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Spin 1 Particle with Anomalous Magnetic Moment in an External Uniform Electric Field

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Within the matrix 10-dimensional Duffin–Kemmer–Petiau formalism applied to the Shamaly–Capri field, we study the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. Separation of variables in the wave equation is performed using projective operator techniques and the theory of DKP-algebras. The whole wave function is decomposed into the sum of three components Ψ_0, Ψ_+, Ψ_+. It is enough to solve an equation for the main component Φ_0, two remaining ones are determined by it uniquely, The problem is reduced to a system of three independent differential equations for three functions, which are of the type of one-dimensional Klein–Fock–Gordon equation in the presence of a uniform electric field modified by the anomalous magnetic moment of the particle. Solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for the solutions, one should impose special restriction on a parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well. In this case, the main manifestation of the anomalous magnetic moment consists in modification of the ordinary plane wave solution along the electric field direction. Again one have to impose special restriction on the parameter related to the anomalous moment of the particle.

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1. Introduction

Commonly, only the simplest wave equations for fundamental particles of spin $0, 1/2, 1$ are used. Meanwhile, it is known that other more complicated equations can be proposed for particles with such spins, which are based on application of the extended sets of Lorentz group representations (see [1]–[18]).

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Such generalized wave equations allow to describe more complicated objects, which have besides mass, spin, and electric charge, other electromagnetic characteristics, like polarizability or anomalous magnetic moment. These additional characteristics manifest themselves explicitly in the presence of external electromagnetic fields.

In particular, within this approach Petras has proposed a 20-component theory for spin 1/2 particle [3], which turns to be equivalent to the Dirac particle theory modified by the presence of the Pauli interaction term after excluding 16 subsidiary components. In other words, this theory describes a spin 1/2 particle with anomalous magnetic moment.

A similar equation has been proposed by Shamaly–Capri [6, 7] for spin 1 particles. In the following, we investigate and solve this wave equation in the presence of an external uniform electric field.

The wave equation for spin 1 particle with anomalous magnetic moment [6, 7] may be formulated in the form

$$\left(\beta_\mu D_\mu + \frac{ie}{M}\lambda F_{[\mu\nu]} P J_{[\mu\nu]} + M\right)\Psi = 0\,, \qquad (1)$$

where the 10-dimensional wave function and the DKP-matrices are used:

$$\Psi = \left| \begin{array}{c} \Psi_\mu \\ \Psi_{[\mu\nu]} \end{array} \right|, \qquad J_{[\mu\nu]} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu\,; \qquad (2)$$

P stands for a projective operator separating from Ψ its vector component Ψ_μ; $D_\mu = \partial_\mu - ieA_\mu$; λ denotes an arbitrary real-valued number. In tensor form, (1) is (in Minkowski space, the metric with imaginary unit is used, since $x_4 = ict$):

$$D_\mu\Psi_\nu - D_\nu\Psi_\mu + M\Psi_{[\mu\nu]} = 0\,,$$
$$D_\nu\Psi_{[\mu\nu]} + 2\frac{ie}{M}\lambda F_{[\mu\nu]}\Psi_\nu + M\Psi_\mu = 0\,. \qquad (3)$$

By using DKP-matrices, we apply the method of generalized Kronecker's symbols [20] (the indexes $A(B,C,D,\ldots)$ take the values $1,2,3,4$, 23, 31, 12, 14, 24, 34):

$$\beta_\mu = e^{\nu,[\nu\mu]} + e^{[\nu\mu],\nu}, \quad P = e^{\nu,\nu}\,,$$
$$(e^{A,B})_{CD} = \delta_{AC}\delta_{BD}\,, \quad e^{A,B}e^{C,D}\delta_{BC}e^{A,D},$$
$$\delta_{[\mu\nu],[\rho\sigma]} = \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\,,$$

and the main relationships in the DKP algebra read:

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\rho + \delta_{\rho\nu}\beta_\mu\,,$$
$$[\beta_\lambda, J_{\rho\sigma]}]_- = \delta_{\lambda\rho}\beta_\sigma - \delta_{\lambda\sigma}\beta_\rho\,; \qquad (4)$$

We use the following representation for DKP-matrices:

$$\beta_1 = \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|,$$

$$\beta_2 = \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|,$$

$$\beta_3 = \left| \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|,$$

$$\beta_4 = \left| \begin{array}{cccccccccc} 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \end{array} \right| . \qquad (5)$$

A uniform electric field is specified by the relations

$$(A_\mu) = (0,0,0,-iEx_3), \quad F_{[34]} = -iE \, .$$

The non-minimal interaction through the anomalous magnetic moment is given by the term

$$\pm \frac{ie}{M} \lambda F_{[\mu\nu]} PJ_{[\mu\nu]} = \pm \frac{2eE}{M} \lambda \, PJ_{[34]} \, .$$

Correspondingly, the main equation (1) is written as

$$\Big[\beta_1 \frac{\partial}{\partial x^1} + \beta_2 \frac{\partial}{\partial x^2} + \beta_3 \frac{\partial}{\partial x^3} + \beta_4 (\frac{\partial}{\partial x^4} - eEx_3) + \Gamma_0 \, PJ_{[34]} + M \Big] \; \Psi = 0 \, , \qquad (6)$$

where

$$\Gamma_0 = \frac{2eE}{M} \lambda \, . \qquad (7)$$

2. Transformations in the wave equation

Let us introduce the matrix

$$Y = iJ_{[34]} = i(\beta_3 \beta_4 - \beta_4 \beta_3) \, ;$$

it satisfies the minimal polynomial equation $Y(Y-1)(Y+1) = 0$, which permits to define tree projective operators:

$$Y^3 = Y \, , \quad P_0 = 1 - Y^2 \, , \qquad P_+ = \frac{1}{2} Y(Y+1) \, , \quad P_- = \frac{1}{2} Y(Y-1) \qquad (8)$$

and resolve the wave function into three components, $\Psi = \Psi_0 + \Psi_- + \Psi_+$:

$$\Psi_0 = P_0 \Psi, \quad \Psi_+ = P_+ \Psi, \quad \Psi_- = P_- \Psi \, . \qquad (9)$$

Acting on eq. (6) by the operator P_0, and taking into account the identities

$$Y \beta_{1,2} = \beta_{1,2} Y, \quad P_0 \beta_{1,2} = \beta_{1,2} P_0, \qquad P_0 PJ_{[34]} = -iP(1 - Y^2) Y \equiv 0 \, ,$$

we get

$$(\beta_1 \partial_1 + \beta_2 \partial_2 + M) \, \Psi_0 + P_0 \beta_3 \, \partial_3 \Psi + (\partial_4 - eEx_3) \, P_0 \beta_4 \Psi = 0 \, . \qquad (10)$$

Let us consider the operator $P_0 \beta_3$ (remembering on the DKP-algebra):

$$P_0 \beta_3 = (1 + 2\beta_3\beta_3\beta_4\beta_4 - \beta_3\beta_3 - \beta_4\beta_4)\beta_3 = \beta_3 + 2\beta_3\beta_3\beta_4\beta_4\beta_3 - \beta_3 - \beta_4\beta_4\beta_3 = 2\beta_3\beta_3(\beta_3 - \beta_3\beta_4\beta_4) - (\beta_3 - \beta_3\beta_4\beta_4) = \beta_3 - \beta_3\beta_4\beta_4 \, .$$

Allowing for the identities

$$\beta_3(1 - P_0) = \beta_3 Y^2 = \beta_3[\beta_3\beta_3 + \beta_4\beta_4 - 2\beta_3\beta_3\beta_4\beta_4] = \beta_3 + \beta_3\beta_4\beta_4 - 2\beta_3\beta_4\beta_4 = \beta_3 - \beta_3\beta_4\beta_4 \, ,$$

we write the previous one in the form

$$P_0 \beta_3 = \beta_3 (1 - P_0) = \beta_3 (P_+ + P_-) \, . \qquad (11)$$

Similarly, one can obtain an identity

$$P_0 \beta_4 = \beta_4 (1 - P_0) = \beta_4 (P_+ + P_-) \, . \qquad (12)$$

With relations (11)–(12) in mind, eq. (10) reduces to the form

$$(\beta_1 \partial_1 + \beta_2 \partial_2 + M) \, \Psi_0 + [\beta_3 \, \partial_3 + \beta_4 (\partial_4 - eEx_3)] \Psi_+ + [\beta_3 \, \partial_3 + \beta_4 (\partial_4 - eEx_3)] \Psi_- = 0 \, . \qquad (13)$$

Let us consider the operator

$$\beta_3 P_+ = \beta_3 \frac{1}{2} (Y + Y^2) = \beta_3 \frac{1}{2} [i(\beta_3\beta_4 - \beta_4\beta_3) - 2\beta_3\beta_3\beta_4\beta_4 + \beta_3\beta_3 + \beta_4\beta_4];$$

whence it follows (note that $\beta_3^3 = \beta_3$, $\beta_3\beta_4\beta_3 = 0$)

$$\beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_3\beta_3\beta_4 - \beta_3\beta_4\beta_4)\,. \qquad (14)$$

Similarly, we derive (using the identities $\beta_4^3 = \beta_4$ and $\beta_4\beta_3\beta_4 = 0$)

$$\beta_4 P_+ = -\frac{i}{2}[\beta_3 - \beta_3\beta_4\beta_4 + i\beta_3\beta_3\beta_4]\,.$$

So, we obtain the algebraic relation

$$\beta_3 P_+ = i\beta_4 P_+ \quad \Longrightarrow \quad (\beta_3 - i\beta_4)P_+ = 0\,. \; (15)$$

Combining the relationships,

$$i\beta_4 P_+ = \frac{1}{2}[\beta_3 - \beta_3\beta_4\beta_4 + i\beta_3\beta_3\beta_4],$$
$$\beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_3\beta_3\beta_4 - \beta_3\beta_4\beta_4)\,;$$

we easily derive that

$$\beta_3 P_+ = \frac{1}{2}(\beta_3 + i\beta_4)P_+\,. \qquad (16)$$

In turn, combining (15)–(16), one derive

$$\beta_4 P_+ = -\frac{i}{2}(\beta_3 + i\beta_4)P_+\,. \qquad (17)$$

Acting in the same manner, we get three identities

$$(\beta_3 + i\beta_4)P_- = 0\,,$$
$$\beta_3 P_- = \frac{1}{2}(\beta_3 - i\beta_4)P_-,$$
$$\beta_4 P_- = \frac{i}{2}(\beta_3 - i\beta_4)P_-\,. \qquad (18)$$

Now, let us turn back to eq. (13); it can be written as

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0$$
$$+(\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]P_+\Psi_+$$
$$+(\beta_3\,\partial_3 + \beta_4(\partial_4 - eEx_3)]P_-\Psi_- = 0\,.$$

Further, with the help of above identities it can be re-written in the form (reminding on $P_\pm^2 = P_\pm$)

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\,\Psi_0$$
$$+\frac{1}{2}(\beta_3 + i\beta_4)\,[(\partial_3 + iEx_3) - i\partial_4]\,\Psi_+$$
$$+\frac{1}{2}(\beta_3 - i\beta_4)\,[(\partial_3 - eEx_3) + i\partial_4]\,\Psi_- = 0\,. \; (19)$$

Now, let us consider eq. (6)

$$[\,\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} + \beta_3\frac{\partial}{\partial x^3}$$
$$+\beta_4(\frac{\partial}{\partial x^4} - eEx_3) - i\Gamma_0 PY + M\,]\;\Psi = 0\,, \; (20)$$

and act on it by the operator $1 - P_0 = P_+ + P_-$; this yields

$$[\beta_1\frac{\partial}{\partial x^1} + \beta_2\frac{\partial}{\partial x^2} - i\Gamma_0 PY + M](\Psi_+ + \Psi_-)$$
$$+(1 - P_0)\beta_3\frac{\partial}{\partial x^3} + (1 - P_0)\beta_4(\frac{\partial}{\partial x^4} - eEx_3)\Psi = 0.$$

Allowing for an identity

$$1 - P_0 = Y^2 = \beta_3\beta_3 + \beta_4\beta_4 - 2\beta_3\beta_3\beta_4\beta_4\,,$$

we get

$$(1 - P_0)\beta_3 = \beta_3 + (\beta_3 - \beta_3\beta_4\beta_4)$$
$$-2\beta_3\beta_3(\beta_3 - \beta_3\beta_4\beta_4) = +\beta_3\beta_4\beta_4\,. \qquad (21)$$

Similarly, we derive

$$\beta_3 P_0 = \beta_3(1 - \beta_3\beta_3 - \beta_4\beta_4$$
$$+2\beta_3\beta_3\beta_4\beta_4) = +\beta_3\beta_4\beta_4\,. \qquad (22)$$

Combining two last relationships, we obtain a commutative relation

$$(1 - P_0)\beta_3 = \beta_3 P_0\,. \qquad (23)$$

In the same manner, we derive three similar relations

$$\beta_4 - \beta_3\beta_3\beta_4 = (1 - P_0)\beta_4,$$
$$\beta_4 - \beta_3\beta_3\beta_4 = \beta_4 P_0,$$
$$(1 - P_0)\beta_4 = \beta_4 P_0\,. \qquad (24)$$

With the help of which, the above equation can be presented as

$$[\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 PY + M](\Psi_+ + \Psi_-)$$
$$+\beta_3\partial_3\Psi_0 + \beta_4(\partial_4 - eEx_3)\Psi_0 = 0\,, \qquad (25)$$

Let us act on eq. (25) by the operator $\frac{1}{2}(1 + Y)$. With the help of easily checked identities

$$\frac{1}{2}(1 + Y)P_+ = \frac{1}{2}(1 + Y)\frac{1}{2}Y(1 + Y) = P_+\,,$$
$$\frac{1}{2}(1 + Y)P_- = \frac{1}{4}(Y + Y^2)(Y - 1) = 0\,,$$

we derive

$$[\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 PY + M]\Psi_+ + \frac{1}{2}(1+Y)\beta_3\partial_3\Psi_0 + \frac{1}{2}(1+Y)\beta_4(\partial_4 - eEx_3)\Psi_0 = 0, \tag{26}$$

We need three auxiliary relations. From the known formula

$$\beta_\lambda J_{[\rho\sigma]} - J_{[\rho\sigma]}\beta_\lambda = \delta_{\rho\sigma}\beta_\lambda - \delta_{\lambda\sigma}\beta_\rho$$

it follows that

$$\beta_3 Y - Y\beta_3 = +i\beta_4 \implies Y\beta_3 = \beta_3 Y - i\beta_4\,,$$
$$\beta_4 Y - Y\beta_4 = -i\beta_3 \implies Y\beta_4 = \beta_4 Y + i\beta_4\,.$$

Therefore, eq. (26) can be transformed to the form

$$[\,\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 PY + M]\Psi_+ + \frac{1}{2}(\beta_3 + \beta_3 Y - i\beta_4)\partial_3\Psi_0 + \frac{1}{2}(\beta_4 + \beta_4 Y + i\beta_4)(\partial_4 - eEx_3)\Psi_0 = 0\,, \tag{27}$$

From this, taking into account $YP_0 \equiv 0$, we obtain a more simple form

$$[\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 PY + M]\Psi_+ + \frac{1}{2}(\beta_3 - i\beta_4)[\partial_3\Psi_0 + i(\partial_4 - eEx_3)]\Psi_0 = 0\,. \tag{28}$$

Now, let us take into account an identity

$$YP_+ = \frac{1}{2}(Y^2 + Y^3 = P_+ \implies Y\Psi_+ = \Psi_+\,,$$

then the previous equation reads

$$(\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 P + M)\Psi_+ + \frac{1}{2}(\beta_3 - i\beta_4)(\partial_3 - ieEx_3 + i\partial_4)\Psi_0 = 0\,. \tag{29}$$

In turn, acting on eq. (25) by the operator $\frac{1}{2}(1-Y)$, after analogous calculations we arrive at a similar equation

$$(\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 P + M)\Psi_- + \frac{1}{2}(\beta_3 + i\beta_4)(\partial_3 + ieEx_3 - i\partial_4)\Psi_0 = 0\,, \tag{30}$$

3. Separation of the variables

We start with three equations

$$(\beta_1\partial_1 + \beta_2\partial_2 + M)\Psi_0 + \frac{1}{\sqrt{2}}\beta_+[(\partial_3 + ieEx_3) - i\partial_4]\Psi_+ + \frac{1}{\sqrt{2}}\beta_-[(\partial_3 - ieEx_3) + i\partial_4]\Psi_- = 0\,, \tag{31}$$

$$(\beta_1\partial_1 + \beta_2\partial_2 - i\Gamma_0 P + M)\Psi_+ + \frac{1}{\sqrt{2}}\beta_-[(\partial_3 - ieEx_3) + i\partial_4]\Psi_0 = 0\,, \tag{32}$$

$$(\beta_1\partial_1 + \beta_2\partial_2 + i\Gamma_0 P + M)\Psi_- + \frac{1}{\sqrt{2}}\beta_+[(\partial_3 + ieEx_3) - i\partial_4]\Psi_0 = 0\,, \tag{33}$$

where

$$\beta_+ = \frac{1}{\sqrt{2}}(\beta_3 + i\beta_4), \quad \beta_- = \frac{1}{\sqrt{2}}(\beta_3 - i\beta_4)\,.$$

Solutions are looking for in the form

$$\Psi_0 = e^{ip_4x_4}e^{ip_1x_1}e^{ip_2x_2}f_0(x_3)\,,$$
$$\Psi_+ = e^{ip_4x_4}e^{ip_1x_1}e^{ip_2x_2}f_+(x_3)\,,$$
$$\Psi_- = e^{ip_4x_4}e^{ip_1x_1}e^{ip_2x_2}f_-(x_3)\,. \tag{34}$$

So, we have the system of three equations in the variable x_3:

$$(ip_1\beta_1 + ip_2\beta_2 + M)\Psi_0 +$$
$$+\frac{1}{\sqrt{2}}\beta_+ \left[\left(\frac{d}{dx_3} + ieEx_3\right) + p_4\right]\Psi_+ + \frac{1}{\sqrt{2}}\beta_- \left[\left(\frac{d}{dx_3} - ieEx_3\right) - p_4\right]\Psi_- = 0\,, \tag{35}$$

$$(ip_1\beta_1 + ip_2\beta_2 - i\Gamma_0 P + M)\Psi_+ + \frac{1}{\sqrt{2}}\beta_- \left[\left(\frac{d}{dx_3} - ieEx_3\right) - p_4\right]\Psi_0 = 0\,, \tag{36}$$

$$(ip_1\beta_1 + ip_2\beta_2 + i\Gamma_0 P + M)\Psi_- + \frac{1}{\sqrt{2}}\beta_+ \left[\left(\frac{d}{dx_3} + ieEx_3\right) + p_4\right]\Psi_0 = 0\,. \tag{37}$$

With the shortening notation

$$\hat{a} = \frac{1}{\sqrt{2}}\left(+\frac{d}{dx_3} + ieEx_3 + p_4\right), \quad \hat{b} = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx_3} + ieEx_3 + p_4\right)\; ; \; i\Gamma_0 = \Gamma, \;\; p_1\beta_1 + p_2\beta_2 = \hat{p}\,; \tag{38}$$

it is written as

$$(i\hat{p} + M)\Psi_0 + \beta_+\hat{a}\Psi_+ - \beta_-\hat{b}\Psi_- = 0\,, \tag{39}$$

$$(i\hat{p} - \Gamma P + M)\Psi_+ - \beta_-\hat{b}\Psi_0 = 0\,, \quad (i\hat{p} + \Gamma P + M)\Psi_- + \beta_+\hat{a}\Psi_0 = 0\,. \tag{40}$$

Let us act on the first equation in (40) by the operator $(M-\Gamma)^{-1}(M-\Gamma\bar{P}$; this results in

$$\left(\frac{M-\Gamma\bar{P}}{M-\Gamma}i\hat{p} + \frac{M-\Gamma\bar{P}}{M-\Gamma}(M-\Gamma P)\right)\Psi_+ - \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_-\hat{b}\Psi_0 = 0\,.$$

With the help of identities

$$\frac{M-\Gamma\bar{P}}{M-\Gamma}(M-\Gamma P) = \frac{1}{M-\Gamma}(M^2 - M\Gamma P - M\Gamma\bar{P}) = M\,;$$

it reads

$$\left(\frac{M-\Gamma\bar{P}}{M-\Gamma}i\hat{p} + M\right)\Psi_+ - \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_-\hat{b}\Psi_0 = 0\,.$$

With the use of special notations, the previous equation is written shorter as

$$\frac{M-\Gamma\bar{P}}{M-\Gamma}i\hat{p} = A, \quad \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_- = \beta'_-, \quad (A+M)\Psi_+ - \beta'_-\hat{b}\Psi_0 = 0\,. \tag{41}$$

Analogously, let us act on the second equation in (40) by the operator $(M+\Gamma)^{-1}(M+\Gamma\bar{P})$; this yields

$$\left(\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p} + \frac{M+\Gamma\bar{P}}{M-\Gamma}(M+\Gamma P)\right)\Psi_- + \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+\hat{a}\Psi_0 = 0\,.$$

Allowing for identities

$$\frac{M+\Gamma\bar{P}}{M+\Gamma}(M+\Gamma P) = \frac{1}{M+\Gamma}(M^2 + M\Gamma P + M\Gamma\bar{P}) = M\,;$$

we derive

$$\left(\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p}+M\right)\Psi_{-}+\frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_{+}\hat{a}\Psi_{0}=0\,.$$

With the special notations, the last equation reads shorter

$$\frac{M+\Gamma\bar{P}}{M+\Gamma}i\hat{p}=C,\quad \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_{+}=\beta'_{+},\qquad (C+M)\Psi_{-}+\beta'_{+}\hat{b}\Psi_{0}=0\,. \tag{42}$$

Let us consider the powers of A

$$A^{2}=\frac{1}{(M-\Gamma)^{2}}(iM\hat{p}-i\Gamma\bar{P}\hat{p})(iM\hat{p}-i\Gamma\bar{P}\hat{p})=\frac{1}{(M-\Gamma)^{2}}[-M^{2}\hat{p}^{2}+M\Gamma\hat{p}\bar{P}\hat{p}+M\Gamma\bar{P}\hat{p}^{2}-\Gamma^{2}\bar{P}\hat{p}\bar{P}\hat{p}]\,.$$

Taking into account the identities $P+\bar{P}=1, P\bar{P}=\bar{P}P=0$ and

$$\beta_{\mu}=P\beta_{\mu}+\beta_{\mu}P=\bar{P}\beta_{\mu}+\beta_{\mu}\bar{P},\beta_{\mu}P=P\beta_{\mu},\bar{P}\beta_{\mu}=\beta_{\mu}\bar{P},$$
$$P\beta_{\mu}P=\bar{P}\beta_{\mu}\bar{P}=0,\beta_{\mu}\beta_{\nu}P=P\beta_{\mu}\beta_{\nu},\beta_{\mu}\beta_{\nu}\bar{P}=\bar{P}\beta_{\mu}\beta_{\nu},$$

we get

$$A^{2}=\frac{1}{(M-\Gamma)^{2}}(-M^{2}\hat{p}^{2}+M\Gamma\hat{p}^{2})=-\frac{M\hat{p}^{2}}{M-\Gamma}\,.$$

Let us calculate A^3

$$A^{3}=-\frac{M}{(M-\Gamma)^{2}}(M-\Gamma\bar{P})(i\hat{p})\hat{p}^{2}=-\frac{Mp^{2}}{(M-\Gamma)}\frac{(M-\Gamma\bar{P})}{M-\Gamma}(i\hat{p})\,, \tag{43}$$

so, the minimal polynomial for A has the form (similarly looks that for C):

$$A^{3}=-\frac{Mp^{2}}{M-\Gamma}A\,,\quad C^{3}=-\frac{Mp^{2}}{M+\Gamma}C\,. \tag{44}$$

The minimal polynomial for $i\hat{p}$ has the form $i\hat{p}\,[(i\hat{p})^{2}+p^{2}]=0\,.$

Thus, the complete set of equations in the variable x_3 is of the form

$$(i\hat{p}+M)\,f_{0}+\beta_{+}\hat{a}f_{+}-\beta_{-}\hat{b}f_{-}=0\,,$$
$$(A+M)f_{+}-\beta'_{-}\hat{b}f_{0}=0\,,\quad (C+M)f_{-}+\beta'_{+}\hat{a}f_{0}=0\,. \tag{45}$$

To proceed with these equations, we introduce the matrices (we take in the account that $p^2=p_1^2+p_2^2$) with the properties

$$\overline{(i\hat{p}+M)}(i\hat{p}+M)=p^{2}+M^{2},\quad \overline{(A+M)}(A+M)=p^{2}+M^{2},\quad \overline{(C+M)}(C+M)=p^{2}+M^{2}\,.$$

In fact these formulas determine the inverse matrices up to numerical factors $(p^2+M^2)^{-1}$. Then the system of radial equations can be rewritten alternatively

$$(i\hat{p}+M)\,(p^{2}+M^{2})f_{0}+\beta_{+}\hat{a}(p^{2}+M^{2})f_{+}-\beta_{-}\hat{b}(p^{2}+M^{2})f_{-}=0\,,$$
$$(p^{2}+M^{2})f_{+}-\overline{(A+M)}\beta'_{-}\hat{b}f_{0}=0\,,\quad (p^{2}+M^{2})f_{-}+\overline{(C+M)}\beta'_{+}\hat{a}f_{0}=0\,. \tag{46}$$

The first equation in (46), with the help of the other two, is transformed into an equation for the component $f_0(r)$:

$$(i\hat{p}+M)(p^2+M^2)^2 f_0 + \beta_+ \hat{a}\overline{(A+M)}\beta'_- \hat{b} f_0 + \beta_- \hat{b}\overline{(C+M)}\beta'_+ \hat{a}_m f_0 = 0\,; \tag{47}$$

while the two remaining ones are not changed

$$(p^2+M^2)f_+ - \overline{(A+M)}\beta'_- \hat{b} f_0 = 0\,, \quad (p^2+M^2)f_- + \overline{(C+M)}\beta'_+ \hat{a} f_0 = 0\,. \tag{48}$$

In fact, the equations (48) mean that it suffices to solve (47) with respect to f_0; two other components f_+ and f_- can be calculated by means of the equations (48).

To proceed further, we need to know the explicit form of the inverse operators (**??**). The needed inverse operators must be quadratic with respect to the relevant matrices. They are given by the formulas:

$$\begin{gathered}
\overline{(M+i\hat{p})} = \frac{1}{M}\ \left[\ (i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\ \right]\ , \\
\overline{(A+M)} = \frac{p^2+M^2}{M}\ \left[\ 1 - \frac{M-\Gamma}{p^2+M^2-M\Gamma}A + \frac{M-\Gamma}{M(p^2+M^2-M\Gamma)}A^2\ \right]\ , \\
\overline{(C+M)} = \frac{p^2+M^2}{M}\ \left[\ 1 - \frac{M+\Gamma}{p^2+M^2+M\Gamma}C + \frac{M+\Gamma}{M(p^2+M^2+M\Gamma)}C^2\ \right]\ .
\end{gathered} \tag{49}$$

Taking into account the explicit form for the inverse operators, we get

$$\begin{gathered}
(p^2+M^2)f_0 + \frac{1}{M^2}\left[\ (i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\ \right]\beta_+ \\
\times\left[\ 1 - \frac{M-\Gamma}{p^2+M^2-M\Gamma}A + \frac{M-\Gamma}{M(p^2+M^2-M\Gamma)}A^2\ \right]\beta'_-\,\hat{a}\hat{b} f_0 \\
+\frac{1}{M^2}\left[\ (i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\ \right]\beta_- \\
\times\left[\ 1 - \frac{M+\Gamma}{p^2+M^2+M\Gamma}C + \frac{M+\Gamma}{M(p^2+M^2+M\Gamma)}C^2\right]\beta'_+\,\hat{b}\hat{a} f_0 = 0\,.
\end{gathered}$$

Now, by taking in mind the formulas

$$\begin{gathered}
A = \frac{M-\Gamma\bar{P}}{M-\Gamma}\,i\hat{p}\,, \quad A^2 = -\frac{M\hat{p}^2}{M-\Gamma}\,, \quad C = \frac{M+\Gamma\bar{P}}{M+\Gamma}\,i\hat{p}\,, \quad C^2 = -\frac{M\hat{p}^2}{M+\Gamma}\,, \\
\beta'_- = \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_-\,, \quad \beta'_+ = \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+\,,
\end{gathered}$$

we transform the above equation into the following one

$$\begin{gathered}
(p^2+M^2)f_0 + \frac{1}{M^2}\ \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\ \right]\ \beta_+ \\
\times\left[\ 1 - \frac{M-\Gamma\bar{P}}{p^2+M^2-M\Gamma}\,i\hat{p} + \frac{(i\hat{p})^2}{p^2+M^2-M\Gamma}\ \right]\ \frac{M-\Gamma\bar{P}}{M-\Gamma}\beta_-\,\hat{a}\hat{b} f_0 \\
+\frac{1}{M^2}\ \left[(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\ \right]\ \beta_- \\
\times\left[\ 1 - \frac{M+\Gamma\bar{P}}{p^2+M^2+M\Gamma}\,i\hat{p} + \frac{(i\hat{p})^2}{p^2+M^2+M\Gamma}\ \right]\ \frac{M+\Gamma\bar{P}}{M+\Gamma}\beta_+\,\hat{b}\hat{a} f_0 = 0\,.
\end{gathered}$$

After some manipulations it becomes

$$\{ (p^2+M^2) + \hat{a}\hat{b}\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma}$$
$$\times[\,(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)]\beta_+[\,(p^2+M^2-M\Gamma) - \,i\hat{p}(M-\Gamma P) + (i\hat{p})^2\,]\,(M-\Gamma\bar{P})\beta_-$$
$$+\hat{b}\hat{a}\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma}$$
$$\times[\,(i\hat{p})^2 - M(i\hat{p}) + (p^2+M^2)\,]\,\beta_-\,[(p^2+M^2+M\Gamma) - \,i\hat{p}(M+\Gamma P) + (i\hat{p})^2\,]\,(M+\Gamma\bar{P})\beta_+\,\}\,f_0 = 0\,.$$

Due to the identity $\hat{p}\beta_+\hat{p} = \hat{p}\beta_-\hat{p} = 0$, it reads simpler

$$\{ (p^2+M^2) + \hat{a}\hat{b}\frac{1}{M^2(p^2+M^2-M\Gamma)}\frac{1}{M-\Gamma} \times [\,(p^2+M^2-M\Gamma)(i\hat{p})^2\beta_+$$
$$-M(p^2+M^2-M\Gamma)i\hat{p}\beta_+ + (p^2+M^2)(p^2+M^2-M\Gamma)\beta_+$$
$$-(p^2+M^2)\beta_+ i\hat{p}(M-\Gamma P) + (p^2+M^2)\beta_+(i\hat{p})^2]\,(M-\Gamma\bar{P})\beta_-$$

$$+\hat{b}\hat{a}\frac{1}{M^2(p^2+M^2+M\Gamma)}\frac{1}{M+\Gamma} \times [\,(p^2+M^2+M\Gamma)(i\hat{p})^2\beta_-$$
$$-M(p^2+M^2+M\Gamma)i\hat{p}\beta_- + (p^2+M^2)(p^2+M^2+M\Gamma)\beta_-$$
$$-(p^2+M^2)\beta_- i\hat{p}(M+\Gamma P) + (p^2+M^2)\beta_-(i\hat{p})^2]\,(M+\Gamma\bar{P})\beta_+\,\}f_0 = 0 \qquad (50)$$

Now we should take into account the explicit form of f_0,

$$f = \left| \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_{[23]} \\ f_{[31]} \\ f_{[12]} \\ f_{[14]} \\ f_{[24]} \\ f_{[34]} \end{array} \right|, \qquad f_0 = \left| \begin{array}{c} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{[12]} \\ 0 \\ 0 \\ f_{[34]} \end{array} \right|.$$

explicit form of $i\hat{p}$ and all involved matrices. After rather cumbersome manipulations, we get equations for constituents of f_0:

$$(p^2+M^2)f_1 + \frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}$$
$$\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_1 - p_2(p^2+M^2-M\Gamma)(p_2f_1 - p_1f_2)$$
$$+Mp_1(p^2+M^2)f_{[34]} - (p^2+M^2)p_1(p_1f_1+p_2f_2)\} + \frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}$$
$$\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_1 - p_2(p^2+M^2+M\Gamma)(p_2f_1 - p_1f_2)$$
$$-Mp_1(p^2+M^2)f_{[34]} - p_1(p^2+M^2)(p_1f_1+p_2f_2)\} = 0\,, \qquad (51)$$

$$
\begin{aligned}
&(p^2+M^2)f_2+\frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\\
&\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_2+p_1(p^2+M^2-M\Gamma)(p_2f_1-p_1f_2)\\
&+M(p^2+M^2)p_2f_{[34]}-p_2(p^2+M^2)(p_1f_1+p_2f_2)\}+\frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\\
&\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_2+p_1(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)\\
&-Mp_2(p^2+M^2)f_{[34]}-p_2(p^2+M^2)(p_1f_1+p_2f_2)\}=0\,,
\end{aligned}
\tag{52}
$$

$$
(p^2+M^2)f_{[12]}+\frac{\hat{a}\hat{b}}{M}\{-i(p_2f_1-p_1f_2)\}+\frac{\hat{b}\hat{a}}{M}\{-i(p_2f_1-p_1f_2)\}=0\,,
\tag{53}
$$

$$
\begin{aligned}
&f_{[34]}+\frac{\hat{a}\hat{b}}{M(M-\Gamma)(p^2+M^2-M\Gamma)}\{(p^2+M^2-M\Gamma)f_{[34]}-(M-\Gamma)(p_1f_1+p_2f_2))p^2f_{[34]}\}\\
&+\frac{\hat{b}\hat{a}}{M(M+\Gamma)(p^2+M^2+M\Gamma)}\{(p^2+M^2+M\Gamma)f_{[34]}+(M+\Gamma)(p_1f_1+p_2f_2))-p^2f_{[34]}\}=0\,.
\end{aligned}
\tag{54}
$$

From (53) we derive

$$
(p^2+M^2)f_{[12]}-\frac{i}{M}(\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2)=0\,.
\tag{55}
$$

From (54) it follows

$$
\begin{aligned}
&f_{[34]}+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\{Mf_{[34]}-(p_1f_1+p_2f_2)\}\\
&+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{Mf_{[34]}+(p_1f_1+p_2f_2)\}=0\,.
\end{aligned}
\tag{56}
$$

From Eqs. (51–52) we get

$$
\begin{aligned}
[1]\qquad &(p^2+M^2)f_1++\frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}\\
&\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_1-p_2(p^2+M^2-M\Gamma)(p_2f_1-p_1f_2)\\
&+Mp_1(p^2+M^2)f_{[34]}-(p^2+M^2)p_1(p_1f_1+p_2f_2)\}+\frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}\\
&\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_1-p_2(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)\\
&-Mp_1(p^2+M^2)f_{[34]}-p_1(p^2+M^2)(p_1f_1+p_2f_2)\}=0\,,
\end{aligned}
\tag{57}
$$

$$[2] \qquad (p^2+M^2)f_2 + \frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma}$$
$$\times\{(p^2+M^2)(p^2+M^2-M\Gamma)f_2 + p_1(p^2+M^2-M\Gamma)(p_2f_1-p_1f_2)$$
$$+M(p^2+M^2)p_2f_{[34]} - p_2(p^2+M^2)(p_1f_1+p_2f_2)\} + \frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma}$$
$$\times\{(p^2+M^2)(p^2+M^2+M\Gamma)f_2 + p_1(p^2+M^2+M\Gamma)(p_2f_1-p_1f_2)$$
$$-Mp_2(p^2+M^2)f_{[34]} - p_2(p^2+M^2)(p_1f_1+p_2f_2)\} = 0\,. \qquad (58)$$

Combining these equations as follows

$$p_1\,[1] + p_2\,[2], \qquad p_2\,[1] - p_1\,[2]\,,$$

we derive

$$(p_1f_1+p_2f_2) + \frac{\hat{a}\hat{b}}{M^2(p^2+M^2-M\Gamma)}$$
$$\times\{(p^2+M^2-M\Gamma)(p_1f_1+p_2f_2) + Mp^2f_{[34]} - p^2(p_1f_1+p_2f_2)\} + \frac{\hat{b}\hat{a}}{M^2(p^2+M^2+M\Gamma)}$$
$$\times\{(p^2+M^2+M\Gamma)(p_1f_1+p_2f_2) - Mp^2f_{[34]} - p^2(p_1f_1+p_2f_2)\} = 0\,,$$

and

$$(p^2+M^2)(p_2f_1-p_1f_2) + \frac{\hat{a}\hat{b}}{M^2}\{(p^2+M^2)(p_2f_1-p_1f_2) - p^2(p_2f_1-p_1f_2)\}$$
$$+\frac{\hat{b}\hat{a}}{M^2}\{(p^2+M^2)(p_2f_1-p_1f_2) - p^2(p_2f_1-p_1f_2)\} = 0\,.$$

After elementary manipulations they read

$$(p_1f_1+p_2f_2)$$
$$+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\,\{(M-\Gamma)(p_1f_1+p_2f_2) + p^2f_{[34]}\}$$
$$+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{(M+\Gamma)(p_1f_1+p_2f_2) - p^2f_{[34]}\} = 0\,, \qquad (59)$$

$$(p^2+M^2)(p_2f_1-p_1f_2) + (\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2) = 0\,. \qquad (60)$$

Let us write down here two other equations as well:

$$(p^2+M^2)f_{[12]} - \frac{i}{M}(\hat{a}\hat{b}+\hat{b}\hat{a})(p_2f_1-p_1f_2) = 0\,, \qquad (61)$$

$$f_{[34]} + \frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\{Mf_{[34]} - (p_1f_1+p_2f_2)\}$$
$$+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}\{Mf_{[34]} + (p_1f_1+p_2f_2)\} = 0\,. \qquad (62)$$

From Eqs. (60–61), we easily derive

$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+(p^2+M^2)](p_2f_1-p_1f_2)=0\,,\quad f_{[12]}=\frac{1}{iM}(p_2f_1-p_1f_2)\,. \tag{63}$$

Thus, we need to investigate only two remaining equations. Let us introduce a shortening notation:

$$F=f_{[34]},\qquad G=p_1f_1+p_2f_2\,;$$

then we have equations in the form

$$F+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}(MF-G)+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}(MF+G)=0\,, \tag{64}$$

$$G+\frac{\hat{a}\hat{b}}{M(p^2+M^2-M\Gamma)}\,[(M-\Gamma)G+p^2F]+\frac{\hat{b}\hat{a}}{M(p^2+M^2+M\Gamma)}[(M+\Gamma)G-p^2F]=0\,. \tag{65}$$

Let us transform the first equation (64) to the form

$$F+\frac{1}{M(p^2+M^2-M\Gamma)(p^2+M^2+M\Gamma)}$$
$$\times\{(p^2+M^2+M\Gamma)\,\hat{a}\hat{b}\,(MF-G))+(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,(MF+G)\}=0\,,$$

which after elementary manipulation yields

$$[1]\qquad F+\frac{1}{M[(p^2+M^2)^2-M^2\Gamma^2]}$$
$$\times\{M(p^2+M^2)\,(\hat{a}\hat{b}+\hat{b}\hat{a})\,F+M^2\Gamma\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,F$$
$$-M\Gamma\,(\hat{a}\,\hat{b}+\hat{b}\hat{a})G-(p^2+M^2)\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,G\}=0\,. \tag{66}$$

Now, let us turn to eq. (65) an rewrite it as follows

$$G+\frac{1}{M[(p^2+M^2)^2-M^2\Gamma^2)}$$
$$\times\{\,(p^2+M^2+M\Gamma)(M-\Gamma)\,\hat{a}\hat{b}\,G+p^2(p^2+M^2+M\Gamma)\,\hat{a}\hat{b}\,F$$
$$+(M+\Gamma)(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,G-p^2(p^2+M^2-M\Gamma)\,\hat{b}\hat{a}\,F\}=0\,,$$

or

$$[2]\qquad G+\frac{1}{M[(p^2+M^2)^2-M^2\Gamma^2)}$$
$$\times\{\,M(p^2+M^2)(\hat{a}\hat{b}+\hat{b}\hat{a})\,G-\Gamma(p^2+M^2)(\hat{a}\hat{b}-\hat{b}\hat{a})G$$
$$-M\Gamma^2(\,\hat{a}\hat{b}+\hat{b}\hat{a})\,G+\Gamma M^2(\hat{a}\hat{b}-\hat{b}\hat{a})G$$
$$+M\Gamma p^2\,(\hat{a}\hat{b}+\hat{b}\hat{a})\,F+p^2(p^2+M^2)\,(\hat{a}\hat{b}-\hat{b}\hat{a})\,F\}=0\,, \tag{67}$$

Let us combine equations (66) and (67) as follows:

$$-\Gamma p^2\,[1]+(p^2+M^2)\,[2]\,,\qquad (p^2+M^2-\Gamma^2)\,[1]+\Gamma\,[2]\,;$$

this results

$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]G-\Gamma p^2F+\frac{p^2}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})F=0\,, \tag{68}$$

$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]F+\Gamma G-\frac{1}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})G-\Gamma^2F+\frac{\Gamma}{M}(\hat{a}\hat{b}-\hat{b}\hat{a})F=0\,. \tag{69}$$

Allowing for two relations

$$\hat{a}=\frac{1}{\sqrt{2}}\left(+\frac{+d}{dx_3}+ieEx_3+i\epsilon\right),\quad \hat{b}=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx_3}+ieEx_3+i\epsilon\right),$$

we get $(\hat{a}\hat{b}-\hat{b}\hat{a})=ieE$; then the last equations are presented in a simpler form as

$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]G=p^2(\Gamma-\frac{ieE}{M})F\,,$$
$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]F=-(\Gamma-\frac{ieE}{M})G+\Gamma(\Gamma-\frac{ieE}{M})F\,. \tag{70}$$

Let is introduce the notation $ieE=E_0$, then the system is written as

$$(\Gamma-\frac{E_0}{M})^{-1}[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]G=p^2F\,,$$
$$(\Gamma-\frac{E_0}{M})^{-1}[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]F=-G+\Gamma F\,. \tag{71}$$

This sub-system is solved through diagonalizing the mixing matrix. With this goal, let us introduce new functions

$$\Phi_1=G-\lambda_1F\,,\quad \Phi_2=G-\lambda_2F\,, \tag{72}$$

$$\lambda_1=\frac{\Gamma+\sqrt{\Gamma^2-4p^2}}{2}\,,\quad \lambda_2=\frac{\Gamma-\sqrt{\Gamma^2-4p^2}}{2}\,. \tag{73}$$

So we get two separate equations:

$$[(\hat{a}\hat{b}+\hat{b}\hat{a})+p^2+M^2]-\lambda'_{1,2}]\,\Phi_{1,2}=0\,,\quad \lambda'_{1,2}=\lambda_{1,2}(\Gamma-\frac{E_0}{M})\,. \tag{74}$$

For the 2-nd order operator we have the explicit form (let it be $x_3=z$)

$$(\hat{a}\hat{b}+\hat{b}\hat{a})=-\frac{d^2}{dz^2}-e^2E^2z^2-2eE\epsilon z-\epsilon^2\,. \tag{75}$$

Thus, we get the equation (we follow both variants)

$$\left[\frac{d^2}{dz^2}+(e^2E^2z^2+2eE\epsilon z+\epsilon^2)-\mu^2p^2-M^2+\lambda'_{1,2}\right]\Phi_{1,2}(z)=0\,, \tag{76}$$

where $\mu^2=p^2-M^2+\lambda'_{1,2}$. This task coincides with that arising for scalar Klein–Fock–Gordon particle in an external uniform electric field modified by an anomalous magnetic moment though the term $\lambda=\lambda_{1,2}$.

4. Solving the differential equation

We start with the equation

$$\left[\frac{d^2}{dz^2} + (\epsilon + eEz)^2 - \mu^2 \right] \Phi(z) = 0 \,, \quad \mu^2 = M^2 + p^2 - \lambda'_{1,2} > 0 \,. \tag{77}$$

It may be seen that this equation after transforming to a new variable x from mathematical point of view is very similar to that arising for non-relativistic quantum harmonic oscillator:

$$x = \frac{\epsilon + eEz}{eE}, \quad \left[\frac{d^2}{dx^2} - \mu^2 + (eE)^2 x^2 \right] \Phi = 0 \,, \quad \left(\frac{d^2}{dx^2} + E - kx^2 \right) f = 0 \,. \tag{78}$$

In eq. (77) let us use a new variable Z (let it be $eE > 0$):

$$Z = i\frac{(\epsilon + eEz)^2}{eE} \,, \; \sigma = \frac{\mu^2}{4eE}, \quad \left(\frac{d^2}{dZ^2} + \frac{1/2}{Z}\frac{d}{dZ} - \frac{1}{4} + \frac{i\sigma}{Z} \right) \Phi(Z) = 0 \,. \tag{79}$$

It has two singular points. The point $Z = 0$ is regular, behavior of solutions in its vicinity may be as follows $Z \to 0, \; \Phi(Z) = Z^A, \; A = 0, 1/2$. The point $Z = \infty$ is irregular singularity of the rank 2. Indeed, in the variable $y = Z^{-1}$ the above equation reads

$$\left(\frac{d^2}{dy^2} + \frac{3/2}{y}\frac{d}{dy} - \frac{1}{4y^4} + \frac{i\sigma}{y^3} \right) \Phi = 0 \,. \tag{80}$$

Asymptotic at $y \to 0$ should have the structure $y \to 0, \; \Phi = y^C e^{D/y}$; the above equation gives $D^2 - 1/4 = 0, \; -2CD + 2D - \frac{3}{2}D + i\sigma = 0$; whence it follows

$$D_1 = +1/2, \quad C_1 = 1/4 + i\sigma \,; \qquad D_2 = -1/2, \quad C_2 = 1/4 - i\sigma \,. \tag{81}$$

Thus, at infinity two asymptotics are possible

$$Z \to \infty, \qquad \Phi = Z^{-C} e^{DZ} = \left\{ \begin{array}{l} Z^{-C_1} e^{D_1 Z} = Z^{-1/4 - i\sigma} e^{+Z/2} \\ Z^{-C_2} e^{D_2 Z} = Z^{-1/4 + i\sigma} e^{-Z/2} \end{array} \right. ; \tag{82}$$

where (we use the main branch of logarithmic function)

$$Z = i\frac{(\epsilon + eEz)^2}{eE} = iZ_0, \qquad Z_0 > 0 \,, \quad e^{\pm Z/2} = e^{\pm iZ_0/2}$$
$$Z^{-1/4 \mp i\sigma} = (e^{\ln iZ_0})^{-1/4 \mp i\sigma} = (e^{\ln Z_0 + i\pi/2})^{-1/4 \mp i\sigma} \,. \tag{83}$$

Now we are to construct solutions in the whole region of Z. We start with the substitution

$$\Phi(Z) = Z^A \, e^{BZ} \, f(Z) \,; \quad \text{let it be} \quad A = 0\,, 1/2\,, \; B = -1/2\,;$$
$$[\, Z \frac{d^2}{dZ^2} + (2A + 1/2 - Z) \frac{d}{dy} - (A + 1/4 - i\sigma) \,] \, f(Z) = 0 \,,$$

the last equation coincides with the confluent hypergeometric one

$$a = A + 1/4 - i\sigma \,, \quad c = 2A + 1/2 \,, \quad f(Z) = Z^A \, e^{-Z/2} \, F(a, c; Z) \,. \tag{84}$$

Without loss of generality, we may take the value $A = 0$:

$$A = 0, \quad a = 1/4 - i\sigma\,, \quad c = +1/2\,, \quad \Phi(Z) = e^{-Z/2} f(Z)\,. \tag{85}$$

Let us consider two definite independent solutions of the confluent hypergeometric equation (note the equivalent representations for each solution):

$$Y_1(Z) = F(a, c; Z) = e^Z F(c - a, c; -Z)\,, \tag{86}$$

$$Y_2(Z) = Z^{1-c} F(a - c + 1, 2 - c; Z) = Z^{1-c} e^Z F(1 - a, 2 - c; -Z)\,. \tag{87}$$

They lead to the corresponding Φ-solutions:

$$\Phi_1 = e^{-Z/2} F(a, c; Z) = e^{+Z/2} F(c - a, c; -Z)\,; \tag{88}$$

$$\Phi_2 = e^{-Z/2} Z^{1-c} F(a - c + 1, 2 - c; Z) = Z^{1-c} e^{+Z/2} F(1 - a, 2 - c; -Z)\,. \tag{89}$$

Allowing for identities

$$c = \frac{1}{2},\ a = \frac{1}{4} - i\sigma,\ c - a = \frac{1}{4} + i\sigma = a^*,\ c = c^* = \frac{1}{2},\ Z^* = -Z\,,$$
$$a - c + 1 = \frac{3}{4} - i\sigma = (1 - a)^*, \quad (2 - c) = (2 - c)^* = \frac{3}{2}\,,$$

we conclude that the first solution $\Phi_1(Z)$ is given by a real-valued function, whereas the second one $\Phi_2(Z)$ has a quite definite property with respect to complex conjugation

$$\Phi_1(Z) = +[\Phi_1(Z)]^*, \qquad \Phi_2(Z) = i[\Phi_2(Z)]^*\,. \tag{90}$$

This behavior of $\Phi_2(Z)$ can be presented as the real-valuedness if one uses another normalizing factor

$$\bar{\Phi}_2(Z) = \frac{1 - i}{\sqrt{2}} \Phi_2(Z) = \left(\frac{1 - i}{\sqrt{2}} \Phi_2(Z) \right)^* = (\bar{\Phi}_2(Z))^*\,. \tag{91}$$

At small values of Z, solutions behave as follows

$$Y_1(Z) \approx 1,\ Y_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0}\,; \qquad \Phi_1(Z) \approx 1,\ \Phi_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0}\,. \tag{92}$$

At large values of $Z = iZ_0, Z_0 \to +\infty$, one can employ the knows asymptotic formulas

$$F(a, c, Z) = \frac{\Gamma(c)}{\Gamma(c - a)} (-Z)^{-a} + + \frac{\Gamma(c)}{\Gamma(a)} e^Z Z^{a-c}\,. \tag{93}$$

In this way, we derive (again we use the main branch of the logarithmic function)

$$(-Z)^{-a} = (-iZ_0)^{-1/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2} \right)^{-1/4+i\sigma} = e^{-(-1/4+i\sigma)i\pi/2}\ e^{(-1/4+i\sigma)\ln Z_0}\,,$$
$$Z^{a-c} = (iZ_0)^{-1/4-i\sigma} = \left(e^{\ln Z_0 + i\pi/2} \right)^{-1/4-i\sigma} = e^{+(-1/4-i\sigma)i\pi/2}\ e^{(-1/4-i\sigma)\ln Z_0}\,,$$

and

$$\frac{\Gamma(c)}{\Gamma(c - a)} = \frac{\Gamma(1/2)}{\Gamma(1/4 + i\sigma)}, \qquad \frac{\Gamma(c)}{\Gamma(a)} = \frac{\Gamma(1/2)}{\Gamma(1/4 - i\sigma)}\,;$$

so we derive

$$Y_1(Z) = F(a,c,Z) = e^{iZ_0/2} \left\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)} \; e^{-(-1/4+i\sigma)i\pi/2} \; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2} \; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} . \tag{94}$$

From (94) it follows the asymptotic form for $\Phi_1(Z)$:

$$\Phi_1(Z) = \left\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)} \; e^{-(-1/4+i\sigma)i\pi/2} \; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2} \; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} ; \tag{95}$$

as it should be, we see the sum of two conjugate terms.

In similar manner, we study at infinity the function $F(a-c+1, 2-c; Z)$:

$$F(a-c+1, 2-c, Z) = \frac{\Gamma(2-c)}{\Gamma(1-a)} (-Z)^{-a+c-1} + \frac{\Gamma(2-c)}{\Gamma(a-c+1)} e^Z Z^{a-1} . \tag{96}$$

Allowing for relationships

$$(-Z)^{-a+c-1} = (-iZ_0)^{-3/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2} \right)^{-3/4+i\sigma} = e^{-(-3/4+i\sigma)i\pi/2} \; e^{(-3/4+i\sigma)\ln Z_0} ,$$

$$Z^{a-1} = (iZ_0)^{-3/4-i\sigma} = \left(e^{\ln Z_0 + i\pi/2} \right)^{-3/4-i\sigma} = e^{+(-3/4-i\sigma)i\pi/2} \; e^{(-3/4-i\sigma)\ln Z_0} ,$$

$$\frac{\Gamma(2-c)}{\Gamma(1-a)} = \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}, \qquad \frac{\Gamma(2-c)}{\Gamma(a-c+1)} = \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)}$$

we derive the asymptotic formula

$$F(a-c+1, 2-c, Z) = e^{iZ_0/2} \times \left\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)} \; e^{-(-3/4+i\sigma)i\pi/2} \; e^{(-3/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2} \; e^{(-3/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} . \tag{97}$$

From this, for the function $\Phi_2(Z)$ we derive (recalling that $\sqrt{Z} = e^{(1/2)(\ln Z_0 + i\pi/2)}$)

$$\Phi_2(Z) = \sqrt{Z} Z^{1/2} F(a-c+1, 2-c, Z) = e^{i\pi/4} \times \left\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)} \; e^{-(-3/4+i\sigma)i\pi/2} \; e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2} \; e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} . \tag{98}$$

This results conforms to the previously noted formula.

We can construct linearly independent solutions which behave at infinity as conjugate functions. To this end, we should employ another pair of linearly independent solutions

$$Y_5(Z) = \Psi(a, c; Z), \qquad Y_7(Z) = e^Z \Psi(c-a, c; -Z) . \tag{99}$$

Two pairs $\{Y_5, Y_7\}$ and $\{Y_1, Y_2\}$ relate to each other by Kummer formulas

$$Y_5 = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, Y_1 + \frac{\Gamma(c-1)}{\Gamma(a)}\, Y_2\,, \quad Y_7 = \frac{\Gamma(1-c)}{\Gamma(1-a)}\, Y_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)}\, e^{i\pi c}\, Y_2\,. \tag{100}$$

At large Z, as $\mid Z \mid \to \infty$, an asymptotic formula is valid

$$Y_5 == Z^{-a} = (iZ_0)^{-1/4+i\sigma} = (e^{\ln Z_0 + i\pi/2})^{-1/4+i\sigma}\,,$$
$$Y_7(Z) = e^{Z}(-iZ_0)^{a-c} = e^{iZ_0}(-iZ_0)^{-1/4-i\sigma} = e^{iZ_0}(e^{\ln Z_0 - i\pi/2})^{-1/4-i\sigma}\,.$$

These formulas after transition to functions $\Phi(Z)$ take the form

$$\Phi_5 = e^{-iZ_0/2}(e^{\ln Z_0 + i\pi/2})^{-1/4+i\sigma},\ \Phi_7 = e^{+iZ_0/2}(e^{\ln Z_0 - i\pi/2})^{-1/4-i\sigma}. \tag{101}$$

We see that they are conjugate to each other functions; just these ones enter in superpositions (95) and (98).

5. Restrictions on the values of anomalous magnetic moment

By physical grounds, the above parameter μ^2 must be positive, for both cases of $\lambda = \lambda'_{1,2}$:

$$\mu^2 = M^2 + p^2 - (\Gamma - \frac{ieE}{M})\frac{\Gamma \pm \sqrt{\Gamma^2 - 4p^2}}{2} > 0\,.$$

Let us take into account that $\Gamma = i\Gamma_0$ (Γ_0 is real-valued):

$$\mu^2 = M^2 + p^2 + (\Gamma_0 - \frac{eE}{M})\frac{\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}}{2} > 0\,.$$

Evidently, the region for Γ_0, given by (for definiteness, let it be $eE > 0$)

$$\Gamma_0 - \frac{eE}{M} > 0 \quad (eE > 0),$$

has no physical sense, because it does not contain the vicinity of the point $\Gamma_0 = 0$. So, in the following we assume that

$$\Gamma_0 - y < 0, \quad y = \frac{eE}{M} > 0\,. \tag{102}$$

Then, the main inequality takes the form

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} > \Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}\,. \tag{103}$$

Let us study the variant $\Gamma_0 < 0$,$(-)$ – lower sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} > \Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2}\,. \tag{104}$$

this is valid without any additional restrictions.

Let us study the variant $\Gamma_0 < 0$, $(+)$ – upper sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} > \Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2}\,. \tag{105}$$

This yields

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} - \Gamma_0 > +\sqrt{\Gamma_0^2 + 4p^2}\,;$$

which after squaring takes the form

$$\frac{4(M^2+p^2)^2}{(y-\Gamma_0)^2} - 2\Gamma_0 \frac{2(M^2+p^2)}{(y-\Gamma_0)} - 4p^2 > 0\,,$$

or

$$(M^2+p^2)^2 - \Gamma_0(M^2+p^2)(y-\Gamma_0) - p^2(y-\Gamma_0)^2 > 0\,.$$

It is convenient to employ the variable x:

$$y - \Gamma_0 = x > 0\,, \tag{106}$$

then we have

$$(M^2+p^2)^2 - (y-x)x(M^2+p^2) - p^2x^2 > 0\,.$$

So, we get

$$x^2 - 2x\frac{(M^2+p^2)y}{2M^2} + \frac{(M^2+p^2)^2}{M^2} > 0 \; ; \quad (107)$$

the roots of the quadratic equations are

$$x_{1,2} = \frac{(M^2+p^2)}{2M^2} \pm \frac{(M^2+p^2)y}{2M^2}\sqrt{y^2-4M^2} \; .$$

The whole parabola lays above the horizontal axes only if the discriminant is negative, this yields

$$y^2 - 4M^2 < 0 \quad \Longrightarrow \quad \frac{eE}{M} < 2M \; . \quad (108)$$

Thus, we get a definite restriction on the magnitude of an electric field (beside we should remember on

$$\Gamma_0 < 0 \; . \quad (109)$$

Let us study the variant $\Gamma_0 > 0$,(−) – lower sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} > \Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2} \; ; \quad (110)$$

which is evidently valid.

Let us study the variant $\Gamma_0 > 0$, (+) – upper sign:

$$\frac{2(M^2+p^2)}{(y-\Gamma_0)} - \Gamma_0 > \sqrt{\Gamma_0^2 + 4p^2} \; . \quad (111)$$

After squaring this inequality, we obtain

$$(M^2+p^2)^2 - \Gamma_0(M^2+p^2)(y-\Gamma_0) - p^2(y-\Gamma_0)^2 > 0 \; .$$

Now, we can repeat the previous analysis. With the help of the variable $y - \Gamma_0 = x$; we get

$$x^2 - 2x\frac{(M^2+p^2)y}{2M^2} + \frac{(M^2+p^2)^2}{M^2} > 0 \; ;$$

the roots are

$$x_{1,2} = \frac{(M^2+p^2)}{2M^2} \pm \frac{(M^2+p^2)y}{2M^2}\sqrt{y^2-4M^2}$$

The whole parabola lays above the horizontal axes only if the discriminant is negative, this yields

$$y^2 - 4M^2 < 0 \quad \Longrightarrow \quad \frac{eE}{M} < 2M \; ;$$

we should remember on assumption $0 < \Gamma_0$, so we derive

$$0 < \Gamma_0 < \frac{eE}{M} \; . \quad (112)$$

Summarizing the results, we conclude that μ^2 will be positive, if the following double inequality is valid

$$\Gamma_0 < \frac{eE}{M} < 2M \; . \quad (113)$$

6. Particle with vanishing electric charge

Let us specify the result for the case of a neutral particle. Formally, it can be reached through the following limiting procedure

$$e \to 0 \, , \; \frac{2E}{M}\lambda \to \infty \, , \; \Gamma = \pm\frac{2eE}{M}\lambda \to \frac{2E}{M}\Lambda \, ,$$

λ has been a dimensionless parameter; the new Λ has the electric charge dimension.

Below we write down only main relations:

$$\hat{a} = \frac{1}{\sqrt{2}}(+\frac{d}{dx_3} + i\epsilon), \; \hat{b} = \frac{1}{\sqrt{2}}(-\frac{d}{dx_3} + i\epsilon) \, ,$$

$$(\hat{a}\hat{b} + \hat{b}\hat{a}) = -\frac{d^2}{dz^2} - \epsilon^2, \quad (\hat{a}\hat{b} - \hat{b}\hat{a}) = 0 \, ;$$

$$f_{[12]} = \frac{1}{iM}(p_2 f_1 - p_1 f_2) \, ,$$

$$(\frac{d^2}{dz^2} + \epsilon^2 - p^2 - M^2)(p_2 f_1 - p_1 f_2) = 0 \, ; \quad (114)$$

$$F = f_{[34]}, \qquad G = p_1 f_1 + p_2 f_2 \, ,$$

$$\Gamma^{-1}[\hat{a}\hat{b} + \hat{b}\hat{a} + p^2 + M^2]G = p^2 F \, ,$$

$$\Gamma^{-1}[\hat{a}\hat{b} + \hat{b}\hat{a} + p^2 + M^2]F = -G + \Gamma F \, ; \quad (115)$$

$$\Phi_1 = G - \lambda_1 F \, , \quad \Phi_2 = G - \lambda_2 F \, ,$$

$$\lambda_{1,2} = \frac{\Gamma + \sqrt{\Gamma^2 \pm 4p^2}}{2} \, ; \quad (116)$$

$$[\frac{d^2}{dz^2} + \epsilon^2 - p^2 - M^2 + \Gamma\lambda_{1,2}]\Phi_{1,2}(z) = 0 \, .$$

Let us introduce the notation

$$\Delta = \epsilon^2 - p^2 - M^2 > 0\,, \qquad \Delta + \Gamma\lambda_{1,2} = p_z^2\,, \tag{117}$$

where

$$\Gamma\lambda_{1,2} = \frac{\Gamma}{2}(\Gamma \pm \sqrt{\Gamma^2 - 4p^2}) = = -\frac{\Gamma_0}{2}(\,\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}\,)\,. \tag{118}$$

Solutions have the form of the plane waves $\Phi_{1,2}(z) = e^{\pm ip_3 z}$, only if

$$p_z^2 = \Delta - \frac{1}{2}\Gamma_0(\,\Gamma_0 \pm \sqrt{\Gamma_0^2 + 4p^2}\,) > 0\,. \tag{119}$$

Let us study this inequality. It is convenient to consider separately four sub-cases.

1. upper sign (+), $\Gamma_0 > 0$;
2. lower sign (−), $\Gamma_0 > 0$;
3. upper sign (+) $\Gamma_0 < 0$;
4. lower sign (−) $\Gamma_0 < 0$.

Consider the variant 1:

$$\Gamma_0 > 0, \qquad 2\Delta > \Gamma_0(\Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2}), \qquad 2\Delta - \Gamma_0^2 > \Gamma_0\sqrt{\Gamma_0^2 + 4p^2}\;;$$

here we should impose evident restriction $\Gamma_0^2 < 2\Delta$; further we derive

$$4\Delta^2 - 4\Delta\Gamma_0^2 + \Gamma_0^4 > \Gamma_0^2(\Gamma_0^2 + 4p^2) \quad \Longrightarrow \quad \Delta^2 - \Delta\Gamma_0^2 > \Gamma_0^2 p^2\;,$$

that is $\Gamma_0^2 < \Delta^2/(\Delta + p^2)$. It readily checked the inequality below $2\Delta > \Delta^2/(\Delta+p^2$; thus we arrive at the restriction

$$1. \qquad 0 < \Gamma_0 < \frac{\Delta}{\sqrt{\Delta + p^2}}, \qquad \Gamma_0 > 0\;,\; \Delta = \epsilon^2 - p^2 - M^2 > 0\,. \tag{120}$$

Now consider the variant 2:

$$2. \qquad \Gamma_0 > 0,\; 2\Delta > \Gamma_0(\Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2})\;;$$

evidently, this relationship is valid always.

Now consider the variant 3:

$$3. \qquad \Gamma_0 < 0,\; 2\Delta > \Gamma_0(\Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2})\;;$$

this relationship is valid always.

Finally, let us consider the variant 4:

$$4.\;\; \Gamma_0 < 0,\; 2\Delta > \Gamma_0(\Gamma_0 - \sqrt{\Gamma_0^2 + 4p^2}) = = (-\Gamma_0)(-\Gamma_0 + \sqrt{\Gamma_0^2 + 4p^2})\;;$$

here we can employ results for the case 1 – thus we obtain

$$\Gamma_0^2 < \frac{\Delta^2}{\Delta + p^2}, \qquad \Gamma_0 < 0\,. \tag{121}$$

Summing the consideration, we conclude that the parameter Γ_0 must lay within the following interval

$$\Gamma_0 < +\frac{\Delta}{\sqrt{\Delta + p^2}},\; \Delta > 0\,. \tag{122}$$

7. Conclusion

We have studied the behavior of a vector particle with anomalous magnetic moment in the presence of an external uniform electric field. The problem is reduced to three independent differential equations, for three functions, which are of the type of one-dimensional Klein-Fock-Gordon equation in the presence of a uniform electric field modified by the anomalous magnetic moment of the particle. Solutions are constructed in terms of confluent hypergeometric functions. For assigning physical sense for solutions, one should impose special restriction on a parameter related to the anomalous moment of the particle. The neutral spin 1 particle is considered as well.

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