INTEGRALS RELATED TO THE MULTIDIMENSIONAL-MATRIX GAUSSIAN DISTRIBUTION

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Abstract

The three integrals, total probability formula and Bayes' formula connected with the multidimensional-matrix Gaussian distribution are presented. These results can be used in the various tasks of the statistical decision theory, particularly in the dual control theory.

1 Introduction

Integrals related to the probability distributions are the part of the statistical decision theory. One example of using of the statistical decision theory is the dual control [4], [2]. Some integrals related to the vector Gaussian distribution are developed in the paper [2]. More complicated problems in the framework of the statistical decision theory require generalizations of the results of the paper [2] in various directions. In this paper, such generalizations for the multidimensional-matrix Gaussian distribution are developed. The three integrals, total probability formula and Bayes' formula related to the multidimensional-matrix Gaussian distribution are presented.

2 The integrals related to the multidimensional-matrix Gaussian distribution

The random q-dimensional matrix $\xi = (\xi_{\bar{i}_q})$, $\bar{i}_q = (i_1, i_2, , ..., i_q)$, $i_\alpha = 1, 2, ..., m_\alpha$, $\alpha = 1, 2, ..., q$, is distributed according to the normal or Gaussian law if its probability density is defined by the following expression [3]:

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{r_q}|d_\xi|}} exp\left(-\frac{1}{2}\,{}^{0,2q}\left(d_\xi^{-1}(\xi - \nu_\xi)^2\right)\right), \ \xi \in E^{r_q}, \tag{1}$$

where: ν_ξ, d_ξ are the parameters of the Gaussian multidimensional-matrix distribution, herewith $\nu_\xi = (\nu_{\xi,\bar{i}_q})$, $\bar{i}_q = (i_1, i_2, , ..., i_q)$, $i_\alpha = 1, 2, ..., m_\alpha$, $\alpha = 1, 2, ..., q$, is the mathematical expectation of the random q-dimensional matrix ξ and $d_\xi = (d_{\xi,\bar{i}_q,\bar{j}_q})$, $\bar{i}_q = (i_1, i_2, , ..., i_q)$, $\bar{j}_q = (j_1, j_2, , ..., j_q)$, $i_\alpha, j_\alpha = 1, 2, ..., m_\alpha$, $\alpha = 1, 2, ..., q$, is the dispersion matrix of the random q-dimensional matrix ξ; d_ξ^{-1} is the matrix $(0, q)$-inverse to the matrix d_ξ; $|d_\xi|$ is the determinant of the matrix d_ξ; $r_q = \prod_{i=1}^{q} m_i$ is the number of the elements of the matrix ξ ; E^{r_q} is the r_q-dimensional Euclidean space; $\bar{i}_q = (i_1, i_2, , ..., i_q)$, $\bar{j}_q = (j_1, j_2, , ..., j_q)$ are the multi-indexes either of which contains q indexes.

The mathematical expectation of the random q-dimensional matrix ξ is the q-dimensional matrix with the same size as the matrix ξ. It is defined by the expression

$$\nu_\xi = E(\xi) = \left(E(\xi_{\bar{i}_q})\right) = (\nu_{\xi,\bar{i}_q}),\ \bar{i}_q = (i_1, i_2, ..., i_q),\ i_\alpha = 1, 2, ..., m_\alpha,\ \alpha = 1, 2, ..., q,$$

so that $\nu_{\xi,\bar{i}_q} = E(\xi_{\bar{i}_q})$, E is the symbol of the mathematical expectation [3].

The dispersion matrix d_ξ of the random q-dimensional matrix ξ is the $2q$-dimensional matrix defined by the expression

$$d_\xi = E\left((\xi - \nu_\xi)^2\right) = \left(E\left((\xi_{\bar{i}_q} - \nu_{\xi,\bar{i}_q})(\xi_{\bar{j}_q} - \nu_{\xi,\bar{j}_q})\right)\right) = (d_{\xi,\bar{i}_q,\bar{j}_q}),$$

$$\bar{i}_q = (i_1, i_2, ..., i_q),\ \bar{j}_q = (j_1, j_2, ..., j_q), i_\alpha, j_\alpha = 1, 2, ..., m_\alpha,\ \alpha = 1, 2, ..., q,$$

so that $d_{\xi,\bar{i}_q,\bar{j}_q} = E\left((\xi_{\bar{i}_q} - \nu_{\xi,\bar{i}_q})(\xi_{\bar{j}_q} - \nu_{\xi,\bar{j}_q})\right)$, E is the symbol of the mathematical expectation, and $(\xi - \nu_\xi)^2 = \left((\xi_{\bar{i}_q} - \nu_{\xi,\bar{i}_q})(\xi_{\bar{j}_q} - \nu_{\xi,\bar{j}_q})\right)$ is the $(0, 0)$-rolled square of the matrix ξ [3].

The determinant $|d_\xi|$ of the matrix d_ξ is defined as the determinant of the two-dimensional matrix $\tilde{d}_{\xi,q,0,q}$ that is the $(q, 0, q)$-associated matrix with the $2q$-dimensional matrix d_ξ [3].

We have proved the following equalities connected with the function (1):

$$\int\limits_{E^{r_q}} exp\left(-\frac{1}{2}\,{}^{0,2q}(A\xi^2) + {}^{0,q}(B\xi)\right) d\xi = \sqrt{(2\pi)^{r_q}|A^{-1}|}\, exp\left(\frac{1}{2}\,{}^{0,2q}(A^{-1}B^2)\right),$$

$$\int\limits_{E^{r_q}} {}^{0,q}(C\xi) exp\left(-\frac{1}{2}\,{}^{0,2q}(A\xi^2) + {}^{0,q}(B\xi)\right) d\xi =$$

$$= \sqrt{(2\pi)^{r_q}|A^{-1}|}\, exp\left(\frac{1}{2}\,{}^{0,2q}(A^{-1}B^2)\right)\ {}^{0,q}\left(C\,{}^{0,q}(A^{-1}B)\right),$$

$$\int\limits_{E^{r_q}} {}^{0,2q}(U\xi^2) exp\left(-\frac{1}{2}\,{}^{0,2q}(A\xi^2) + {}^{0,q}(B\xi)\right) d\xi =$$

$$= \sqrt{(2\pi)^{r_q}|A^{-1}|}\, exp\left(\frac{1}{2}\,{}^{0,2q}(A^{-1}B^2)\right)\ {}^{0,2q}\left(U\left(A^{-1} + {}^{0,0}({}^{0,q}(A^{-1}B))^2\right)\right),$$

where: $\xi = (\xi_{\bar{i}_q})$, $\bar{i}_q = (i_1, i_2, , ..., i_q)$, is the q-dimensional $(m_1 \times m_2 \times ... \times m_q)$-matrix; $r_q = \prod\limits_{i=1}^{q} m_i$ is the number of the elements of the matrix ξ; $B = (b_{\bar{i}_q})$, $C = (c_{\bar{i}_q})$ are the q-dimensional $(m_1 \times m_2 \times ... \times m_q)$-matrices of the parameters; $A = (a_{\bar{i}_q,\bar{j}_q})$, $U = (u_{\bar{i}_q,\bar{j}_q})$ are the $2q$-dimensional $(m_1 \times m_2 \times ... \times m_q \times m_1 \times m_2 \times ... \times m_q)$-matrices of the parameters that are positive-definite and symmetric relative their q-multi-indexes $\bar{i}_q$, $\bar{j}_q$; A^{-1} is the matrix $(0, q)$-inverse to the matrix A; $|A^{-1}|$ is the determinant of the matrix A^{-1}.

3 The total probability formula for the multidimensional-matrix Gaussian distributions

Theorem 1 *(the total probability formula for multidimensional-matrix Gaussian distributions). Let* ξ *is the* q*-dimensional* $(m_1 \times m_2 \times ... \times m_q)$*-matrix,* x *is the* p*-dimensional* $(s_1 \times s_2 \times ... \times s_p)$*-matrix,* $r_q = \prod_{i=1}^{q} m_i$ *is the numbers of the elements of the matrix* ξ*,* $r_p = \prod_{i=1}^{p} s_i$ *is the numbers of the elements of the matrix* x*,* $f(\xi)$ *is the probability density of the matrix* ξ*,* $f(x/\xi)$ *is the conditional probability density of the matrix* x*,* R^{r_q} *is the* r_q*-dimensional Euclidean space. If in the total probability formula*

$$f(x) = \int\limits_{E^{r_q}} f(x/\xi) f(\xi) d\xi \qquad (2)$$

the probability density $f(x/\xi)$ *is represented in the form*

$$f(x/\xi) = \frac{1}{\sqrt{(2\pi)^{r_p}|d_x|}} exp\left(-\frac{1}{2}\,{}^{0,2q}(S\xi^2) + {}^{0,q}(V\xi) - \frac{1}{2}W\right),$$

and probability density $f(\xi)$ *is represented in the form*

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{r_q}|d_\xi|}} exp\left(-\frac{1}{2}\,{}^{0,2q}(d_\xi^{-1}\xi^2) + {}^{0,q}\left({}^{0,q}(d_\xi^{-1}\nu_\xi)\xi\right) - \frac{1}{2}\,{}^{0,2q}(d_\xi^{-1}\nu_\xi^2)\right),$$

then the probability density $f(x)$ *(2) (total probability formula) determine by the expression*

$$f(x) = \frac{1}{\sqrt{(2\pi)^{r_p}|d_x||d_\xi||A|}} exp\left(-\frac{1}{2}\,{}^{0,2q}(A^{-1}B^2) - \frac{1}{2}C\right),$$

where

$$A = d_\xi^{-1} + S,$$

$$B = {}^{0,q}(d_\xi^{-1}\nu_\xi) + V,$$

$$C = {}^{0,2q}(d_\xi^{-1}\nu_\xi^2) + W,$$

$|d_x|$, $|d_\xi|$, $|A|$ *are the determinants of the corresponding multidimensional matrices, and* A^{-1} *is the matrix* $(0, q)$*-inverse to the matrix* A*.*

We denote that the matrices d_ξ, d_ξ^{-1}, S, A, A^{-1} are $2q$-dimensional symmetrical relative their q-multi-indexes, ν_ξ, V, B are q-dimensional, d_x is the $2p$-dimensional symmetrical relative its p-multi-indexes, W, C are zero-dimensional (scalars).

4 The Bayes' formula for the multidimensional-matrix Gaussian distributions

Theorem 2 *(Bayes' formula for multidimensional-matrix Gaussian distributions). Let* ξ *is the q-dimensional* $(m_1 \times m_2 \times ... \times m_q)$*-matrix,* x *is the p-dimensional* $(s_1 \times s_2 \times ... \times s_p)$*-matrix,* $r_q = \prod_{i=1}^{q} m_i$ *is the numbers of the elements of the matrix* ξ*,* $r_p = \prod_{i=1}^{p} s_i$ *is the numbers of the elements of the matrix* x*,* $f(\xi)$ *is the probability density of the matrix* ξ*,* $f(x/\xi)$ *is the conditional probability density of the matrix* x*,* R^{r_q} *is the* r_q*-dimensional Euclidean space. If in the Bayes' formula*

$$f(\xi/x) = \frac{f(x/\xi)f(\xi)}{\int\limits_{E^{r_q}} f(x/\xi)f(\xi)d\xi} \tag{3}$$

the probability density $f(x/\xi)$ *is represented in the form*

$$f(x/\xi) = \frac{1}{\sqrt{(2\pi)^{r_p}|d_x|}} exp\left(-\frac{1}{2}\,{}^{0,2q}(S\xi^2) + {}^{0,q}(V\xi) - \frac{1}{2}W\right),$$

and the probability density $f(\xi)$ *is represented in the form*

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{r_q}|d_\xi|}} exp\left(-\frac{1}{2}\,{}^{0,2q}(d_\xi^{-1}\xi^2) + {}^{0,q}\left({}^{0,q}(d_\xi^{-1}\nu_\xi)\xi\right) - \frac{1}{2}\,{}^{0,2q}(d_\xi^{-1}\nu_\xi^2)\right),$$

then the posteriori probability density $f(\xi/x)$ *of the random vector* ξ *determined by the Bayes' formula (3), haves the following form*

$$f(\xi/x) = \frac{1}{\sqrt{(2\pi)^{r_q}|A^{-1}|}} exp\left(-\frac{1}{2}\,{}^{0,2q}\left(A\left(\xi - {}^{0,q}(A^{-1}B)\right)^2\right)\right),$$

where $A = d_\xi^{-1} + S$, $B = {}^{0,q}(d_\xi^{-1}\nu_\xi) + V$.

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