Spin 1/2 Particle with Two Masses in External Magnetic Field

[1]E.M. Ovsiyuk, [2]O.V. Veko, [3]Y.A. Voynova, [4]V.M. Red'kov, [5]V.V. Kisel, [6]*N.V. Samsonenko

[1]Mozyr State Pedagogical University named after I.P. Shamyakin, Mozyr, Belarus

[2 3 4] B.I. Stepanov Institute of Physics of NAS of Belarus, Minsk, Belarus

[5] Belarusian State University of Informatics and Radioelectronics, Minsk, Belarus

[6]* Department of Theoretical Physics, RUDN University, Moscow, Russia

Corresponding Author: N.V. Samsonenko

Email: nsamson@bk.ru

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Abstract

Equation for spin 1/2 particle with two mass states is investigated in presence of magnetic field. The problem reduces to a system of 4 linked 2-nd order differential equations. After diagonalization of the mixing term, separate equations for four different functions are derived, in which the spectral parameters coincide with the roots of a 4-th order polynomial. Solutions are constructed in terms of confluent hyper-geometric functions; four series of energy spectrum are found. Numerical study of the spectra is performed. Physical energy levels for the two mass fermion differ from those for the ordinary Dirac fermion.

Keywords: spin 1/2 particle, two mass parameters, external magnetic field

I Introduction

In (Kisel, 2017), a relativistic model for a spin 1/2 particle with two masses was developed (in general, existence of more general wave equations than commonly used ones is well known within the so-called Gel'fand–Yaglom formalism (Gel'fand and Yaglom, 1948); also see (Bhabha, 1952), (Fedorov, 1952),(Shimazu, 1956); and the books (Pletjukhov et al, 2015), (Kisel et al, 2018).

In (Kisel et al, 2017) it was shown that in absence of external fields the main equation for the fermion with two masses, M_1 and M_2, is split into two separate Dirac-like equation. Also it was shown that in presence of external electromagnetic fields there arises a more complicated wave equation which mixes two bi-spinor components.

This wave equation with respect to bi-spinor functions $\Psi_1(x)$ and $\Psi_2(x)$ has the structure

$$(i\not{D} - M_1 + b\Lambda_1\Sigma)\Psi_1 - a\Lambda_1\Sigma\Psi_2 = 0,$$
$$(i\not{D} - M_2 - a\Lambda_2\Sigma)\Psi_2 + b\Lambda_2\Sigma\Psi_1 = 0; \qquad (1)$$

where (Red'kov, 2009)

$$\not{D} = \gamma^{\alpha}(\partial_{\alpha} + \Gamma_{\alpha} + ieA_{\alpha}), \quad \gamma^{\alpha} = e^{\alpha}_{(b)}\gamma^{b},$$

$$\Sigma = -ieF_{\alpha\beta}\sigma^{\alpha\beta}, \quad \sigma^{\alpha\beta} = \frac{\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha}}{4};$$

we employ the tetrad formalism and use the denotation $e/\hbar c \Rightarrow e$.

We use the following parameters (the quantities ρ, σ appear when constricting Eq. (1)):

$$\sin^2\gamma = \frac{4\sigma^2}{\rho^2}, \quad \gamma \in (0, \pi/2),$$

$$\lambda_1 = \frac{\rho}{2}(1 + \cos\gamma), \quad \lambda_2 = \frac{\rho}{2}(1 - \cos\gamma),$$

$$M_1 = \frac{M}{\lambda_1} = \frac{M/\rho}{(1+\cos\gamma)/2},$$

$$M_2 = \frac{M}{\lambda_2} = \frac{M/\rho}{(1-\cos\gamma)/2},$$

$$A' = \frac{\lambda_1 + \lambda_2}{2}, \quad B' = \frac{\sqrt{(\lambda_1+\lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2},$$

$$a = \frac{5A' - 3B' - \lambda_1}{M}, \quad b = \frac{5A' - 3B' - \lambda_2}{M},$$

$$\Lambda_1 = (A' + B')\frac{\lambda_1 - A' - B'}{\lambda_1(\lambda_1 - \lambda_2)},$$
$$\Lambda_2 = (A' + B')\frac{\lambda_2 - A' - B'}{\lambda_2(\lambda_2 - \lambda_1)}. \qquad (2)$$

II. Main Part

In the present paper, this equation is solved in presence of uniform magnetic field. In cylindrical coordinates the field is specified by potential

$$A_\phi = -Br/2\,;$$

further we use shortening notations (additionally note physical dimensions of the quantities)

$$\frac{eB}{\hbar c} \Rightarrow B, \ [B] = L^{-2}, \quad \frac{mc}{\hbar} \Rightarrow M, \ [M] = L^{-1}.$$

Then the extended derivative is

$$\not{D} = i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \left(\frac{i\partial_\phi}{r} + \frac{Br}{2} \right) + i\gamma^3 \frac{\partial}{\partial z},$$

$$e\sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) = iB\gamma^2\gamma^1 = iB\Sigma_3, \tag{3}$$

final system taking the form:

$$(i\not{D} - M_1 + 2b\Lambda_1 B\Sigma_3)\psi_1 - a\Lambda_1 B\Sigma_3 \psi_2 = 0,$$

$$(i\not{D} - M_2 - 2a\Lambda_2 B\Sigma_3)\psi_2 + b\Lambda_2 B\Sigma_3 \psi_1 = 0. \tag{4}$$

Below we use shortening notations

$$2b\Lambda_1 B = \Gamma_1, \quad a\Lambda_1 B = R_1,$$

$$2a\Lambda_2 B = -\Gamma_2, \quad b\Lambda_2 B = -R_2,$$

$$\frac{m}{r} - \frac{Br}{2} \Rightarrow \mu(r), \quad \frac{\varepsilon}{\hbar c} \Rightarrow \varepsilon, \quad [\varepsilon] = L^{-1}; \tag{5}$$

parameters $\Gamma_{1,2}$ and $R_{1,2}$ have physical dimension of length.

For the wave function $\psi = \{\psi_1 \otimes \psi_2\}$ the following substitution is used

$$\psi_1 = e^{-i\varepsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{vmatrix}, \ \psi_2 = e^{-i\varepsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} g_1(r) \\ g_2(r) \\ g_3(r) \\ g_4(r) \end{vmatrix}, \quad (6)$$

and further get 8 equations (let $i(d/dr \pm \mu) = D_{\pm}$)

$$(\varepsilon + k) f_3 + (\Gamma_1 - M_1) f_1 - R_1 g_1 = +D_+ f_4,$$

$$(\varepsilon - k) f_1 + (\Gamma_1 - M_1) f_3 - R_1 g_3 = -D_+ f_2,$$

$$(\varepsilon + k) g_3 + (\Gamma_2 - M_2) g_1 - R_2 f_1 = +D_+ g_4, \quad (7)$$

$$(\varepsilon - k) g_1 + (\Gamma_2 - M_2) g_3 - R_2 f_3 = -D_+ g_2;$$

$$(\varepsilon - k) f_4 - (\Gamma_1 + M_1) f_2 + R_1 g_2 = +D_- f_3,$$

$$(\varepsilon + k) f_2 - (\Gamma_1 + M_1) f_4 + R_1 g_4 = -D_- f_1,$$

$$(\varepsilon - k) g_4 - (\Gamma_2 + M_2) g_2 + R_2 f_2 = +D_- g_3,$$

$$(\varepsilon + k) g_2 - (\Gamma_2 + M_2) g_4 + R_2 f_4 = -D_- g_1. \quad (8)$$

These two sub-systems are solved with respect to f_1, f_3, g_1, g_3 and f_2, f_4, g_2, g_4 (for brevity we write down only its structure):

$$\begin{vmatrix} f_1 \\ f_3 \\ g_1 \\ g_3 \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} \begin{vmatrix} D_+ f_2 \\ D_+ f_4 \\ D_+ g_2 \\ D_+ g_4 \end{vmatrix}.$$

$$\begin{vmatrix} f_2 \\ f_4 \\ g_2 \\ g_4 \end{vmatrix} = \begin{vmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{vmatrix} \begin{vmatrix} D_- f_1 \\ D_- f_3 \\ D_- g_1 \\ D_- g_3 \end{vmatrix}.$$

Combining these relations, $C = AB$, we derive

$$\begin{vmatrix} f_1 \\ f_3 \\ g_1 \\ g_3 \end{vmatrix} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{vmatrix} \begin{vmatrix} D_+D_- f_1 \\ D_+D_- f_3 \\ D_+D_- g_1 \\ D_+D_- g_3 \end{vmatrix}, \tag{9}$$

or in the matrix form

$$(D_+D_-)\, F = \Lambda\, F\,, \quad \Lambda = C^{-1}. \tag{10}$$

The matrix Λ can be found in the explicit form, but for brevity we omit it. Omitting details of diagonalizing the matrix Λ (see below), we write down only relations determining the eigenvalues λ_i :

$$\begin{aligned} &\pm 2[(16R+4y^2)\,E^2 - 4[4Rx + y(yx-\nu\mu)]E + \\ &+4\,R(x^2-\nu^2) + (yx-\nu\mu)^2\,] = \\ &= [4E^2 - 4Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_{1,2}\,]^2\,, \end{aligned} \tag{11}$$

$$\begin{aligned} &\pm 2[(16R+4y^2)E^2 + 4[4Rx + y(yx-\nu\mu)]E + \\ &+4\,R(x^2-\nu^2) + (yx-\nu\mu)^2\,] = \\ &= [4E^2 + 4Ex + 4R + x^2 + y^2 - \mu^2 - \nu^2 - 4\lambda_{3,4}\,]^2\,. \end{aligned} \tag{12}$$

Here we use the following denotations:

$$\begin{aligned} &\varepsilon^2 - k^2 = E^2,\;\; M_1 + M_2 = \mu,\;\; M_1 - M_2 = \nu, \\ &\varepsilon^2 - k^2 = E^2,\;\; M_1 + M_2 = \mu,\;\; M_1 - M_2 = \nu. \end{aligned} \tag{13}$$

In this way, we have 4 second order equations ofthe same type:

$$D_+D_- F'_{(i)} = \lambda_{(i)}\, F'_{(i)}\,, \quad i = 1,2,3,4\,; \tag{14}$$

or in explicit form ($i = 1,2,3,4$)

$$\left[\frac{d^2}{dr^2} + \lambda_i + \frac{m}{r^2} + B - \left(\frac{m}{r} - Br \right)^2 \right] \Phi_i = 0. \tag{15}$$

The problem is reduced to confluent hyper-geometric equations; their solutions and energy spectra being known (by physical reasons assume $\lambda > 0$):

$$\lambda_i = 4bn_i, \quad n_i \in \{0,1,2...\}. \tag{16}$$

In order to examine equations(11) and (12) (with spectra (16) in mind) we need explicit form of all involved parameters. So we probe (let $M_0 = M/\rho$):

$$x = -\frac{4}{3}\frac{B\sin^2\gamma}{M_0\cos\gamma}, \quad y = 0, \quad R = \frac{1}{9}\frac{B^2\sin^4\gamma}{{M_0}^2\cos^2\gamma},$$

$$\mu = \frac{4M_0}{\sin^2\gamma}, \quad \nu = -\frac{4M_0\cos\gamma}{\sin^2\gamma}. \tag{17}$$

In the sequel we use thedimensionless parameters:

$$\frac{E}{M_0} \Rightarrow E, \quad \frac{x}{M_0} \Rightarrow x, \quad \frac{y}{M_0} \Rightarrow y, \quad \frac{\mu}{M_0} \Rightarrow \mu,$$

$$\frac{\nu}{M_0} \Rightarrow \nu, \quad \frac{B}{M_0^2} \Rightarrow B, \quad \frac{R}{M_0^2} \Rightarrow R. \tag{18}$$

For E we have the following 4-th order algebraic equation (recall that $\lambda = \lambda_i = 4Bn_i$):

$$\begin{aligned} &E^4 + \frac{8}{3}\frac{B\sin^2\gamma E^3}{\cos\gamma} + \\ &+\left(\frac{22}{9}\frac{B^2\sin^4\gamma}{\cos^2\gamma} - \frac{8(1+\cos^2\gamma)}{\sin^4\gamma} - 8Bn\right)E^2 + \\ &+\left(\frac{8}{9}\frac{B^3\sin^6\gamma}{\cos^3\gamma} - \frac{32}{3}\frac{B(1+\cos^2\gamma)}{\sin^2\gamma\cos\gamma} - \right. \\ &\left.-\frac{32}{3}\frac{B^2\sin^2\gamma n}{\cos\gamma}\right)E + \frac{16}{\sin^4\gamma} - \frac{8}{3}B^2 - \\ &-\frac{40}{9}\frac{B^2}{\cos^2\gamma} + \frac{1}{9}\frac{B^4\sin^8\gamma}{\cos^4\gamma} - \frac{40}{9}\frac{B^3\sin^4\gamma n}{\cos^2\gamma} + \\ &+\frac{32Bn(1+\cos^2\gamma)}{\sin^4\gamma} + 16B^2n^2 = 0. \end{aligned} \tag{19}$$

Let us write down several examples of numerical results:

$$B = 1, \quad \sin\gamma = 1/10,$$

$$n = 1,$$

$$E_1 = 2.23,\ E_2 = 399.00,\ E_3 = 2.24,\ E_4 = 399.01;$$
$$n = 5,$$

$$E_1 = 4.58,\ E_2 = 399.02,\ E_3 = 4.59,\ E_4 = 399.03;$$
$$n = 10,$$

$$E_1 = 6.40,\ E_2 = 399.04,\ E_3 = 6.41,\ E_4 = 399.05.$$

$$B = 1, \quad \sin\gamma = 9/10,$$

$$n = 1,$$

$$E_1 = 1.00,\ E_2 = 3.03,\ E_3 = 3.48,\ E_4 = 5.51;$$

$$n = 5,$$

$$E_1 = 3.16,\ E_2 = 4.75,\ E_3 = 5.64,\ E_4 = 7.23;$$

$$n = 10,$$

$$E_1 = 4.90,\ E_2 = 6.35,\ E_3 = 7.38,\ E_4 = 8.83.$$

Evidently, the energy levels make up 4 series.

In the end, taking into account relations $y = 0$ and $\Gamma_1 = \Gamma_2 = \Gamma$, let us simplify the form of the generalized equation in presence of magnetic field Indeed, with the help of elementary change in notations

$$\sqrt{R_1}\psi_1 --> \Psi_1,\ \sqrt{R_2}\psi_2 --> \Psi_2,$$

$$\Gamma = -\frac{2}{3}\frac{B\sin^2\gamma}{M_0\cos\gamma}, \sqrt{R} = \sqrt{R_1 R_2} = \frac{B}{3}\frac{\sin^2\gamma}{\cos\gamma}$$

the system reduces to more simple and symmetrical form

$$(i\not{D} - M_1 + \Gamma\Sigma_3)\Psi_1 - \sqrt{R}\Sigma_3\Psi_2 = 0,$$

$$(i\not{D} - M_2 + \Gamma\Sigma_3)\Psi_2 - \sqrt{R}\Sigma_3\Psi_1 = 0. \qquad (20)$$

III. Results and Discussions

In the paper, a new wave equation for a spin ½ fermion, which is characterized by two mass parameters, is solved in presence of external uniform magnetic field. Obtained results indicate that such a particle will manifest itself differently in comparison with the ordinary Dirac particle. This property might be tested experimentally. It would bedesirable to get explicit solutions of such a generalized equation in presence of other electromagnetic potentials. For instance, exact solutions may be found in presence of a uniform electric field, this problem will be studied in a separate work. Also interesting is the case of Coulomb potential, because far from the origin we must see two free particles with different masses, whereas near the origin such a separation appears to be impossible because of the mixing interaction ofthe two components.

IV. Conclusion

Generalized equation for spin 1/2 particle with two mass states is investigated in presence of external uniform magnetic field. After separation of the variables in cylindrical coordinates the problemreduces to a system of eight first-order differential equations, whence it follows the system of four equivalent second-order differential equations. After diagonalization of the mixing term, separate equations for four functions are derived, in which the spectral parameters coincide with the roots of 4-th order polynomial. Solutions to the system in question are constructed in terms of confluent hyper-geometric functions, analytical formulas for two series of energy spectrum can be found in explicit form as solutions of the 4-th order algebraic equations, however they are cumbersome and useless. Numerical study of the energy levels is performed depending the parameter γ, determining the mass values, on the magnitude of the magnetic field and the magnetic and main quantum numbers: $E = E_{1,2}(\gamma, B; m, n)$. So, physical energy spectrum for a two mass fermion differs significantly from the energy spectrum of the ordinary Dirac particle.

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