ON BI-DECOMPOSITION OF INCOMPLETELY SPECIFIED BOOLEAN FUNCTIONS

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A method for bi-decomposition of incompletely specified (partial) Boolean functions is suggested. The problem of bi-decomposition is reduced to the problem of a weighted two-block covering the edge set of the orthogonality graph of the truth table rows by complete bipartite subgraphs (bicliques). Every biclique is assigned with a set of variables in a definite way. The weight of a biclique is the size of that set. According to each biclique of the obtained cover, a Boolean function is constructed whose arguments are the variables from the assigned set

Introduction

The bi-decomposition problem is considered here in the following statement. Given a partial (incompletely specified) Boolean function $f(\boldsymbol{x})$ where the components of the vector $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ are Boolean variables forming a set X. A superposition $\phi(g_1(\boldsymbol{z}_1), g_2(\boldsymbol{z}_2)) \geq f(\boldsymbol{x})$ must be found where the components of the vectors z_1 and z_2 are the variables in the sets $Z_1 \subset X$ and $Z_2 \subset X$, respectively, and $\geq$ denotes the realization relation, i.e. the values of the function ϕ coincide with the values of the values of f everywhere they are specified. The type of the function ϕ of two variables is given, as well. It can be any of ten Boolean functions depending essentially on two variables. Usually, the sets Z_1 and Z_2 are given and $Z_1 \cap Z_2 = \emptyset$ and such bi-decomposition is called disjoint in contrast to non-disjoint bi-decomposition when $Z_1 \cap Z_2 = \emptyset$ condition is not obligatory. At that, the cardinalities of Z_1 and Z_2 must be less than n.

There are examples of application of bi-decomposition for increasing the performance of a logic circuit [1, 2] and for synthesis of circuits on the base of field programmable gate array (FPGA) [3]. The problem of bi-decomposition with the output function ϕ expressed by XOR operation is considered in [4] that suggests to use logical equations to solve the problem In [5], the disjoint bi-decomposition with a given partition (Z_1, Z_2) is investigated in detail. An approach to solving this problem using Boolean differential calculus is suggested in [6]. Here, the approach to solving the general problem of parallel decomposition for partial Boolean function described in [7] is suggested to solve the bi-decomposition problem.

I. The approach to the problem

The approach uses the specification of a partial Boolean function in truth table that is considered as a pair of binary matrices: matrix $\boldsymbol{X}$ of dimension $l \times n$ and one-column matrix $\boldsymbol{F}$ with l elements. The rows of the matrix $\boldsymbol{X}$ give the values of the arguments $x_1, x_2, ..., x_n$ and the elements of the matrix $\boldsymbol{F}$ the corresponding values of the function. The value of the function is not specified at the set of argument values if this set does not exist among the rows of $\boldsymbol{X}$. The rows of $\boldsymbol{X}$ and $\boldsymbol{F}$ have common numeration.

The graphs $G_X = (V, E_X)$ and $G_F = (V, E_F)$ are considered where V is the set of common numbers of rows of the matrices $\boldsymbol{X}$ and $\boldsymbol{F}$, and E_X and E_F are the sets of pairs of orthogonal rows of the matrices $\boldsymbol{X}$ and $\boldsymbol{F}$, respectively. Two row-vectors are orthogonal if there is a component being 0 in one vector and 1 in the other. The approach from [7] is intended for a system of partial Boolean functions given by a pair of ternary matrices, and the graphs G_X and G_F can have an arbitrary type. In the case of binary matrices, G_X is complete graph and G_F a complete bipartite graph. A system of Boolean functions is given with matrices $\boldsymbol{X}$ and $\boldsymbol{F}$ correctly if $E_F \subseteq E_X$, i.e. G_F is a spanned subgraph of G_X. Every edge in E_X is assigned with the variables from the set $X = \{x_1, x_2, ..., x_n\}$, according to which the corresponding rows of $\boldsymbol{X}$ are orthogonal. A complete bipartite subgraph (*biclique*) of G_X is assigned with the set of variables in X taken one by one from each edge of the biclique. A biclique is called *admissible* if the number of variables assigned to it is less than n, and it contains at least one edge in E_F.

S t a t e m e n t. There exists a superposition $\phi(g_1(\boldsymbol{z}_1), g_2(\boldsymbol{z}_2))$ realizing the given partial Boolean function $f(\boldsymbol{x})$ if a cover of the set E_F by two admissible bicliques of the graph G_X exists.

Let B_1, B_2 be bicliques covering the set E_F. Any biclique B_i can be given by a pair of vertex sets $<V_i', V_i''>$. Every function $g_i(z_i)$ of the required superposition is specified by matrices $\boldsymbol{X}_i$ and $\boldsymbol{F}_i$. The matrix $\boldsymbol{X}_i$ is the minor of $\boldsymbol{X}$ formed by the columns corresponding to the variables assigned to the biclique B_i. The matrix $\boldsymbol{F}_i$ consists of one column where the element with number corresponding to the vertex in V_i' is 0, and the element with number corresponding to the vertex in V_i'' is 1 (or vice versa). The function ϕ is given by matrices $\boldsymbol{U}$ and $\boldsymbol{\Phi}$. The matrix $\boldsymbol{U}$ consists of the columns that are the one-column matrices $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and the matrix $\boldsymbol{\Phi}$ coincide with $\boldsymbol{F}$.

II. A method for bi-decomposition

To give the type of the function ϕ one must assign to each pair of rows of matrices $\boldsymbol{X}$ and $\boldsymbol{F}$ the set of the pairs of the possible values of g_1 and g_2 required according to the value of the given f. For example, when $\phi = g_1 \wedge g_2$ (ϕ is a conjunction), this set is $\{(1,\ 1)\}$ if $f = 1$, and $\{(0,\ 0),(0,\ 1),(1,\ 0)\}$, if $f = 0$.

Let bicliques B_1 and B_2 constitute a two-block cover of the set E_F, according to which functions g_1, g_2 and ϕ can be constructed. Such covers have the following two properties.

P r o p e r t y A. A set of vertices of graph G_X (denote it W), each of which corresponds to equal values of functions g_1 and g_2, is a subset of partites in the bipartite subgraphs B_1 and B_2.

P r o p e r t y B. If vertices v_i and v_j from the set $V \setminus W$ are in different partites of one of the bicliques, then they are in different partites of the other one. If they are in the same partite of one of the bicliques, then they are in the same partite of the other one.

For linear functions that are exclusive disjunction and equivalence, both the properties are present. For other functions, Property B is not obligatory. We call two-block cover of E_V by admissible bicliques of graph G_X admissible for non-linear function ϕ if it has Property A, and admissible for linear function ϕ if it has properties A and B.

The process of solution of the considered problem with minimizing the sum of the numbers of arguments of the functions g_1 and g_2 consists of the following stages.

1. Finding all maximal admissible bicliques with Property A in graph G_X. The method suggested in [8] and described later in [9] can be used for that. Note that graph G_F in the problem is a biclique of G_X that is not admissible because it is an one-block cover resulting in trivial solution where one of the functions g_1 or g_2 is a constant. The obtained bicliques are assigned with weights equal to the numbers of variables assigned to the bicliques.

2. Obtaining a two-block cover of the set E_F by the found bicliques. The cover must have minimal weight and, for a linear function ϕ, have Property B. The weight of a cover is the sum of the weights of bicliques that constitute the cover. At this stage, the requirement for non-crossing sets Z_1 and Z_2 can be satisfied when a disjoint decomosition is demanded.

3. Constructing Boolean functions $g_1(\boldsymbol{z}_1), g_2(\boldsymbol{z}_2)$ and ϕ. The functions $g_1(\boldsymbol{z}_1)$ and $g_2(\boldsymbol{z}_2)$ are specified by the pairs of matrices $\boldsymbol{X}_1$, $\boldsymbol{F}_1$ and $\boldsymbol{X}_2$, $\boldsymbol{F}_2$, and the function ϕ by a formula with the given operation of logic algebra.

The "bottle-neck"of the described approach is the search for all the maximum bicliques in a graph. The number of them in a complete graph with n vertices is $2^{n-1}-1$. Evidently, graph G_X is complete if all the rows of matrix $\boldsymbol{X}$ are mutually orthogonal. This takes place always when $\boldsymbol{X}$ is a Boolean matrix. Property A gives a possibility to decrease the number of the considered bicliques significantly with the help of replacing all the vertices of the set W by one vertex by contraction. The number of maximum bicliques in such a graph is $2^{n-|W|} - 1$ where $|W|$ is the size of W. It should be noted although that the complexity of the problem of the search for all the two-block covers is expressed by a polynomial of the second power. So, enumeration of all the two-block covers is not considered as laborious problem.

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