1}{4}\,R + BN - \frac{1}{16}\,y^2)^2 = 0\,. \tag{3.26}$$

There exist evident relations satisfied by the roots:

$$E_1 + E_2 + E_3 + E_4 = -\frac{a_1}{a_0} = 2x\,,$$

$$E_1E_2 + E_1E_3 + E_1E_4 + E_2E_3 + E_2E_4 + E_3E_4 = \frac{a_2}{a_0}\,,$$

$$E_1E_2E_3 + E_1E_2E_4 + E_1E_3E_4 + E_2E_3E_4 = -\frac{a_3}{a_0}\,,$$

$$E_1E_2E_3E_4 = \frac{a_4}{a_0}\,,$$

where

$$a_0 = 16\,, \quad a_1 = -32x\,, \quad a_2 = -8\,\mu^2 - 8\,\nu^2 + 24\,x^2 - 8\,y^2 - 32\,R - 128\,BN\,,$$

$$a_3 = -8\,x^3 + (128\,BN + 8\,\nu^2 + 32\,R + 8\,y^2 + 8\,\mu^2)x - 16\,y\nu\,\mu\,,$$
$$a_4 = x^4 + (-2\,\nu^2 - 8\,R - 2\,y^2 - 32\,BN - 2\,\mu^2)x^2$$
$$+8\,yx\nu\,\mu + \mu^4 + (-8\,R - 2\,\nu^2 + 32\,BN - 2\,y^2)\mu^2 +$$
$$+\nu^4 + (8\,R - 2\,y^2 + 32\,BN)\nu^2 + 256\,(-\frac{1}{4}\,R + BN - \frac{1}{16}\,y^2)^2\,.$$

In particular, we have the identity

$$a_4 = x^4 - (2\nu^2 + 8R + 32BN + 2\mu^2)x^2 + \mu^4$$

[8] We remind here that $\lambda = 4BN$.

$$+(-8R-2\nu^2+32BN)\mu^2+\nu^4+(8R+32BN)\nu^2+256(-\frac{1}{4}R+BN)^2 .$$

We see that for large N, the inequality $a_4>0$ holds, and therefore $E_1E_2E_3E_4>0$. Moreover, we can prove the identity

$$a_4=\big[16BN-(x^2+4R-\mu^2-\nu^2)\big]^2+16R(\nu^2-x^2)-4\mu^2\nu^2 ,$$

which shows that $a_4>0$ for all the values of N.

In order to study possible energy levels, let us take into account the expressions for initial parameters

$$\sin^2\gamma=\frac{4\sigma^2}{\rho^2}, \quad \gamma\in(0,\pi/2), \quad \lambda_1=\frac{\rho}{2}(1+\cos\gamma), \quad \lambda_2=\frac{\rho}{2}(1-\cos\gamma),$$

$$M_1=\frac{M/\rho}{(1+\cos\gamma)/2}, \qquad M_2=\frac{M/\rho}{(1-\cos\gamma)/2},$$

$$a=\frac{5A'-3B'-\lambda_1}{M}=\frac{\rho}{2}\frac{1}{M}(4-3\sqrt{1+(1/3)\sin^2\gamma}-\cos\gamma),$$

$$b=\frac{5A'-3B'-\lambda_2}{M}=\frac{\rho}{2}\frac{1}{M}(4-3\sqrt{1+(1/3)\sin^2\gamma}+\cos\gamma),$$

$$\Lambda_1=(1+\sqrt{1+(1/3)\sin^2\gamma})\ \frac{\cos\gamma-\sqrt{1+(1/3)\sin^2\gamma}}{\cos\gamma(1+\cos\gamma)} ,$$

$$\Lambda_2=(1+\sqrt{1+(1/3)\sin^2\gamma})\ \frac{\cos\gamma+\sqrt{1+(1/3)\sin^2\gamma}}{\cos\gamma(1-\cos\gamma)} .$$

First, we find the expressions of Γ_1, R_1 and Γ_2, R_2:

$$\Gamma_1=2b\Lambda_1B$$

$$=\frac{B}{M/\rho}2(4-3\sqrt{1+(1/3)\sin^2\gamma}+\cos\gamma)\frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma-\sqrt{1+(1/3)\sin^2\gamma})}{4\cos\gamma\ (1+\cos\gamma)},$$

$$R_1=a\Lambda_1B$$

$$=\frac{B}{M/\rho}(4-3\sqrt{1+(1/3)\sin^2\gamma}-\cos\gamma)\frac{(1+\sqrt{1+(1/3)\sin^2\gamma})\ (\cos\gamma-\sqrt{1+(1/3)\sin^2\gamma})}{4\cos\gamma(1+\cos\gamma)},$$

$$\Gamma_2=-2a\Lambda_2B$$

$$=-\frac{B}{M/\rho}2(4-3\sqrt{1+(1/3)\sin^2\gamma}-\cos\gamma)\frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma+\sqrt{1+(1/3)\sin^2\gamma})}{4\cos\gamma(1-\cos\gamma)},$$

$$R_2=-b\Lambda_2B$$

$$=-\frac{B}{M/\rho}(4-3\sqrt{1+(1/3)\sin^2\gamma}+\cos\gamma)\ \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma+\sqrt{1+(1/3)\sin^2\gamma})}{4\cos\gamma(1-\cos\gamma)} .$$

Now, we can write down explicit form of all parameters entering the 4-th order equation[9]:

$$x = \Gamma_1 + \Gamma_2$$

$$= \frac{B}{4M_0} 2(4 - 3\sqrt{1+(1/3)\sin^2\gamma} + \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma - \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma(1+\cos\gamma)}$$
$$- \frac{B}{4M_0} 2(4 - 3\sqrt{1+(1/3)\sin^2\gamma} - \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma + \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma\ (1-\cos\gamma)} ,$$

$$y = \Gamma_1 - \Gamma_2$$

$$= \frac{B}{4M_0} 2(4 - 3\sqrt{1+(1/3)\sin^2\gamma} + \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma - \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma\ (1+\cos\gamma)}$$
$$+ \frac{B}{4M_0} 2(4 - 3\sqrt{1+(1/3)\sin^2\gamma} - \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma + \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma\ (1-\cos\gamma)} ,$$

$$R = R_1 R_2$$

$$= -\frac{B}{4M_0} (4 - 3\sqrt{1+(1/3)\sin^2\gamma} - \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma - \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma(1+\cos\gamma)}$$
$$\times \frac{B}{4M_0} (4 - 3\sqrt{1+(1/3)\sin^2\gamma} + \cos\gamma) \frac{(1+\sqrt{1+(1/3)\sin^2\gamma})(\cos\gamma + \sqrt{1+(1/3)\sin^2\gamma})}{\cos\gamma\ (1-\cos\gamma)} ,$$

(3.27)

After simplifying these formulas, we get

$$x = -\frac{4}{3}\frac{B\sin^2\gamma}{M_0\ \cos\gamma}, \qquad y = 0, \qquad R = \frac{1}{9}\frac{B^2\sin^4\gamma}{{M_0}^2\cos^2\gamma} . \tag{3.28}$$

The functions $x(\gamma), R(\gamma)$ may be illustrated by Fig. 1 and Fig. 2 (the multipliers $B/M_0, B^2/M_0^2$ are ignored).

The parameters μ, ν are

$$\mu = M_1 + M_2 = \frac{M_0}{(1+\cos\gamma)/2} + \frac{M_0}{(1-\cos\gamma)/2} = \frac{4M_0}{\sin^2\gamma} ,$$
$$\nu = M_1 - M_2 = \frac{M_0}{(1+\cos\gamma)/2} - \frac{M_0}{(1-\cos\gamma)/2} = -\frac{4M_0\cos\gamma}{\sin^2\gamma} , \tag{3.29}$$

and are illustrated by Fig. 3 and Fig. 4 (the multipliers $B/M_0, B^2/M_0^2$ are ignored).

Taking into account the relations (3.28) and (3.29), we get the final form of the 4-th order equation for E:

$$E^4 + \frac{8}{3}\frac{B\sin^2\gamma E^3}{M_0\ \cos\gamma} + (\frac{22}{9}\frac{B^2\sin^4\gamma}{{M_0}^2\cos^2\gamma} - \frac{8\,{M_0}^2(1+\cos^2\gamma)}{\sin^4\gamma} - 8\,BN)E^2$$

[9] Let here $M_0 = M/\rho$.

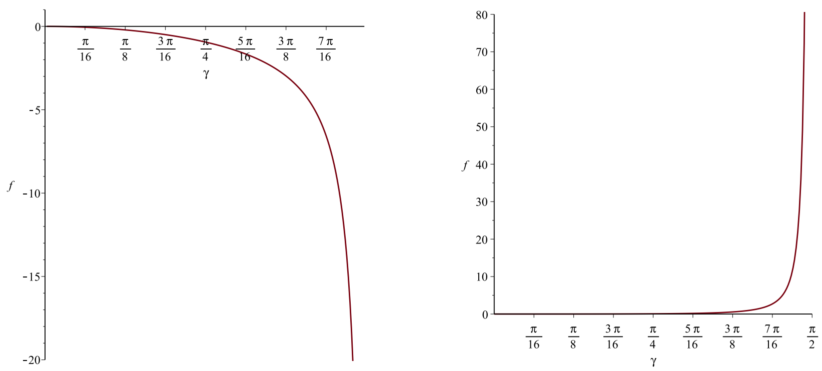

Figure 1: a) Parameter $x(\gamma)$. b) Parameter $R(\gamma)$.

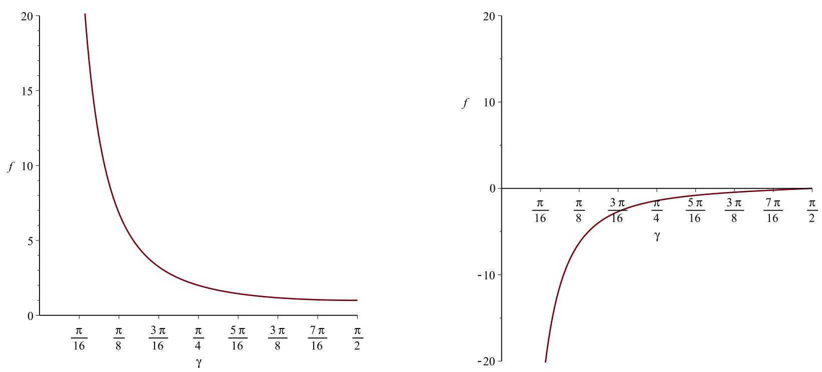

Figure 2: c) Parameter $\mu(\gamma)$. d) Parameter $\nu(\gamma)$.

$$+(\frac{8}{9}\,\frac{B^3\sin^6\gamma}{{M_0}^3\cos^3\gamma}-\frac{32}{3}\,\frac{M_0\,B(1+\cos^2\gamma)}{\sin^2\gamma\cos\gamma}-\frac{32}{3}\,\frac{B^2\sin^2\gamma N}{M_0\,\cos\gamma})E$$

$$+\frac{16\,{M_0}^4}{\sin^4\gamma}-\frac{8}{3}B^2-\frac{40}{9}\,\frac{B^2}{\cos^2\gamma}+\frac{1}{9}\,\frac{B^4\sin^8\gamma}{{M_0}^4\cos^4\gamma}-\frac{40}{9}\,\frac{B^3\sin^4\gamma N}{{M_0}^2\cos^2\gamma}$$

$$+\frac{32\,{M_0}^2BN(1+\cos^2\gamma)}{\sin^4\gamma}+16\,B^2N^2=0\,. \tag{3.30}$$

The energy and all the parameters may become dimensionless by means of the following procedure

$$\frac{E}{M_0}-->E,\ \frac{x}{M_0}-->x,\ \frac{y}{M_0}-->y,\ \frac{\mu}{M_0}-->\mu,\ \frac{\nu}{M_0}-->\nu\,,$$

$$\frac{B}{M_0^2}-->B,\ \frac{R}{M_0^2}-->R\,; \tag{3.31}$$

in this way we obtain a simpler equation,

$$E^4 + \frac{8}{3}\,\frac{B\sin^2\gamma E^3}{\cos\gamma} + (\frac{22}{9}\,\frac{B^2\sin^4\gamma}{\cos^2\gamma} - \frac{8\,(1+\cos^2\gamma)}{\sin^4\gamma} - 8\,BN)E^2$$

$$+(\frac{8}{9}\,\frac{B^3\sin^6\gamma}{\cos^3\gamma} - \frac{32}{3}\,\frac{B(1+\cos^2\gamma)}{\sin^2\gamma\cos\gamma} - \frac{32}{3}\,\frac{B^2\sin^2\gamma N}{\cos\gamma})E$$

$$+\frac{16}{\sin^4\gamma} - \frac{8}{3}B^2 - \frac{40}{9}\,\frac{B^2}{\cos^2\gamma} + \frac{1}{9}\,\frac{B^4\sin^8\gamma}{\cos^4\gamma} - \frac{40}{9}\,\frac{B^3\sin^4\gamma N}{\cos^2\gamma}$$

$$+\frac{32\,BN(1+\cos^2\gamma)}{\sin^4\gamma} + 16\,B^2N^2 = 0\,. \tag{3.32}$$

The numerical study of several such typical cases gives

$$B=1\,,\quad \sin\gamma=\frac{1}{10}\,,$$
$$N=1\,,\quad E_1=2.23\,,\quad E_2=399.00\,,\quad E_3=-2.24\,,\quad E_4=-399.01\,;$$
$$N=5\,,\quad E_1=4.58\,,\quad E_2=399.02\,,\quad E_3=-4.59\,,\quad E_4=-399.03\,;$$
$$N=10\,,\quad E_1=6.40\,,\quad E_2=399.04\,,\quad E_3=-6.41\,,\quad E_4=-399.05\,.$$

$$B=1\,,\quad \sin\gamma=\frac{1}{2}\,,$$
$$N=1\,,\quad E_1=2.08\,,\quad E_2=14.87\,,\quad E_3=-2.46\,,\quad E_4=-15.25\,;$$
$$N=5\,,\quad E_1=4.41\,,\quad E_2=15.39\,,\quad E_3=-4.79\,,\quad E_4=-15.78\,;$$
$$N=10\,,\quad E_1=6.22\,,\quad E_2=16.02,\,,\quad E_3=-6.61\,,\quad E_4=-16.41\,.$$

$$B=1\,,\quad \sin\gamma=\frac{9}{10}\,,$$
$$N=1\,,\quad E_1=1.00\,,\quad E_2=3.03\,,\quad E_3=-3.48\,,\quad E_4=-5.51\,;$$
$$N=5\,,\quad E_1=3.16\,,\quad E_2=4.75\,,\quad E_3=-5.64\,,\quad E_4=-7.23\,;$$
$$N=10\,,\quad E_1=4.90\,,\quad E_2=6.35\,,\quad E_3=-7.38\,,\quad E_4=-8.83\,.$$

$$B=5\,,\quad \sin\gamma=\frac{1}{10}\,,$$
$$N=1\,,\quad E_1=4.55\,,\quad E_2=398.99\,,\quad E_3=-4.62\,,\quad E_4=-399.06\,;$$
$$N=5\,,\quad E_1=10.02\,,\quad E_2=399.09\,,\quad E_3=-10.08\,,\quad E_4=-399.16\,;$$
$$N=10\,,\quad E_1=14.14\,,\quad E_2=399.21\,,\quad E_3=-14.21\,,\quad E_4=-399.28\,.$$

$$B=5\,,\quad \sin\gamma=\frac{1}{2}\,,$$
$$N=1\,,\quad E_1=3.62\,,\quad E_2=14.64\,,\quad E_3=-5.55\,,\quad E_4=-16.56\,;$$
$$N=5\,,\quad E_1=9.07\,,\quad E_2=17.03\,,\quad E_3=-11.00\,,\quad E_4=-18.96\,;$$
$$N=10\,,\quad E_1=13.19\,,\quad E_2=19.63\,,\quad E_3=-15.11\,,\quad E_4=-21.56\,.$$

$$B = 5\,, \quad \sin\gamma = \frac{9}{10}\,,$$
$$N = 1\,, \quad E_1 = -4.01\,, \quad E_2 = 2.11\,, \quad E_3 = -8.38\,, \quad E_4 = -14.50\,;$$
$$N = 5\,, \quad E_1 = 1.07\,, \quad E_2 = 7.25\,, \quad E_3 = -13.46\,, \quad E_4 = -19.64\,;$$
$$N = 10\,, \quad E_1 = 5.11\,, \quad E_2 = 11.30\,, \quad E_3 = -17.50\,, \quad E_4 = -23.69\,.$$

$$B = 10\,, \quad \sin\gamma = \frac{1}{10}\,,$$
$$N = 1\,, \quad E_1 = 6.34\,, \quad E_2 = 398.98\,, \quad E_3 = -6.47\,, \quad E_4 = -399.11\,;$$
$$N = 5\,, \quad E_1 = 14.11\,, \quad E_2 = 399.18\,, \quad E_3 = -14.24\,, \quad E_4 = -399.32\,;$$
$$N = 10\,, \quad E_1 = 19.96\,, \quad E_2 = 399.43\,, \quad E_3 = -20.09\,, \quad E_4 = -399.57\,.$$

$$B = 10\,, \quad \sin\gamma = \frac{1}{2}\,,$$
$$N = 1\,, \quad E_1 = 4.43\,, \quad E_2 = 14.37\,, \quad E_3 = -8.28\,, \quad E_4 = -18.22\,;$$
$$N = 5\,, \quad E_1 = 12.14\,, \quad E_2 = 18.77\,, \quad E_3 = -15.99\,, \quad E_4 = -22.62\,;$$
$$N = 10\,, \quad E_1 = 17.94\,, \quad E_2 = 23.20\,, \quad E_3 = -21.79\,, \quad E_4 = -27.05\,.$$

$$B = 10\,, \quad \sin\gamma = \frac{9}{10}\,,$$
$$N = 1\,, \quad E_1 = 0.65\,, \quad E_2 = -11.23\,, \quad E_3 = -13.55\,, \quad E_4 = -25.42\,;$$
$$N = 5\,, \quad E_1 = 8.19\,, \quad E_2 = -4.16\,, \quad E_3 = -20.62\,, \quad E_4 = -32.97\,;$$
$$N = 10\,, \quad E_1 = 13.98\,, \quad E_2 = 1.61\,, \quad E_3 = -26.39\,, \quad E_4 = -38.76\,.$$

The values of energy – these are divided into four series – may be foreseen, from physical point of view. The analytical expressions for all the four series of energy levels may be written down in explicit form as well; however their expressions are very cumbersome and can hardly be useful.

It should be pointed out that, due to the identity $y = 0$, we have the equality $\Gamma_1 = \Gamma_2 = \Gamma$, and the system (2.7)–(2.8) reads

$$(i\hat{D} - M_1 + \Gamma\Sigma_3)\psi_1 - R_1\Sigma_3\ \psi_2 = 0\ ,$$
$$(i\hat{D} - M_2 + \Gamma\Sigma_3)\psi_2 - R_2\Sigma_3\ \psi_1 = 0\ ; \tag{3.33}$$

This equation means that two Dirac particles with different masses (M_1 and M_2) and equal anomalous magnetic moments Γ are linked to each other through the terms $R_1\Sigma_3$ and $R_2\Sigma_3$. Evidently, by means of the elementary change of variables

$$\sqrt{R_1}\ \psi_1 \rightsquigarrow \Psi_1\,, \qquad \sqrt{R_2}\ \psi_2 = \Psi_2\,,$$

the system may get the more symmetrical form

$$(i\hat{D} - M_1 + \Gamma\Sigma_3)\Psi_1 - \sqrt{R}\ \Sigma_3\ \Psi_2 = 0\ ,$$
$$(i\hat{D} - M_2 + \Gamma\Sigma_3)\Psi_2 - \sqrt{R}\ \Sigma_3\ \Psi_1 = 0\ ; \tag{3.34}$$

where

$$\Gamma = -\frac{2}{3}\,\frac{B\sin^2\gamma}{M_0\,\cos\gamma}\,, \quad \sqrt{R} = \sqrt{R_1 R_2} = \frac{B}{3}\,\frac{\sin^2\gamma}{\cos\gamma}\ .$$

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