On describing bound states for a spin 1 particle in the external Coulomb field

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Abstract. The system of 10 radial equations, derived from the Duffin–Kemmer–Petiau equation for a spin 1 particle in the external Coulomb field, is studied. With the use of the space reflection operator, the whole system is split to independent subsystems, consisting of 4 and 6 equations, respectively. The most simple subsystem of 4 equations is solved in terms of hypergeometric functions, which gives the known energy spectrum. Also, the solutions and energy spectrum are found for the minimal value of the total angular momentum, $j = 0$. The second subsystem is expected to give the description of the other two series of bound states. With the use of the Lorentz generalized condition in presence of the Coulomb field, we prove that one of 6 radial function turns to be equal to zero. This simplifies the explicit form of the system of 6 equations, which contains only 5 unknown functions. Combining this system, we derive a new separated of 2-nd order system of differential equations for three radial functions. In particular, one of the equations turns out to be a rather simple one, and may be recognized as a confluent Heun equation. A series of bound states is constructed in terms of the so called transcendental confluent Heun functions, which provides us with solutions for the second class of bound states, with corresponding formula for energy levels. The subsystem of 6 equations, with no use of additional constraints due to the Lorentz condition, after excluding two non-differential relations reduces to the system of 1-st order differential equations for 4 functions $f_i, i = 1, 2, 3, 4$. We derive the explicit form of a corresponding of 4-th order equation for each function. Among them, there are equations with two substantially different sets of singular points: 3 regular (or 2) and 2 irregular of rank 2. Any of these functions may be considered as a main one, and all remaining functions may be found in explicit form, in terms of the main one. From the four independent solutions of each 4-th order equation, only two solutions may be referred to series of bound states.

M.S.C. 2010: 33E10, 34B30.
Key words: spin one particle; Coulomb potential; separation of variables; exact solutions; hypergeometric functions; Heun functions; energy levels; first order equations; fourth order equations.

BSG Proceedings, vol. 25, 2018, pp. 59-78.

1 Introduction

Till now, the known quantum-mechanical problem of a spin 1 particle in presence of the external Coulomb field remains unsolved. The first of three expected sub-classes of bound states was described by I.E. Tamm [1]. The incompleteness of the study relates to two other sub-classes of expected bound states. The main conclusion from I.E. Tamm considerations consists in the statement that in two remaining sub-classes of states, the particle should fall to the center. However, the study of the non-relativistic problem of spin 1 particle in the Coulomb field showed [2] that there exist three correctly defined series of bound states, which are described by the formulas similar to the one of the spinless Schrödinger particle in Coulomb field. In [3, 4], it was shown a possibility for some radial functions to confine to different 2-nd order differential equations, instead the expected equations of the 4-th order[1].

In Section 2, the system of 10 radial equations, derived from the Duffin–Kemmer–Petiau equation for a spin 1 particle in the external Coulomb field, is studied. With the use of the space reflection operator, the whole system is split to independent subsystems, consisting of 4 and 6 equations, respectively. The most simple subsystem of 4 equations is solved in terms of hypergeometric functions, which gives the known energy spectrum. Also, the solutions and energy spectrum are found for the minimal value of total angular momentum, $j = 0$. The second subsystem should give a description of the two other series of bound states.

In Section 3, with the use of the Lorentz generalized condition in presence of the Coulomb field, we prove that one of the 6 radial functions turns to equal to zero. This simplifies the explicit form of the system of 6 equations, which contain only 5 unknown functions.

In Section 4, by combining the equations in the 6-equation system, we derive for several radial functions the 2-nd order differential equations, and we derive a more simple 2-nd order equation for one of the radial functions. The qualitative analysis of this equation indicates that it may have solutions which describe bound states.

In Section 5, this simple equation is related to a confluent Heun equation. Its Frobenius solutions have been constructed and convergence of the involved power series is proved. The functions relevant to bound states are constructed in terms of the so called transcendental confluent Heun functions. This provides us with the second class of bound states for spin 1 particle in the external Coulomb field, with a corresponding formula for energy levels.

In Section 6, the subsystem of 6 equations, with no use of additional constraints due to the Lorentz condition, after excluding two non-differential relations reduces to the system of 1-st order differential equations for four functions, $f_i, i = 1, 2, 3, 4$. We elaborate a method which permits to examine projections of the whole set of solutions – the curve $\{f_1(r), f_2(r), f_3(r), f_4(r)\}$ in the 4-dimensional space – on the different planes $f_i = 0$ of the space. In each case, such a projection consists of two parts (branches), which are determined by different 2-nd order differential equations. In particular, the constraint $f_1(x) = 0$ coincides with that derived previously from the Lorentz condition. In this way we obtain explicit form of four pairs of 2-nd order

[1] Unfortunately, one technical error appeared in [3, 4], so some part of intermediate formulas turns to be incorrect, though the main final result is the right one. In the present paper, we repair this error.

equations, all of them having the structure of singularities more complicated than the class of hypergeometric functions.

In section 7, we derive the explicit form of the 4-th order equation for each function. Among them there arise two substantially different sets of singular points: 3 regular (or 2) and 2 irregular of the rank 2. Any of these four functions may be considered as a main one, then all the remaining functions may be found in explicit form through the main one. From four independent solutions of the main 4-th order equation, only two solutions may be referred to series of bound states.

Section 8 contains a general discussion and prospects the further research of the problem.

2 The separation of variables

We apply the matrix Duffin–Kemmer–Petiau form of the wave equation for spin 1 particle, adjusted to tetrad based formalism [5]; in spherical coordinates and tetrads, the main equation reads

$$\left\{ \beta^0(\epsilon + \frac{\alpha}{r}) + i \left[\beta^3 \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32})\right] + \frac{1}{r}\Sigma_{\theta,\phi} - m \right\} \Phi(x) = 0 \,, \tag{2.1}$$

where $\epsilon = E/c\hbar,\ m = Mc/\hbar,\ \alpha = e^2/(c\hbar) = 1/137$; and $\Sigma_{\theta,\phi}$ stands for the angular operator

$$\Sigma_{\theta,\phi} = \ i\, \beta^1 \partial_\theta \ + \ \beta^2 \ \frac{i\partial_\phi \ + \ i\, j^{12} \cos\theta}{\sin\theta} \,. \tag{2.2}$$

Wave functions $\Psi(x) = \{ \ \Phi_0(x),\ \vec{\Phi}(x),\ \vec{E}(x), \vec{H}(x) \ \}$ with the quantum numbers (ϵ, j, m) are searched in the form [5]:

$$\Phi_0(x) = e^{-i\epsilon t}\Phi_0(r) D_0 \,, \quad \vec{\Phi}(x) = e^{-i\epsilon t} \begin{pmatrix} \Phi_1(r) D_{-1} \\ \Phi_2(r) D_0 \\ \Phi_3(r) D_{+1} \end{pmatrix},$$

$$\vec{E}(x) = e^{-i\epsilon t} \begin{pmatrix} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \end{pmatrix}, \quad \vec{H}(x) = e^{-i\epsilon t} \begin{pmatrix} H_1(r) D_{-1} \\ H_2(r) D_0 \\ H_3(r) D_{+1} \end{pmatrix}; \tag{2.3}$$

The Wigner functions are defined as follows: $D_\sigma = D^j_{-m,\sigma}(\phi,\ \theta,\ 0)\ ,\ \ \sigma = 0,\ +1,\ -1$; quantum number j takes on the values $0, 1, 2, ...$ Applying the known recurrent formulas for Wigner functions, after simple calculation we arrive at the radial system of 10 equations [3,4]:

$$-\ (\frac{d}{dr} + \frac{2}{r})\ E_2 - \frac{\nu}{r}\ (E_1 + E_3) = m\Phi_0\ , \quad +i(\epsilon + \frac{\alpha}{r})\ E_1 \ \ + i\ (\frac{d}{dr} + \frac{1}{r})\ H_1 + i\frac{\nu}{r}\ H_2 = m\Phi_1\ ,$$

$$+i(\epsilon + \frac{\alpha}{r})\ E_2 \ \ - i\frac{\nu}{r}\ (H_1 - H_3) = m\Phi_2\ , \quad +i(\epsilon + \frac{\alpha}{r})\ E_3 \ \ - i\ (\frac{d}{dr} + \frac{1}{r})\ H_3 - i\frac{\nu}{r}\ H_2 = m\ \Phi_3\ ,$$

$$-i(\epsilon + \frac{\alpha}{r})\ \Phi_1 \ \ + \frac{\nu}{r}\ \Phi_0 - mE_1 = 0\ , \quad -i(\epsilon + \frac{\alpha}{r})\ \Phi_2 \ \ - \ \frac{d}{dr}\ \Phi_0 - mE_2 = 0\ ,$$

$$-i(\epsilon + \frac{\alpha}{r})\ \Phi_3 \ \ + \frac{\nu}{r}\ \Phi_0 \ - \ mE_3 = 0\ , \quad -i\ (\frac{d}{dr} + \frac{1}{r})\ \Phi_1 - i\frac{\nu}{r}\ \Phi_2 \ - \ mH_1 = 0\ ,$$

$$+i\frac{\nu}{r}\ (\Phi_1 - \Phi_3) - mH_2 = 0\ , \quad +i\ (\frac{d}{dr} + \frac{1}{r})\ \Phi_3 + i\frac{\nu}{r}\Phi_2 - mH_3 = 0\ , \tag{2.4}$$

where $\nu = \sqrt{j(j+1)/2}$, $j = 1, 2, \ldots$.

Together with the operators $\vec{J}^{\,2}$, J_3 we will diagonalize the space reflection operator $\hat{\Pi}$. After transforming it from usual Cartesian tetrad to a spherical one and to a cyclic representation of the matrices β^a, for this discrete operator we get the following expression

$$\hat{\Pi} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \Pi_3 & 0 & 0 \\ 0 & 0 & \Pi_3 & 0 \\ 0 & 0 & 0 & -\Pi_3 \end{array} \right) \hat{P} \, , \qquad \Pi_3 = \left(\begin{array}{ccc} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{array} \right) . \tag{2.5}$$

The eigenvalue equation $\hat{\Pi}\Psi = P\,\Psi$ gives two possibilities:

$$P = (-1)^{j+1}, \quad \Phi_0 = \Phi_2 = 0 \, , \quad \Phi_3 = -\Phi_1, \; E_3 = -E_1 \, , \; E_2 = 0 \, , \; H_3 = H_1 \, ; \tag{2.6}$$

$$P = (-1)^{j}, \quad \Phi_3 = \Phi_1 \, , \; E_3 = +E_1 \, , \; H_3 = -H_1 \, , \; H_2 = 0 \, . \tag{2.7}$$

Correspondingly, the system of 10 equation (2.4) gives two more simple subsystems.

The first is

$$P = (-1)^{j+1}, \qquad +i(\epsilon + \frac{\alpha}{r})\; E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 + i\frac{\nu}{r}H_2 = m\Phi_1 \, ,$$
$$-i(\epsilon + \frac{\alpha}{r})\; \Phi_1 = mE_1 \, , \quad -i(\frac{d}{dr} + \frac{1}{r})\Phi_1 = mH_1 \, , \quad 2i\frac{\nu}{r}\Phi_1 = mH_2 \, . \tag{2.8}$$

After excluding the variables E_1, H_1, H_2, we get a 2-nd order equation for Φ_1:

$$[\, \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (\epsilon + \frac{\alpha}{r})^2 - m^2 - \frac{j(j+1)}{r^2}]\Phi_1 = 0 \, . \tag{2.9}$$

In fact, this coincides with the radial equation arising for scalar Klein–Fock–Gordon particle in external Coulomb field. Solutions are constructed in terms of confluent hypergeometric functions, and we can write down only the energy spectrum

$$E = \frac{Mc^2}{\sqrt{1 + \alpha^2/N^2}} \, , \qquad N = n + \frac{1}{2} + \sqrt{(j+1/2)^2 - \alpha^2} \, . \tag{2.10}$$

For states with parity $P = (-1)^j$, we have the system of 6 equations:

$$1) \; (\frac{d}{dr} + \frac{2}{r})E_2 + 2\frac{\nu}{r}E_1 + m\Phi_0 = 0 \, , \quad 2) \; + i(\epsilon + \frac{\alpha}{r})\; E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 - m\Phi_1 = 0 \, ,$$
$$3) \; + i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 - m\Phi_2 = 0 \, , \quad 4) \; - i(\epsilon + \frac{\alpha}{r})\; \Phi_1 + \frac{\nu}{r}\Phi_0 - mE_1 = 0 \, ,$$
$$5) \; i(\epsilon + \frac{\alpha}{r})\Phi_2 + \frac{d}{dr}\Phi_0 + mE_2 = 0 \, , \quad 6) \; i(\frac{d}{dr} + \frac{1}{r})\Phi_1 + i\frac{\nu}{r}\Phi_2 + mH_1 = 0 \, . \tag{2.11}$$

The states with $j = 0$ should be considered separately, because in this case we should start with the more simple substitution

$$\Phi_0(x) = e^{-i\epsilon t}\Phi_0(r) \, , \qquad \vec{\Phi}(x) = e^{-i\epsilon t} \left(\begin{array}{c} 0 \\ \Phi_2(r) \\ 0 \end{array} \right) ,$$
$$\vec{E}(x) = e^{-i\epsilon t} \left(\begin{array}{c} 0 \\ E_2(r) \\ 0 \end{array} \right) , \qquad \vec{H}(x) = e^{-i\epsilon t} \left(\begin{array}{c} 0 \\ H_2(r) \\ 0 \end{array} \right) . \tag{2.12}$$

The angular operators $\Sigma_{\theta,\phi}$ act as follows: $\Sigma_{\theta,\phi}\Psi = 0$; the parity is $P = (-1)^{0+1} = -1$. In order to exclude the imaginary unit from the arising four equations, we will use the variables $\Phi_0 = \varphi_0,\ -i\Phi_1 = \varphi_1,\ -i\Phi_2 = \varphi_2$. Then the radial system reads

$$H_2 = 0\ ,\ -(\frac{d}{dr} + \frac{2}{r})E_2 = m\varphi_0\ ,\ (\epsilon + \frac{\alpha}{r})E_2 = m\varphi_2\ ,\ (\epsilon + \frac{\alpha}{r})\varphi_2 - \frac{d}{dr}\varphi_0 = mE_2, \tag{2.13}$$

whence it follows a 2-nd order equation for the main function E_2:

$$[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + (\epsilon + \frac{\alpha}{r})^2 - m^2]E_2 = 0\ . \tag{2.14}$$

Its solutions are constructed in terms of confluent hypergeometric functions. We write down only the relevant energy spectrum

$$E = Mc^2\left(1 + \frac{\alpha^2}{(n+\Gamma)^2}\right)^{-1/2}\ , \qquad \Gamma = \frac{1+\sqrt{9-4\alpha^2}}{2},\ \ n = 0,1,2,... \tag{2.15}$$

3 Lorentz-like condition for particle wave functions

It is known that for a massive spin 1 particle in external electromagnetic field there exists a generalized Lorentz constraint for the wave function of the particle. To get its form, it is convenient to start with the tensor equations in Proca form [5]:

$$D_\alpha\ \Phi_\beta - D_\beta\ \Phi_\alpha = m\ \Phi_{\alpha\beta}\ , \qquad D^\alpha\ \Phi_{\alpha\beta} = m\ \Phi_\beta\ , \tag{3.1}$$

where $D_\alpha = \nabla_\alpha + ieA_\alpha$. Acting on the second equation in (3.1) by the operator D_α, we derive the following relationship

$$(\nabla_\alpha + ieA_\alpha)\ \Phi^\alpha = \frac{i\alpha}{2m}\ F_{\alpha\beta}\ \Phi^{\alpha\beta}\ . \tag{3.2}$$

This can be transformed to the usual form of the wave functions, and leads to the following radial relationship [5]:

$$-i\ (\epsilon + \frac{\alpha}{r})\ \Phi_0 - \ (\frac{d}{dr} + \frac{2}{r})\ \Phi_2 - \frac{\nu}{r}\ (\Phi_1 + \Phi_3) = \frac{i\alpha}{2mr^2}\ E_2\ . \tag{3.3}$$

For states with parity $P = (-1)^{j+1}$, the relation (3.3) holds identically. For states with parity $P = (-1)^j$, it reads

$$-i\ (\epsilon + \frac{\alpha}{r})\ \Phi_0 - \ (\frac{d}{dr} + \frac{2}{r})\ \Phi_2 - \frac{2\nu}{r}\ \Phi_1 = \frac{i\alpha}{2mr^2}\ E_2\ . \tag{3.4}$$

With the use of relation (3.4), from the system (2.11) we can derive a more simple constraint on the radial functions. To this end, from eq. (3.4) let us exclude the function Φ_2 with the help of the third equation in (2.11); this yields

$$i\ (\epsilon + \frac{\alpha}{r})\ m\Phi_0 + i(\epsilon + \frac{\alpha}{r})(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{2i\nu}{r}(\frac{d}{dr} + \frac{1}{r})H_1 + \frac{2m\nu}{r}\ \Phi_1 = \frac{i\alpha}{2r^2}\ E_2\ . \tag{3.5}$$

Transforming the second and the third terms with the help of the first and second equations from (2.11), we obtain

$$i\ (\epsilon + \frac{\alpha}{r})\ m\Phi_0 - im(\epsilon + \frac{\alpha}{r})\Phi_0 = \frac{i\alpha}{2r^2}\ E_2 \quad \Longrightarrow \quad E_2 = 0\ .$$

Thus, we have the following constraint:

$$E_2 = 0\,. \tag{3.6}$$

In fact, it means that in the subsystem of 6 equations we have only 5 independent functions.

4 Differential 2-order equation for Φ_0

From the system (2.11) with the relation $E_2 = 0$ taken into account:

$$
(4.1) \qquad \begin{array}{ll}
1)\; mE_1 = -\frac{m^2}{2\nu}\, r\Phi_0 \,, & 2)\; (\epsilon + \frac{\alpha}{r})\, mE_1 + (\frac{d}{dr} + \frac{1}{r})mH_1 + m^2\varphi_1 = 0 \,, \\
3)\; \frac{2\nu}{mr} H_1 = \varphi_2 \,, & 4)\; -(\epsilon + \frac{\alpha}{r})\varphi_1 + \frac{\nu}{r}\Phi_0 = mE_1 \,, \\
5)\; (\epsilon + \frac{\alpha}{r})\varphi_2 + \frac{d}{dr}\Phi_0 = 0 \,, & 6)\; (\frac{d}{dr} + \frac{1}{r})\varphi_1 + \frac{\nu}{r}\varphi_2 + mH_1 = 0,
\end{array}
$$

we can derive a rather simple equation for the function Φ_0.

To this end, first by applying eqs. 3) and 4), we exclude the functions φ_2 and E_1:

$$
(4.2) \qquad \begin{array}{l}
1)\; -\frac{2\nu}{r}(\epsilon + \frac{\alpha}{r})\varphi_1 + (\frac{2\nu^2}{r^2} + m^2)\Phi_0 = 0 \,, \\
2)\; (\frac{d}{dr} + \frac{1}{r})H_1 + (\epsilon + \frac{\alpha}{r})\frac{1}{m}\frac{\nu}{r}\Phi_0 + \frac{1}{m}[m^2 - (\epsilon + \frac{\alpha}{r})^2]\varphi_1 = 0 \,, \\
5)\; \frac{d}{dr}\Phi_0 + \frac{2\nu}{mr}(\epsilon + \frac{\alpha}{r})H_1 = 0 \,, \\
6)\; (\frac{d}{dr} + \frac{1}{r})\varphi_1 + \frac{1}{m}[(m^2 + \frac{2\nu^2}{r^2})H_1 = 0 \,.
\end{array}
$$

By acting over eq. 5) in (4.2) by the operator $\frac{d}{dr}$, we infer:

$$
\frac{d^2}{dr^2}\Phi_0 - \frac{2\nu}{mr^2}(\epsilon + \frac{\alpha}{r})H_1 - \frac{2\nu}{mr}\frac{\alpha}{r^2}H_1 + \frac{2\nu}{mr}(\epsilon + \frac{\alpha}{r})\frac{d}{dr}H_1 = 0 \,.
$$

Now, with the help of eq. 2) in (4.2), we get

$$
\frac{d}{dr}H_1 = -\left[\frac{1}{r}H_1 + (\epsilon + \frac{\alpha}{r})\frac{1}{m}\frac{\nu}{r}\Phi_0 + \frac{1}{m}[m^2 - (\epsilon + \frac{\alpha}{r})^2]\varphi_1\right] \,,
$$

and

$$
\frac{d^2}{dr^2}\Phi_0 - \frac{2\nu}{mr^2}(\epsilon + \frac{\alpha}{r})H_1 - \frac{2\nu}{mr}\frac{\alpha}{r^2}H_1 -
$$
$$
-\frac{2\nu}{mr}(\epsilon + \frac{\alpha}{r})\left[\frac{1}{r}H_1 + (\epsilon + \frac{\alpha}{r})\frac{1}{m}\frac{\nu}{r}\Phi_0 + \frac{1}{m}[m^2 - (\epsilon + \frac{\alpha}{r})^2]\varphi_1\right], = 0
$$

or

$$
\left[\frac{d^2}{dr^2} - \frac{2\nu^2}{m^2r^2}(\epsilon + \frac{\alpha}{r})^2\right]\Phi_0 -
$$
$$
-\left[\frac{2\nu}{mr^2}(\epsilon + \frac{\alpha}{r}) + \frac{2\nu}{mr}\frac{\alpha}{r^2} + \frac{2\nu}{mr^2}(\epsilon + \frac{\alpha}{r})\right]H_1 \frac{2\nu}{m^2r}(\epsilon + \frac{\alpha}{r})\left[m^2 - (\epsilon + \frac{\alpha}{r})^2\right]\varphi_1 = 0 \,.
$$

In order to exclude the function φ_1, we use eq. 1) in (4.2):

$$
\frac{2\nu}{mr}(\epsilon + \frac{\alpha}{r})\varphi_1 = \frac{1}{m}(m^2 + \frac{2\nu^2}{r^2})\Phi_0 \,;
$$

hence producing

$$
(4.3) \qquad \left[\frac{d^2}{dr^2} + (\epsilon + \frac{\alpha}{r})^2 - m^2 - \frac{2\nu^2}{r^2}\right]\Phi_0 - \frac{4\nu}{mr^2}(\epsilon + \frac{\alpha}{r})H_1 - \frac{2\nu}{mr}\frac{\alpha}{r^2}H_1 = 0 \,.
$$

Now, we use eq. 5) from (4.2)

$$
\frac{d}{dr}\Phi_0 + \frac{2\nu}{mr}(\epsilon + \frac{\alpha}{r})H_1 = 0 \,;
$$

and further obtain

$$
\left[\frac{d^2}{dr^2} + (\epsilon + \frac{\alpha}{r})^2 - m^2 - \frac{2\nu^2}{r^2} + \frac{2}{r}\frac{d}{dr}\right]\Phi_0 - \frac{2\nu}{mr}\frac{\alpha}{r^2}H_1 = 0 \,.
$$

Finally, we apply again eq. 5) from (4.2), and yield:

$$-\frac{mr}{2\nu}\frac{1}{(\epsilon+\frac{\alpha}{r})}\frac{d}{dr}\Phi_0 = H_1 \ ,$$

so arriving at the equation for a single function Φ_0:

$$\left[\frac{d^2}{dr^2}+(\frac{3}{r}-\frac{\epsilon}{\epsilon r+\alpha})\frac{d}{dr}+\epsilon^2-m^2+\frac{2\epsilon\alpha}{r}+\frac{\alpha^2-2\nu^2}{r^2}\right]\Phi_0 = 0 \ . \tag{4.4}$$

In the variable

$$z=-\frac{\epsilon}{\alpha}\ r\ <0\ , \qquad r=-\frac{\alpha}{\epsilon}\ z \tag{4.5}$$

the last equation takes the form

$$\frac{d^2\Phi_0}{dz^2}+(\frac{3}{z}-\frac{1}{z-1})\frac{d\Phi_0}{dz}+(\alpha^2-\frac{\alpha^2}{E_0^2}-\frac{2\alpha^2}{z}-\frac{2\nu^2-\alpha^2}{z^2})\Phi_0=0\ , \tag{4.6}$$

where all quantities are dimensionless: $m^2/\epsilon^2 = M^2c^4/E^2 = 1/E_0^2$. For shortness, we will apply the notations

$$\Gamma^2 = 2\nu^2-\alpha^2 = j(j+1)-\alpha^2>0\ , \qquad -\Lambda^2 = -(-\alpha^2+\frac{\alpha^2}{E_0^2}) = -\alpha^2\frac{1-E_0^2}{E_0^2}<0\ ;$$

then eq. (4.6) reads

$$\frac{d^2\Phi_0}{dz^2}+(\frac{3}{z}-\frac{1}{z-1})\frac{d\Phi_0}{dz}+(-\Lambda^2-\frac{2\alpha^2}{z}-\frac{\Gamma^2}{z^2})\Phi_0=0\ . \tag{4.7}$$

Let us define the squared linear momentum

$$P^2(z) = (-\Lambda^2-\frac{2\alpha^2}{z}-\frac{\Gamma^2}{z^2}); \tag{4.8}$$

in physical singular points it behaves as follows

$$z\to 0 \qquad P^2(x)\sim -\frac{\Gamma^2}{z^2}\sim -\infty, \qquad z\to\infty \qquad P^2(x)\sim -\Lambda^2<0\ .$$

Two turning points, the root of the equation $\Lambda^2z^2+2\alpha^2z+\Gamma^2=0$, are

$$z_{1,2} = \frac{-\alpha^2\pm\sqrt{\alpha^4-\Gamma^2\Lambda^2}}{\Lambda^2}\ . \tag{4.9}$$

They both are negative, and belong to the physical region, if

$$\alpha^4-\Gamma^2\Lambda^2<0 \quad \Longrightarrow \quad E_0^2<1-\frac{\alpha^2}{\Gamma^2+\alpha^2}\ . \tag{4.10}$$

This qualitative consideration shows that we may expect existence of solutions associated with bound states.

5 Analytical study of the 2-nd order ODE for Φ_0

Equation (4.7) has two regular singularities, $z=0$ and $z=1$ and one irregular singularity, $z=\infty$ of the rank 2. In the vicinity of the point $z=0$, we have

$$\frac{d^2\Phi_0}{dz^2}+\frac{3}{z}\frac{d\Phi_0}{dz}-\frac{\Gamma^2}{z^2}\Phi_0=0\,,\qquad \Phi_0\sim z^A,$$

(5.1) $$A_1=-1+\sqrt{1+\Gamma^2}>0\,,\ A_2=-1-\sqrt{1+\Gamma^2}<0\,;$$

to bound state may correspond only solutions with positive A. When $z\to\infty$, the solutions behave in accordance with the formulas

(5.2) $$\frac{d^2\Phi_0}{dz^2}+\frac{2}{z}\frac{d\Phi_0}{dz}-\Lambda^2\Phi_0=0\,,\quad \Phi_0=e^{+\sqrt{\Lambda^2}\,z}=e^{-\sqrt{M^2c^4-E^2}\,r/\hbar c}\,;$$

to bound state may correspond only solutions which vanish at infinity. Near the nonphysical point $z=1$, the solutions behave in a quite regular manner:

(5.3) $$\Phi_0(z)\sim(z-1)^\sigma,\quad \sigma=0,2\,.$$

The general solutions of eq. (4.6) should be constructed in the form $\Phi_0(z)=z^Ae^{Bz}f(z)$; the equation for f is

$$f''+(2B+\frac{2A+3}{z}-\frac{1}{z-1})f'+$$
$$+\left[(B^2-\Lambda^2)+\frac{A^2+2A-\Gamma^2}{z^2}+\frac{2AB+A+3B-2\alpha^2}{z}-\frac{A+B}{z-1}\right]f=0\,.$$

We take A and B as shown

(5.4) $$A=-1+\sqrt{1+\Gamma^2}\,,\quad B=+\sqrt{\Lambda^2}\,;$$

then the last equation reduces to the more simple structure

(5.5) $$f''+(2B+\frac{2A+3}{z}-\frac{1}{z-1})f'+(\frac{2AB+A+3B-2\alpha^2}{z}-\frac{A+B}{z-1})f=0\,.$$

This can be recognized as a confluent Heun equation [7]

(5.6) $$H''+(-t+\frac{c}{z}+\frac{d}{z-1})H'+\frac{\lambda-ta\,z}{z(z-1)}H=0\,,$$

with parameters

$$t=-2B\,,\qquad c=2A+3\,,\quad d=-1\,,$$
(5.7) $$-\lambda=2AB+3B+A-2\alpha^2\,,\quad -ta=2BA+2B-2\alpha^2\,.$$

In particular, $a=A+1-\alpha^2/B$; and further (see (5.4)) we get

(5.8) $$a=+\sqrt{1+\Gamma^2}-\alpha^2/\Lambda\,.$$

Solutions for function f may be searched in the form of power series: $f=\sum_{k=0}^{\infty}d_kz^k$. Taking in mind the equation

$$\sum_{n=2}^{\infty}n(n-1)d_nz^n-\sum_{n=1}^{\infty}(n+1)nd_{n+1}z^n-$$

$$-t\sum_{n=2}^{\infty}(n-1)d_{n-1}z^n+(t+d+c)\sum_{n=1}^{\infty}nd_nz^n-c\sum_{n=0}^{\infty}(n+1)d_{n+1}z^n+$$

$$+\lambda \sum_{n=0}^{\infty} d_n z^n - ta \sum_{n=1}^{\infty} d_{n-1} z^n = 0 \; ,$$

we arrive at the 3-term recurrent relations

$$n = 0 \; , \qquad c\, d_1 + \lambda\, d_0 = 0 \; ;$$

$$n \geq 1, 2, 3, ... \qquad\qquad t\,(n-1+a)\, d_{n-1}$$

$$-[\, n(n-1+t+d+c) + \lambda \,]\, d_n + (n+1)\,(n+c)\, d_{n+1} = 0 \; . \tag{5.9}$$

The main recurrent formula may be re-written as

$$n = 0, \qquad c\, d_1 + \lambda\, d_0 = 0 \; ,$$

$$n = 1, 2, ... \qquad P_n d_n - (Q_n + \lambda)\, d_{n+1} + R_n\, d_{n+2} = 0 \; , \tag{5.10}$$

where

$$P_n = t\,(n-1+a) \; , \quad Q_n = n(n-1+t+d+c) \; , \quad R_n = (n+1)\,(n+c) \; . \tag{5.11}$$

The relations (5.10) are equivalent to

$$\frac{1}{n^2} P_n - \frac{1}{n^2}(Q_n + \lambda)\, \frac{d_{n+1}}{d_n} + \frac{1}{n^2} R_n\, \frac{d_{n+2}}{d_{n+1}} \frac{d_{n+1}}{d_n} = 0 \; ;$$

whence for $n \to \infty$ we get a simple algebraic equation,

$$-r + r^2 = 0, \qquad \lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \lim_{n\to\infty} \frac{d_{n+2}}{d_{n+1}} = r \; .$$

According to the Poincaré–Perrone method, we yield that the minimal convergence radius is $R_{conv} = 1$. Another possibility is $R'_{conv} = \infty$. We may expect that the series converges in the domain with $R'_{conv} = \infty$, because near the third singular point $z = 1$ on the bound of the circle with radius 1 solutions behave themselves regularly[2].

It is known the possibility to get solutions of the confluent Heun equation in terms of polynomials [6]. To this end, we need to impose the first restriction

$$P_{n+1} = 0 \implies a = -n \; , \quad n \in \{0, 1, 2, ...\} \; . \tag{5.12}$$

and the second restriction $d_{n+1} = 0$; in this way, from the recurrent formulas there follows the breaking of the series to polynomials of power n

$$0 \cdot d_n - (Q_{n+1} + \lambda)\; \cdot 0 + R_{n+1}\, d_{n+2} = 0 \quad \Longrightarrow \quad d_{n+2} = 0 \; ;$$

Indeed, two above restrictions lead to the linear system

$$\begin{pmatrix} -\lambda & c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ P_1 & \nu_1 & R_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & P_2 & \nu_2 & R_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & P_3 & \nu_3 & R_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & P_4 & \nu_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \nu_{n-2} & R_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & P_{n-1} & \nu_{n-1} & R_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_n & \nu_n \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_n \end{pmatrix} = 0 \; ,$$

[2]However, this issue should be studied more accurately.

where $\nu_k = -(Q_k + \lambda)$, $k = \overline{1,n}$, and its solutions exist if the determinant of the system equals to zero; in this way we get an algebraic equation of power n with respect to the parameter λ.

When imposing only the first constraint, $a = -n$, and ignoring the second one $d_{n+1} = 0$, we obtain the so called transcendental confluent Heun functions (which are not polynomials):

$$a = -n, \quad n = 0, 1, 2, 3... \tag{5.13}$$

Using the notation

$$N \equiv n + \sqrt{1 + \Gamma^2} = n + \sqrt{1 + j(j+1) - \alpha^2},$$

the constraint (5.13) leads to

$$N = \frac{\alpha^2}{\Lambda} \equiv \alpha \sqrt{\frac{E_0^2}{1 - E_0^2}} \,,$$

whence it follows the quantization rule for energy levels:

$$E_0 = \frac{1}{\sqrt{1 + \alpha^2 / N^2}}, \qquad N = n + \sqrt{1 + j(j+1) - \alpha^2} \,. \tag{5.14}$$

This formula seems to be reasonable from physical point of view. It may be considered as representing the second series of bound stated from expected three.

Further analysis shows that proceeding combining 6 equations from the system we are able to get 2-order equations for separate functions, which are characterized by only a few sets of singular points[3]. It may be understood as a good feature. However, still remains a number of questions with no replies. We do not know which spectra may arise from studying various 2-nd order equations for different functions. Variety of spectra should not be considered as a good result. Another difficult point consists in the following: in general we have no reliable method to derive the quantization rules for equation with complicated sets of singularities. Also, we should get an answer to the question – which forms of presenting the Lorentz condition are possible, does the form $E_2 = 0$ is unique.

In the following, we will turn back to the initial system of 6 equation, ignoring the above constraint $E_2 = 0$, in fact this established form may be rather accidental.

6 The system of 4 differential equations

Let us turn back to the system of 6 equations for states with the parity $P = (-1)^j, j = 1, 2, ...$:

$$+i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 - M\Phi_2 = 0 \,, \quad -i(\epsilon + \frac{\alpha}{r})\,\Phi_1 + \frac{\nu}{r}\Phi_0 - ME_1 = 0 \,;$$

$$(\frac{d}{dr} + \frac{2}{r})E_2 + 2\frac{\nu}{r}E_1 + M\Phi_0 = 0 \,, \quad +i(\epsilon + \frac{\alpha}{r})\,E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 - M\Phi_1 = 0 \,,$$

$$i\,(\epsilon + \frac{\alpha}{r})\Phi_2 + \frac{d}{dr}\Phi_0 + ME_2 = 0 \,, \quad i(\frac{d}{dr} + \frac{1}{r})\Phi_1 + i\frac{\nu}{r}\Phi_2 + MH_1 = 0 \,. \tag{6.1}$$

We note the physical dimensions of the involved quantities

$$M = \frac{mc}{\hbar} = \frac{1}{\lambda}, \quad [M] = \frac{1}{L} \,, \quad \epsilon = \frac{E}{\hbar c}, \quad [\epsilon] = \frac{1}{L} \,, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137} \,.$$

[3]Within this article we cannot detail these various equations.

It is convenient to work with the equations in dimensionless form, by taking the Compton wave length λ as a unit for the length, and the rest energy of a particle as a unit for energy. We further denote

$$rM \rightsquigarrow x, \quad \epsilon/M = E/mc^2 \rightsquigarrow \epsilon,$$

$$i(\epsilon + \frac{\alpha}{x})E_2 - 2i\frac{\nu}{x}H_1 - \Phi_2 = 0 \,, \quad -i(\epsilon + \frac{\alpha}{x})\ \Phi_1 + \frac{\nu}{x}\Phi_0 - E_1 = 0 \,;$$
$$(\frac{d}{dx} + \frac{2}{x})E_2 + 2\frac{\nu}{x}E_1 + \Phi_0 = 0 \,, \quad +i(\epsilon + \frac{\alpha}{x})\ E_1 + i(\frac{d}{dx} + \frac{1}{r})H_1 - \Phi_1 = 0 \,,$$
$$i(\epsilon + \frac{\alpha}{x})\Phi_2 + \frac{d}{dx}\Phi_0 + E_2 = 0 \,, \quad i(\frac{d}{dx} + \frac{1}{x})\Phi_1 + i\frac{\nu}{x}\Phi_2 + H_1 = 0 \,. \tag{6.2}$$

With the use of the substitutions

$$\Phi_1 = \frac{1}{x}\varphi_1, \qquad E_2 = \frac{1}{x^2}e_2, \quad H_1 = \frac{1}{x}h_1$$

the system reduces to a more simple and symmetrical form

$$\Phi_2 = i(\epsilon + \frac{\alpha}{x})\frac{1}{x^2}e_2 - 2i\frac{\nu}{x^2}h_1 \,, \quad E_1 = -i(\epsilon + \frac{\alpha}{x})\frac{1}{x}\varphi_1 + \frac{\nu}{x}\Phi_0 \,;$$
$$\frac{d}{dx}e_2 = -2\nu x E_1 - x^2\Phi_0 \,, \quad \frac{d}{dx}h_1 = -(x\epsilon + \alpha)\ E_1 - i\varphi_1 \,,$$
$$\frac{d}{dx}\Phi_0 = -i(\epsilon + \frac{\alpha}{x})\Phi_2 - \frac{1}{x^2}e_2 \,, \quad \frac{d}{dx}\varphi_1 = -\nu\Phi_2 + ih_1 \,. \tag{6.3}$$

Applying the two first (non-differential) equations, we exclude the functions Φ_2 and E_1:

$$\frac{d}{dx}e_2 = 2i\nu(\epsilon + \frac{\alpha}{x})\varphi_1 - (2\nu^2 + x^2)\Phi_0 \,,$$
$$\frac{d}{dx}h_1 = +i[(\epsilon + \frac{\alpha}{x})^2 - 1]\varphi_1 - \nu(\epsilon + \frac{\alpha}{x})\Phi_0 \ ; \tag{6.4}$$

$$\frac{d}{dx}\varphi_1 = -\frac{i\nu}{x^2}(\epsilon + \frac{\alpha}{x})e_2 + i(\frac{2\nu^2}{x^2} + 1)h_1 \,,$$
$$\frac{d}{dx}\Phi_0 = \frac{1}{x^2}[(\epsilon + \frac{\alpha}{x})^2 - 1]e_2 - \frac{2\nu}{x^2}(\epsilon + \frac{\alpha}{x})h_1 \,. \tag{6.5}$$

It is convenient to use the following notations

$$a = 2i\nu\frac{\epsilon x + \alpha}{x} \,, \quad c = -(2\nu^2 + x^2) \,, \quad d = i\frac{(\epsilon x + \alpha)^2 - x^2}{x^2}, \quad b = -\frac{\nu(\epsilon x + \alpha)}{x} \,,$$
$$A = -i\frac{\nu(\epsilon x + \alpha)}{x^3} \,, \qquad C = +i\frac{(2\nu^2 + x^2)}{x^2} \,, \quad D = \frac{(\epsilon x + \alpha)^2 - x^2}{x^4} \,, \qquad B = -\frac{2\nu(\epsilon x + \alpha)}{x^3} \,,$$
$$ab - cd = i\ p(x) \,, \qquad AB - CD = -i\frac{p(x)}{x^4}, \quad p(x) = [(\epsilon^2 - 1)x^2 + 2\alpha\epsilon x - (2\nu^2 - \alpha^2)] \,.$$

Also, we re-designate the functions: $e_2 = f_1,\ h_1 = f_2,\ \varphi_1 = f_3,\ \Phi_0 = f_4$. Then the system under consideration reads

$$\frac{d}{dx}f_1 = af_3 + cf_4 \,, \quad \frac{d}{dx}f_2 = df_3 + bf_4 \,;$$
$$\frac{d}{dx}f_3 = Af_1 + Cf_2 \,, \quad \frac{d}{dx}f_4 = Df_1 + Bf_2 \,. \tag{6.6}$$

In Section 3, with the use of the Lorentz condition, we derived the simple constraint $E_2 = 0$ ($f_1 = 0$). We examine the same constraint again, and extend this approach by imposing similar constraints on the other functions.

By requiring for the system (6.6) the restriction $f_1 = 0$, we get[4]

(6.7) $af_3 + cf_4 = 0 \, , \qquad df_3 + bf_4 = \frac{d}{dx} f_2 \, , \qquad \frac{d}{dx} f_3 = C f_2 \, , \qquad \frac{d}{dx} f_4 = B f_2 \, .$

Considering tho first equations as a linear system with respect to f_3 and f_4, we get

(6.8) $$f_3 = \frac{-c}{ab - cd} \frac{d}{dx} f_2, \qquad f_4 = \frac{a}{ab - cd} \frac{d}{dx} f_2 \, .$$

Substituting these formulas into the two remaining equations in (6.7), we obtain two different 2-nd order equations for the variable f_2:

(6.9) $$\frac{d}{dr} \frac{-c}{ab - cd} \frac{d}{dr} f_2^I = C f_2^I \, , \quad \left(\frac{d}{dx} \frac{2\nu^2 + x^2}{p(x)} \frac{d}{dx} + \frac{2\nu^2 + x^2}{x^2} \right) f_2^I = 0 \, ;$$

(6.10) $$\frac{d}{dr} \frac{a}{ab - cd} \frac{d}{dr} f_2^{II} = B f_2^{II} \, , \quad \left(\frac{d}{dx} \frac{\epsilon x + \alpha}{x \, p(x)} \frac{d}{dx} + \frac{(\epsilon x + \alpha)}{x^3} \right) f_2^{II} = 0 \, ;$$

Thus, the system (6.7), describing the projection of the whole solution $\{f_1, ..., f_4\}$ onto the plane $f_1 = 0$, may be solved on the base of the two main functions $f_1 = f_1^I, f_1^{II}$; they obey different 2-order equations, which lead to the different non-zero remaining functions f_3, f_4. In other words, the projection of the whole solution – the curve $\{f_i(x)\}$ onto the plane $f_1 = 0$ consists of the parts (branches), related respectively to the functions f_2^I and f_2^{II}. In fact, the concept of projection is determined by definition, and this definition permits us to get additional information about the needed whole solutions $\{f_i(x)\}$.

Similarly, by imposing the constraint $f_2 = 0$, we get the equations

(6.11) $af_3 + cf_4 = \frac{d}{dx} f_1, \quad 0 = df_3 + bf_4 \, , \quad \frac{d}{dx} f_3 = A f_1, \qquad \frac{d}{dx} f_4 = D f_1 \, .$

which result in

(6.12) $$f_3 = \frac{b}{ab - cd} \frac{d}{dx} f_1 \, , \quad f_4 = \frac{-d}{ab - cd} \frac{d}{dx} f_1 \, , \quad \frac{d}{dx} f_3 = A f_1, \quad \frac{d}{dx} f_4 = D f_1 \, ,$$

and the two equations for f_1:

(6.13) $$\frac{d}{dx} \frac{b}{ab - cd} \frac{d}{dx} f_1^I = A f_1^I, \quad \left(\frac{d}{dx} \frac{(\epsilon x + \alpha)}{x \, p(x)} \frac{d}{dx} + \frac{(\epsilon x + \alpha)}{x^3} \right) f_1^I = 0 \, ;$$

(6.14) $$\frac{d}{dx} \frac{-d}{ab - cd} \frac{d}{dx} f_1^{II} = D f_1^{II} \, , \left(\frac{d}{dx} \frac{(\epsilon x + \alpha)^2 - x^2}{x^2 p(x)} \frac{d}{dx} + \frac{(\epsilon x + \alpha)^2 - x^2}{x^4} \right) f_1^{II} = 0 \, .$$

By imposing the constraint $f_3 = 0$, we get the equations

$$\frac{d}{dx} f_1 = c f_4, \quad \frac{d}{dx} f_2 = b f_4 \, , \quad A f_1 + C f_2 = 0, \qquad D f_1 + B f_2 = \frac{d}{dx} f_4 \, ,$$

which result in

(6.15) $$f_1 = \frac{-C}{AB - CD} \frac{d}{dx} f_4, \quad f_2 = \frac{A}{AB - CD} \frac{d}{dx} f_4 \quad \frac{d}{dx} f_1 = c f_4, \quad \frac{d}{dx} f_2 = b f_4,$$

and the following two equations for f_4:

(6.16) $$\frac{d}{dx} \frac{-C}{AB - CD} \frac{d}{dx} f_4^I = c f_4^I \, , \quad \left(\frac{d}{dx} \frac{(2\nu^2 + x^2) x^2}{p(x)} \frac{d}{dx} + (2\nu^2 + x^2) \right) f_4^I = 0 \, ;$$

[4]Thereby we examine the projection of the curve $\{f_1, ..., f_4\}$ on the plane $f_1 = 0$.

$$\frac{d}{dx}\frac{A}{AB-CD}\frac{d}{dx}f_4^{II} = b\ f_4^{II}\ , \quad (\frac{d}{dx}\frac{(\epsilon x+\alpha)x}{p(x)}\frac{d}{dx} + \frac{(\epsilon x+\alpha}{x})f_4^{II} = 0 \tag{6.17}$$

By imposing the constraint $f_4 = 0$, we get the equations

$$Af_1 + Cf_2 = \frac{d}{dr}f_3, \quad Df_1 + Bf_2 = 0\ , \quad \frac{d}{dx}f_1 = af_3, \quad \frac{d}{dr}f_2 = df_3\ .$$

these yield

$$f_1 = \frac{B}{AB-CD}\frac{d}{dx}f_3, \quad f_2 = \frac{-D}{AB-CD}\frac{d}{dx}f_3\ , \quad \frac{d}{dx}f_1 = af_3, \quad \frac{d}{dx}f_2 = df_3\ , \tag{6.18}$$

and the two equations for f_3:

$$\frac{d}{dx}\frac{B}{AB-CD}\frac{d}{dx}f_3^I = a\ f_3^I, \quad (\frac{d}{dx}\frac{2\nu(\epsilon x+\alpha)x}{p(x)}\frac{d}{dx} + \frac{2\nu(\epsilon x+\alpha}{x})f_3^I = 0\ ; \tag{6.19}$$

$$\frac{d}{dx}\frac{-D}{AB-CD}\frac{d}{dx}f_3^{II} = d\ f_3^{II}\ , \quad (\frac{d}{dx}\frac{(\epsilon x+\alpha)^2-x^2}{p(x)} + \frac{(\epsilon x+\alpha)^2-x^2}{x^2})f_3^{II} = 0\ . \tag{6.20}$$

Let us write down the explicit form of all the derived 2-order differential equations, and fix their singular points. We recall that

$$p(x) = (\epsilon^2-1)x^2 + 2\epsilon\alpha x - (2\nu^2-\alpha^2) \equiv (\epsilon^2-1)(x-x_1)(x-x_2)\ .$$

$$x_{1,2} = \frac{\epsilon \pm \sqrt{2\nu^2\epsilon^2-(2\nu^2-\alpha^2)}}{1-\epsilon^2}\ ;$$

these roots are complex-valued in the case of bound states: $0<\epsilon<1$.

The projection $f_1 = 0$.

We have

$$\left(\frac{d}{dx}\ \frac{2\nu^2+x^2}{p(x)}\ \frac{d}{dx} + \frac{2\nu^2+x^2}{x^2}\right)f_2^I = 0,$$

$$[\frac{d^2}{dx^2} + (\frac{2x}{x^2+2\nu^2} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}]\,f_2^I = 0\ , \tag{6.21}$$

and the singular points $x_1,\ x_2,\ x_{3,4} = \pm i\sqrt{2\nu^2},\ 0,\ \infty_{[2]}$.

Further, the equation

$$[\frac{d^2}{dx^2} + (\frac{\epsilon}{\epsilon x+\alpha} - \frac{1}{x} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}]\,f_2^{II} = 0\ ; \tag{6.22}$$

has the singular points $x_1,\ x_2,\ x_5 = -\frac{\alpha}{\epsilon},\ 0,\ \infty_{[2]}$.

The projection $f_2 = 0$.

We have

$$[\frac{d^2}{dx^2} + (\frac{\epsilon}{\epsilon x+\alpha} - \frac{1}{x} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}]\,f_1^I = 0\ . \tag{6.23}$$

and

$$[\frac{d^2}{dx^2} + (\frac{2(\epsilon x+\alpha)\epsilon - 2x}{(\epsilon x+\alpha)^2-x^2} - \frac{2}{x} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}]\,f_1^{II} = 0\ , \tag{6.24}$$

$$(\epsilon x+\alpha)^2 - x^2 = 0 \quad \Longrightarrow \quad x_{3,4} = -\frac{\alpha}{\epsilon+1}, \frac{\alpha}{1-\epsilon} \ ,$$

with the singular points $x_1, x_2, \quad x_3, x_4, \quad 0, \quad \infty_{[2]}$.

The projection $f_3 = 0$.

We get

$$[\frac{d^2}{dx^2} + \left(\frac{2x}{2\nu^2+x^2} + \frac{2}{x} - \frac{p'}{p} \right) \frac{d}{dx} - \frac{p}{x^2}\,]\, f_4^I = 0 \ , \qquad (6.25)$$

and the singular points $x_1, x_2,\ x_{3,4} = \pm i\sqrt{2\nu^2},\ 0,\ \infty_{[2]}$.

Further, we get

$$[\frac{d^2}{dx^2} + (\frac{\epsilon}{\epsilon x+\alpha} + \frac{1}{x} - \frac{p'}{p}) + \frac{p}{x^2}\,]\, f_4^{II} = 0, \qquad (6.26)$$

and the singular points $x_1,\ x_2,\ x_5 = -\frac{\alpha}{\epsilon},\ 0,\ \infty_{[2]}$.

The Projection $f_4 = 0$.

We have

$$[\frac{d^2}{dx^2} + (\frac{\epsilon}{\epsilon x+\alpha} + \frac{1}{x} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}\,]\, f_3^I = 0 \ , \qquad (6.27)$$

and the singular points $x_1,\ x_2,\ x_5 = -\frac{\alpha}{\epsilon},\ 0,\ \infty_{[2]}$.

Further, we obtain

$$[\{\frac{d^2}{dx^2} + (\frac{2(\epsilon x+\alpha)\epsilon - 2x}{(\epsilon x+\alpha)^2 - x^2} - \frac{p'}{p})\frac{d}{dx} + \frac{p}{x^2}\,]\, f_3^{II} = 0 \ , \qquad (6.28)$$

and the singular points $x_1, x_2,\ x_3, x_4,\ 0,\ \infty_{[2]}$.

7 The 4-th order differential equations

We start with the system

$$\frac{d}{dx} f_1 = a f_3 + c f_4 \ , \qquad \frac{d}{dx} f_2 = d f_3 + b f_4 \ ,$$

$$\frac{d}{dx} f_3 = A f_1 + C f_2 \ , \qquad \frac{d}{dx} f_4 = D f_1 + B f_2 \ . \qquad (7.1)$$

It is equivalent to the following

$$f_1 = \frac{B f_3' - C f_4'}{AB - CD}, \quad f_2 = \frac{-D f_3' + A f_4'}{AB - CD} \ , \quad f_3 = \frac{b f_1' - c f_2'}{ab - cd}, \quad f_4 = \frac{-d f_1' + a f_2'}{ab - cd} \ . \qquad (7.2)$$

First, we exclude functions f_3 and f_4:

$$f_1 = \frac{B}{(AB-CD)} \frac{d}{dx} \frac{b f_1' - c f_2'}{ab - cd} - \frac{C}{(AB-CD)} \frac{d}{dx} \frac{-d f_1' + a f_2'}{ab - cd},$$

$$f_2 = -\frac{D}{(AB-CD)} \frac{d}{dx} \frac{b f_1' - c f_2'}{ab - cd} + \frac{A}{(AB-CD)} \frac{d}{dx} \frac{-d f_1' + a f_2'}{ab - cd} \ ,$$

Taking in mind the expressions for $a(x), ..., D(x)$, the last equations may be written as:

$$\left(K_2(x) \frac{d^2}{dx^2} + K_1(x) \frac{d}{dx} + K_0(x) \right) f_1 = \frac{df_2}{dx} \ ,$$

$$\left(L_2(x) \frac{d^2}{dx^2} + L_1(x) \frac{d}{dx} + L_0(x) \right) f_2 = \frac{df_1}{dx} \ , \qquad (7.3)$$

where the following notations are used

$$K_2(x) = \frac{1}{2}\,\frac{-x^5\epsilon^2 - 2\,x^4\alpha\,\epsilon + 2\,\nu^2 x^3 + x^5 - x^3\alpha^2}{x\,(2\,\epsilon\,x^3 + 3\,\alpha\,x^2 + 2\,\alpha\,\nu^2)\,\nu}\,,$$

$$K_1(x) = \frac{1}{2}\,\frac{\epsilon^2 - 1}{\epsilon\,\nu} + \frac{1}{2}\,\frac{\alpha\,\left(3\,x^2 - x^2\epsilon^2 - \epsilon\,x\alpha + 2\,\nu^2\right)}{\epsilon\,\nu\,(2\,\epsilon\,x^3 + 3\,\alpha\,x^2 + 2\,\alpha\,\nu^2)} + \frac{1}{2}\,\frac{\alpha}{x\nu}\,,$$

$$K_0(x) = -\frac{1}{4}\,\frac{\left(\left(\epsilon^2 - 1\right)x^2 + 2\,\epsilon\,x\alpha - 2\,\nu^2 + \alpha^2\right)^2}{\nu\,(\epsilon\,x^3 + 3/2\,\alpha\,x^2 + \alpha\,\nu^2)}\,,$$

$$L_2(x) = \frac{\left(x^5\epsilon^2 + 2\,x^4\alpha\,\epsilon - x^5 + x^3\alpha^2 - 2\,\nu^2x^3\right)x}{(x^2 + x^2\epsilon^2 + 2\,\epsilon\,x\alpha + \alpha^2)\,\nu\,\alpha}\,,$$

$$L_1(x) = \frac{\left(2\,\epsilon\,x\alpha\,\nu^2 + 2\,x^3\epsilon\,\alpha + 2\,x^2\alpha^2 + 2\,\nu^2\alpha^2\right)x}{(x^2 + x^2\epsilon^2 + 2\,\epsilon\,x\alpha + \alpha^2)\,\nu\,\alpha}\,,$$

$$L_0(x) = \frac{\left(\left(\epsilon^2 - 1\right)x^2 + 2\,\epsilon\,x\alpha - 2\,\nu^2 + \alpha^2\right)^2 x^2}{\nu\,(x^2 + x^2\epsilon^2 + 2\,\epsilon\,x\alpha + \alpha^2)\,\alpha}\,.$$

Let us exclude the function f_2 from equations (7.3):

$$f_2(x) = \int \left(K_2(x)\frac{d^2}{dx^2} + K_1(x)\frac{d}{dx} + K_0(x)\right) f_1\,,$$

$$(L_2\frac{d}{dx} + L_1)(K_2\frac{d^2}{dx^2} + K_1\frac{d}{dx} + K_0)\;f_1 + L_0\int dx(K_2\frac{d^2}{dx^2} + K_1\frac{d}{dx} + K_0)f_1 = 0$$

The second relation should be divided by $L_0(x)$ and the result be differentiated. In this way, we obtain a 4-order equation for $f_1(x)$:

$$\left\{\frac{d}{dx}(\frac{L_2}{L_0}\frac{d}{dx} + \frac{L_1}{L_0})(K_2\frac{d^2}{dx^2} + K_1\frac{d}{dx} + K_0) + (K_2\frac{d^2}{dx^2} + K_1\frac{d}{dx} + K_0)\right\} f_1(x) = 0\,. \qquad (7.4)$$

Similarly, we obtain a 4-order equation for f_2:

$$\left\{\frac{d}{dx}(\frac{K_2}{K_0}\frac{d}{dx} + \frac{K_1}{K_0})(L_2\frac{d^2}{dx^2} + L_1\frac{d}{dx} + L_0) + (L_2\frac{d^2}{dx^2} + L_1\frac{d}{dx} + L_0)\right\} f_2(x) = 0\,. \qquad (7.5)$$

Now, we shall exclude the functions f_1 and f_2 from the equations

$$f_1 = \frac{Bf_3' - Cf_4'}{AB - CD}, \quad f_2 = \frac{-Df_3' + Af_4'}{AB - CD}\,, \quad f_3 = \frac{bf_1' - cf_2'}{ab - cd}, \quad f_4 = \frac{-df_1' + af_2'}{ab - cd}.$$

This results in

$$f_3 = \frac{b}{ab - cd}\,\frac{d}{dx}\,\frac{Bf_3' - Cf_4'}{AB - CD} - \frac{c}{ab - cd}\,\frac{d}{dx}\,\frac{-Df_3' + Af_4'}{AB - CD}\,,$$

$$f_4 = -\frac{d}{ab - cd}\,\frac{d}{dx}\,\frac{Bf_3' - Cf_4'}{AB - CD} + \frac{a}{ab - cd}\,\frac{d}{dx}\,\frac{-Df_3' + Af_4'}{AB - CD}\,.$$

Taking into account the expressions for $a(x), ..., D(x)$, we reduce the last equations to the form

$$\left(P_2(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_0(x)\right) f_3 = \frac{df_4}{dx}\,,$$

$$\left(Q_2(x)\frac{d^2}{dx^2} + Q_1(x)\frac{d}{dx} + Q_0(x)\right) f_4 = \frac{df_3}{dx}\,, \qquad (7.6)$$

where the following notations were used

$$P_2(x) = \frac{ix^2\left(2\,\nu^2 - \epsilon^2x^2 - 2\,\epsilon\,x\alpha - \alpha^2 + x^2\right)}{\nu\,\left(2\,x^3\epsilon + 2\,\nu^2\alpha + 3\,x^2\alpha\right)}\,,$$

$$P_1(x) = \frac{2\,i\nu\,\left(\epsilon\,x\alpha + \alpha^2 + 2\,x^2\right)}{x\left(2\,x^3\epsilon + 2\,\nu^2\alpha + 3\,x^2\alpha\right)}\,,$$

$$P_0(x) = \frac{-i\left(\left(\epsilon^2-1\right)x^2 + 2\,\epsilon\,x\alpha - 2\,\nu^2 + \alpha^2\right)^2}{2\,\nu\,\epsilon\,x^3 + 3\,\nu\,\alpha\,x^2 + 2\,\nu^3\alpha}\,,$$

$$Q_2(x) = \frac{1}{2}\,\frac{ix^4\left(2\,\nu^2 - \epsilon^2x^2 - 2\,\epsilon\,x\alpha - \alpha^2 + x^2\right)}{\nu\,\alpha\,\left(2\,\epsilon\,x\alpha + x^2 + \epsilon^2x^2 + \alpha^2\right)}\,,$$

$$Q_1(x) = \frac{ix\left(2\,\nu^2x^2 - \nu^2\alpha^2 - \nu^2\epsilon\,x\alpha - x^4\epsilon^2 - 2\,x^2\alpha^2 + x^4 - 3\,x^3\epsilon\,\alpha\right)}{\nu\,\alpha\,\left(2\,\epsilon\,x\alpha + x^2 + \epsilon^2x^2 + \alpha^2\right)}\,,$$

$$Q_0(x) = \frac{-1/2\,i\left(\left(\epsilon^2-1\right)x^2 + 2\,\epsilon\,x\alpha - 2\,\nu^2 + \alpha^2\right)^2x^2}{\nu\,\alpha\,\left(x^2 + \epsilon^2x^2 + 2\,\epsilon\,x\alpha + \alpha^2\right)}\,.$$

By acting in accordance with the above method (see (7.4)–(7.5)) we derive 4-order equations for functions f_1, f_2, f_3, f_4.

The equations for f_1 and f_3 have the same set of singular points (3 regular and 2 irregular of the rank 3, and 2, respectively):

$$(2\epsilon x^3 + 3\alpha x^2 + 2\nu^2\alpha) = 2\epsilon(x-x_1)(x-x_2)(x-x_3)\quad,\quad x = 0_{[2]},\quad x = \infty_{[2]}\,; \tag{7.7}$$

$$f_1'''' + \left[-\frac{12x\,(\epsilon x + \alpha)}{2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2} + \frac{6}{x}\right] f_1''' +$$

$$+\left[-2 + 2\epsilon^2 - \frac{18\alpha\left(2\nu^2\alpha + 4\epsilon\nu^2x - \alpha x^2\right)}{\left(2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2\right)^2} + \frac{6 + 2\alpha^2 - 4\nu^2}{x^2} + \frac{-30\alpha - 12\epsilon x}{2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2} + \frac{4\epsilon\alpha}{x}\right] f_1'' +$$

$$+\Bigg[\frac{72\alpha x\,(\epsilon x + \alpha)}{\left(2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2\right)^2} + \frac{8\epsilon\alpha}{x^2} + \frac{-4\nu^2 + 2\alpha^2}{x^3} + \frac{6\nu^2 - 6\alpha^2 - 12 + 6\epsilon^2\nu^2}{x\nu^2} +$$

$$+\frac{24\nu^4\epsilon - 36\epsilon\alpha^2\nu^2 - 24x\nu^2\alpha + 18\alpha^3x - 36\alpha x\epsilon^2\nu^2 + 36\alpha x - 12x^2\nu^2\epsilon^3 + 24x^2\epsilon - 12x^2\epsilon\nu^2 + 12x^2\epsilon\alpha^2}{\left(2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2\right)\nu^2}\Bigg] f_1' +$$

$$+\Bigg[1 - 2\epsilon^2 + \epsilon^4 + \frac{-6\alpha^2 + 6\epsilon^2\nu^2 + 6\epsilon^2\alpha^2\nu^2 + 6\nu^2 - 4\epsilon^2\nu^4 + 4\nu^4 - 2\alpha^2\nu^2}{x^2\nu^2} - \frac{4\epsilon\alpha\left(2\nu^2 - \alpha^2\right)}{x^3} +$$

$$+\frac{-18\,\epsilon^2\alpha^2\nu^2 + 18\alpha^4 - 18\alpha^2\nu^2 - 84\alpha\epsilon x\nu^2 + 120\alpha^3\epsilon x - 12\alpha\epsilon^3x\nu^2 - 48x^2\epsilon^2\nu^2 + 72x^2\epsilon^2\alpha^2}{\left(2\epsilon x^3 + 2\nu^2\alpha + 3\alpha x^2\right)\nu^2\alpha} +$$

$$+\frac{72\alpha^4 - 180\epsilon^2\alpha^2\nu^2 - 108\alpha^2\nu^2 - 72\alpha\epsilon^3x\nu^2 - 216\alpha\epsilon x\nu^2 + 288\alpha^3\epsilon x - 144x^2\epsilon^2\nu^2 + 162x^2\epsilon^2\alpha^2 - 18x^2\alpha^2}{\left(2\epsilon x^3 + 2,\,\nu^2\alpha + 3\alpha x^2\right)^2} +$$

$$+\frac{-4\alpha^2 + \alpha^4 + 4\nu^4 - 4\alpha^2\nu^2}{x^4} - \frac{2\alpha^2\nu^2}{x^6} + \frac{4\epsilon\left(-9\alpha^2 - \alpha^2\nu^2 + 6\nu^2 + \epsilon^2\alpha^2\nu^2\right)}{x\nu^2\alpha}\Bigg] f_1 = 0\,, \tag{7.8}$$

$$f_3'''' + \left[-\frac{12x\,(\epsilon x + \alpha)}{2x^3\epsilon + 3\alpha x^2 + 2\nu^2\alpha} + \frac{10}{x}\right] f_3''' +$$

$$+\left[2\epsilon^2 - 2 + \frac{2\alpha^2 - 4\nu^2 + 24}{x^2} + \frac{-66\alpha - 48\epsilon x}{2x^3\epsilon + 3\alpha x^2 + 2\nu^2\alpha} - \frac{18\alpha\left(2\nu^2\alpha + 4\,x\nu^2\epsilon - \alpha x^2\right)}{\left(2x^3\epsilon + 3\alpha x^2 + 2\nu^2\alpha\right)^2} + \frac{4\alpha\epsilon}{x}\right] f_3'' +$$

$$+\Bigg[\frac{16\alpha\epsilon}{x^2} + \frac{12 + 6\alpha^2 - 12\nu^2}{x^3} - \frac{72\alpha\left(2\epsilon\nu^2 - 3\alpha x - 2\epsilon x^2\right)}{\left(2x^3\epsilon + 3\alpha x^2 + 2\nu^2\alpha\right)^2} +$$

$$+\frac{24\epsilon\nu^4 - 36\epsilon\alpha^2\nu^2 - 24\epsilon\nu^2 + 18\alpha^3x - 24\alpha x\nu^2 - 36\alpha x\epsilon^2\nu^2 + 162\alpha x - 12x^2\nu^2\epsilon^3 + 108\epsilon x^2 + 12x^2\epsilon\alpha^2 - 12x^2\epsilon\nu^2}{\left(2x^3\epsilon + 3\alpha x^2 + 2\nu^2\alpha\right)\nu^2} +$$

$$+\frac{-54-6\alpha^2+2\nu^2+10\epsilon^2\nu^2}{x\nu^2}\Bigg] f_3'+$$

$$+\Bigg[-2\epsilon^2+\epsilon^4+1+\frac{-12\alpha^2+12\epsilon^2\nu^2+6\epsilon^2\alpha^2\nu^2+4\nu^4-2\alpha^2\nu^2-4\nu^4\epsilon^2+12\nu^2}{x^2\nu^2}+$$

$$+\frac{-54\epsilon^2\alpha^2\nu^2-42\alpha^2\nu^2+36\alpha^4-132\epsilon x\alpha\,\nu^2+186\alpha^3\epsilon x-24\epsilon^3 x\alpha\nu^2-72x^2\epsilon^2\nu^2+108\alpha^2\epsilon^2x^2}{\left(2x^3\epsilon+3\alpha x^2+2\nu^2\alpha\right)\nu^2\alpha}-$$

$$-\frac{4\alpha\epsilon\left(-2-\alpha^2+2\nu^2\right)}{x^3}-\frac{2\alpha^2\nu^2}{x^6}+$$

$$+\frac{72\alpha^4-180\epsilon^2\alpha^2\nu^2-108\alpha^2\nu^2-72\epsilon^3x\alpha\nu^2-216\epsilon x\alpha\nu^2+288\alpha^3\epsilon x+162\alpha^2\epsilon^2x^2-18\alpha^2x^2-144x^2\epsilon^2\nu^2}{\left(2x^3\epsilon+3\alpha x^2+2\nu^2\alpha\right)^2}+$$

$$+\frac{2\epsilon\left(-27\alpha^2-2\alpha^2\nu^2+18\nu^2+2\epsilon^2\alpha^2\nu^2\right)}{\alpha x\nu^2}+\frac{-2\alpha^2-4\alpha^2\nu^2-4\nu^2+\alpha^4+4\nu^4}{x^4}\Bigg] f_3=0\,, \tag{7.9}$$

The equations for f_2 and f_4 have the same set of singular points (2 regular and 2 irregular of the rank 3 and 2, respectively):

$$(1+\epsilon^2)x^2+2\epsilon\alpha x+\alpha^2=(1+\epsilon^2)(x-x_5)(x-x_6)\quad,\quad x=0,\quad x=\infty\,; \tag{7.10}$$

$$f_2''''+\left[\frac{-4\epsilon\alpha-4\,x\epsilon^2-4x}{2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2}+\frac{10}{x}\right]f_2'''+\Bigg[-2+2\epsilon^2+\frac{22-4\nu^2+2\alpha^2}{x^2}-$$

$$-\frac{8\alpha^2}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)^2}+\frac{32\epsilon^2\alpha-16\alpha+24x\epsilon^3+24\epsilon x}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)\alpha}+4\,\frac{\epsilon\left(-6+\alpha^2\right)}{\alpha x}\Bigg]f_2''+$$

$$+\Bigg[\frac{4\epsilon\left(2\nu^2-6+3\alpha^2\right)}{\alpha x^2}+\frac{24\epsilon\alpha-8\epsilon^3\alpha-8x\epsilon^4+8x}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)^2}+$$

$$+\frac{-72\epsilon^3\alpha+8\nu^2\epsilon^3\alpha+56\epsilon\alpha+8\epsilon\alpha^3-24\nu^2\epsilon\alpha-48x\epsilon^4+8x\nu^2\epsilon^4-32x\epsilon^2+8\epsilon^2\alpha^2x-8x\nu^2+16x+8\alpha^2x}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)\alpha^2}+$$

$$+\frac{8\nu^2-8\nu^2\epsilon^2-16+48\epsilon^2-14\alpha^2+6\epsilon^2\alpha^2}{\alpha^2x}+\frac{-12\nu^2+8+6\alpha^2}{x^3}\Bigg]f_2'+$$

$$+\Bigg[\epsilon^4-2\epsilon^2+1+\frac{24\nu^2+6\epsilon^2\alpha^2+6\epsilon^2\alpha^4-24\nu^2\epsilon^2-4\nu^2\epsilon^2\alpha^2-30\alpha^2-2\alpha^4+4\alpha^2\nu^2}{x^2\alpha^2}+$$

$$+\frac{16\alpha^3-16\alpha\nu^2+48\alpha\nu^2\epsilon^2+32\epsilon x\nu^2+32\epsilon^3x\nu^2}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)^2\alpha}-\frac{2\alpha^2\nu^2}{x^6}+$$

$$+\frac{-40\alpha\nu^2\epsilon^4-40\alpha^3\epsilon^2+192\alpha\nu^2\epsilon^2+24\alpha^3-24\alpha\nu^2-32x\nu^2\epsilon^5+64\epsilon^3x\nu^2-32x\epsilon^3\alpha^2-32\epsilon\alpha^2x+96\epsilon x\nu^2}{\left(2\epsilon x\alpha+x^2+\alpha^2+\epsilon^2x^2\right)\alpha^3}+$$

$$+\frac{4\epsilon\left(-24\nu^2+\epsilon^2\alpha^4+8\nu^2\epsilon^2+8\alpha^2-\alpha^4\right)}{\alpha^3x}+\frac{-8\nu^2-4\alpha^2\nu^2+\alpha^4+4\nu^4}{x^4}-$$

$$-\frac{4\epsilon\left(2\alpha^2\nu^2-\alpha^4-4\nu^2-2\alpha^2\right)}{\alpha x^3}\Bigg]f_2=0\,, \tag{7.11}$$

$$\frac{d^4f_4}{dx^4}+\left[\frac{-4\epsilon\alpha-4\,x\epsilon^2-4x}{x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2}+14x^{-1}\right]\frac{d^3f_4}{dx^3}+$$

$$+\Bigg[2\epsilon^2-2-\frac{8\alpha^2}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)^2}+\frac{-4\nu^2+2\alpha^2+52}{x^2}+$$

$$+\frac{44\alpha\epsilon^2-28\alpha+36x\epsilon^3+36\epsilon x}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)\alpha}+4\,\frac{\epsilon\left(\alpha^2-9\right)}{\alpha x}\Bigg]\frac{d^2f_4}{dx^2}+$$

$$+\Bigg[\frac{-8\epsilon^3\alpha+56\epsilon\alpha-8x\epsilon^4+16x\epsilon^2+24x}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)^2}+\frac{4\epsilon\left(-16+5\alpha^2+2\nu^2\right)}{\alpha x^2}+\frac{48+10\alpha^2-20\nu^2}{x^3}+$$

$$+\frac{8\nu^2\epsilon^3\alpha - 136\epsilon^3\alpha - 24\epsilon\alpha\nu^2 + 8\alpha^3\epsilon + 184\epsilon\alpha - 100x\epsilon^4 + 8x\nu^2\epsilon^4 - 40x\epsilon^2 + 8\alpha^2x\epsilon^2 + 60x + 8\alpha^2x - 8x\nu^2}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)\alpha^2}+$$

$$+\frac{100\epsilon^2 - 60 + 10\alpha^2\epsilon^2 - 18\alpha^2 - 8\epsilon^2\nu^2 + 8\nu^2}{\alpha^2 x}\Bigg]\frac{df_4}{dx}+$$

$$+\Bigg[\epsilon^4 - 2\epsilon^2 + 1 + \frac{-20\nu^2 + \alpha^4 + 6\alpha^2 + 4\nu^4 - 4\alpha^2\nu^2}{x^4}+$$

$$+\frac{48\alpha\epsilon^2\nu^2 + 16\alpha^3 - 16\,\alpha\nu^2 + 32\epsilon^3x\nu^2 + 32\epsilon\,x\nu^2}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)^2\alpha}+$$

$$+\frac{32\nu^2 - 44\alpha^2 + 12\alpha^2\epsilon^2 + 4\alpha^2\nu^2 - 2\alpha^4 - 4\alpha^2\epsilon^2\nu^2 - 32\epsilon^2\nu^2 + 6\epsilon^2\alpha^4}{x^2\alpha^2}-$$

$$-\frac{4\epsilon\left(-\alpha^4 - 5\alpha^2 - 6\nu^2 + 2\alpha^2\nu^2\right)}{\alpha x^3} - \frac{2\alpha^2\nu^2}{x^6}+$$

$$+\frac{-48\alpha\nu^2\epsilon^4 - 48\alpha^3\epsilon^2 + 240\alpha\epsilon^2\nu^2 + 32\alpha^3 - 32\alpha\nu^2 - 40x\epsilon^5\nu^2 - 40x\epsilon^3\alpha^2 + 80,\epsilon^3x\nu^2 - 40\epsilon x\alpha^2 + 120\epsilon x\nu^2}{\left(x^2+\alpha^2+2\epsilon x\alpha+\epsilon^2x^2\right)\alpha^3}+$$

$$+\frac{4\,\epsilon\left(-30\,\nu^2 + 10\,\alpha^2 - \alpha^4 + 10\,\epsilon^2\nu^2 + \epsilon^2\alpha^4\right)}{\alpha^3x}\Bigg]f_4 = 0\,, \tag{7.12}$$

Any of the four functions f_1, f_2, f_3, f_4 may be considered as a main functions, and then all the remaining ones can be calculated in straightforward manner.

Let the function f_1 be the main one. We should take into account the 6 equations (7.2), (7.3):

$$f_1 = \frac{Bf_3' - Cf_4'}{AB - CD}, \quad f_2 = \frac{-Df_3' + Af_4'}{AB - CD}\,, \quad f_3 = \frac{bf_1' - cf_2'}{ab - cd}, \quad f_4 = \frac{-df_1' + af_2'}{ab - cd}\,,$$

$$(K_2\frac{d^2}{dx^2} + K_1\frac{d}{dx} + K_0)f_1 = \frac{df_2}{dx}\,, \quad (L_2\frac{d^2}{dx^2} + L_1\frac{d}{dx} + L_0)f_2 = \frac{df_1}{dx}\,. \tag{7.13}$$

From the fifth equation we find f_2; then from the 3-rd and 4-th, we express f_3 and f_4.

If we choose f_2 as a main function, then from the 6-th equation we express f_1 and, after that, from the 3-rd and 4-th equations we obtain the functions f_3 and f_4. Let the main function be f_3; then we use the 6 equations

$$f_1 = \frac{Bf_3' - Cf_4'}{AB - CD}, \quad f_2 = \frac{-Df_3' + Af_4'}{AB - CD}\,, \quad f_3 = \frac{bf_1' - cf_2'}{ab - cd}, \quad f_4 = \frac{-df_1' + af_2'}{ab - cd}\,,$$

$$(P_2\frac{d^2}{dx^2} + P_1\frac{d}{dx} + P_0)f_3 = \frac{df_4}{dx}\,, \quad (Q_2\frac{d^2}{dx^2} + Q_1\frac{d}{dx} + Q_0)f_4 = \frac{df_3}{dx}\,, \tag{7.14}$$

from the fifth equation, we get f_4; then from the 1-st and 2-nd equations we obtain expressions for f_1, f_2. If the main function is f_4, we get f_3 from the 6-th equation, and after that, from from the 1-st and 2-nd equations we find f_1, f_2.

We conclude that the formal Frobenius solutions of each of the derived 4-order differential equations are constructed, and the convergence of the involved power series, is studied. From four independent solutions of any 4-order equations, only two solutions may be referred to independent series of bound states.

8 Conclusions

In fact, in the paper it is shown that in the system which is described completely by 4-th order differential equations, some 2-order differential equations associated with this system play an important role as well.

The system of 10 radial equations, derived from the Duffin–Kemmer–Petiau equation for a spin 1 particle in the external Coulomb field, is studied. With the use of the space reflection operator, the whole system is split to independent subsystems, consisted of 4 and 6 equations respectively. The most simple subsystem of 4 equations is solved in terms of hypergeometric functions, which gives yet known energy spectrum. Also solutions and energy spectrum are found for minimal value of the total angular momentum, $j = 0$.

The second subsystem should give description of two other series of bound states. With the use of the Lorentz generalized condition in presence of the Coulomb field, we prove that one of 6 radial functions turns to be identically zero. This simplifies the explicit form of the system of 6 equations, which contain only 5 unknown functions. Combining this system, we derive a 2-nd order differential equation for one radial function, which may be recognized as a confluent Heun equation. A series of bound states is constructed in terms of so called transcendental confluent Heun functions, which provides us with the second class of bound states for spin 1 particle in the external Coulomb field, with corresponding formula for energy levels.

The subsystem of 6 equations, with no use of additional constraints due to the Lorentz condition, after excluding two non-differential relations reduces to a system of 1-st order differential equation for 4 independent functions $f_i, i = 1, 2, 3, 4$. We derive the explicit form of 4-th order equation for each function. Among them there arise two substantially sets of singular points: 3 regular (or 2) and 2 irregular of the rank 2. Their formal Frobenius solutions have been constructed, and convergence of the involved power series is studied. Any of these four functions may be considered as a main one, and then all remaining functions may be found in explicit form through the main one. From four independent solutions of the main function, governed by 4-th order equation, only two solutions may be referred to independent series of bound states.

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