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# INTEGRALS AND INTEGRAL TRANSFORMATIONS CONNECTED WITH VECTOR GAUSSIAN DISTRIBUTION

**Abstract.** The paper is devoted to the integrals and integral transformations related to the multivariate Gaussian probability density function. Such integrals and integral transformations arise in probability applications. In the paper, three integrals are presented which allow calculation the moments of the multivariate Gaussian distribution. Besides, the total probability formula and Bayes formula for vector Gaussian distributions are given. The proofs of the obtained results are given. The proof of the integrals is performed on the base of Gauss elimination method. The total probability formula and Bayes formula are obtained on the base of the proved integrals. These integrals and integral transformations could be used, for instant, in the statistical decision theory, particularly, in the dual control theory, and as the table integrals in various areas of research. On the basis of the obtained results, Bayesian estimations of the coefficients of the multiple regression function are calculated.

**Keywords:** vector Gaussian distribution, multidimensional integrals, total probability formula, Bayes formula, multiple regression function, Bayesian estimations

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# ИНТЕГРАЛЫ И ИНТЕГРАЛЬНЫЕ ПРЕОБРАЗОВАНИЯ, СВЯЗАННЫЕ С ВЕКТОРНЫМ ГАУССОВСКИМ РАСПРЕДЕЛЕНИЕМ

**Аннотация**. Рассматриваются интегралы и интегральные преобразования, относящиеся к функции плотности вероятности векторного гауссовского распределения. Такие интегралы и интегральные преобразования возникают в вероятностных приложениях. В статье представлены три интеграла, позволяющие рассчитывать моменты векторного гауссовского распределения, а также формулы полной вероятности и Байеса. Приводятся доказательства полученных результатов. Вывод интегралов выполнен на основе метода исключения Гаусса. Формулы полной вероятности и Байеса получены на основе доказанных интегралов. Представленные интегралы и интегральные преобразования могут быть использованы в различных вероятностных приложениях, например, в теории статистических решений, в частности, в теории дуального управления, а также как табличные интегралы в различных областях исследований. На основе полученных результатов рассчитаны байесовские оценки коэффициентов множественной функции регрессии.

**Ключевые слова**: векторное гауссовское распределение, многомерные интегралы, формула полной вероятности, формула Байеса, множественная функция регрессии, байесовские оценки

**Introduction.** The integrals and integral transformations connected with the probability distributions use in many applications, one of them is statistical decision theory. The statistical decision theory attracts attention due to the ability to formulate the problems in a strict mathematical form. One of technical problems solved by the statistical decision theory is the problem of dual control [1] that requires calculation of integrals connected with the multivariate probability distributions. In this paper, we present three integrals connected with the vector Gaussian distribution, total probability formula and Bayes formula for the vector Gaussian distributions.

**1. Integrals connected with the vector Gaussian distribution.** The random vector with $k$ components $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_k)$ is distributed according to the normal or Gaussian law if its probability density has the form

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^k \, |d_\Xi|}} \exp\left( -\frac{1}{2} (\xi - \nu_\Xi)^T d_\Xi^{-1} (\xi - \nu_\Xi) \right), \ \xi \in E^k, \tag{1}$$

where $\xi^T = (\xi_1, \xi_2, ..., \xi_k)$ is vector-row of the arguments of the probability density $f(\xi)$, $\nu_\Xi^T = (\nu_{\Xi,1}, \nu_{\Xi,2}, ..., \nu_{\Xi,k})$ is the vector-row of the parameters of the probability density $f(\xi)$, $d_\Xi = (d_{\Xi,i,j})$, $i, j = \overline{1,k}$, is the symmetric positive-definite matrix of the parameters of the probability density $f(\xi)$, $d_\Xi^{-1}$ is the matrix inverse to the matrix $d_\Xi$, $|d_\Xi|$ is the determinant of the matrix $d_\Xi$, $E^k$ is the $k$-dimensional Euclidean space, symbol $T$ means transpose. The parameters $\nu_\Xi$ and $d_\Xi$ of the distribution (1) are mathematical expectation and dispersion (variance-covariance) matrix of the random vector $\Xi$ respectively [2].

The following equalities connected with the function (1) hold:

$$\int_{E^k} \exp\left( -\frac{1}{2} \xi^T A \xi + B^T \xi \right) d\xi = \sqrt{(2\pi)^k \, |A^{-1}|} \exp\left( \frac{1}{2} B^T A^{-1} B \right), \tag{2}$$

$$\int_{E^k} C^T \xi \exp\left( -\frac{1}{2} \xi^T A \xi + B^T \xi \right) d\xi = \sqrt{(2\pi)^k \, |A^{-1}|} \exp\left( \frac{1}{2} B^T A^{-1} B \right) C^T A^{-1} B, \tag{3}$$

$$\int_{E^k} \xi^T U \xi \exp\left( -\frac{1}{2} \xi^T A \xi + B^T \xi \right) d\xi = \sqrt{(2\pi)^k \, |A^{-1}|} \left( Tr(A^{-1}U) + B^T A^{-1} U A^{-1} B \right) \exp\left( \frac{1}{2} B^T A^{-1} B \right), \tag{4}$$

where $A = (a_{i,j})$, $i, j = \overline{1,k}$, is the symmetric positive-definite matrix, $\xi^T = (\xi_1, \xi_2, ..., \xi_k)$, $B^T = (b_1, b_2, ..., b_k)$, $C^T = (c_1, c_2, ..., c_k)$ are the vector-rows, $A^{-1}$ is the matrix inverse to the matrix $A$, $|A^{-1}|$ is the determinant of the matrix $A^{-1}$, $U = (u_{i,j})$, $i, j = \overline{1,k}$, is the symmetric positive-definite matrix, $Tr(A^{-1}U)$ is the trace of the matrix $A^{-1}U$.

Integrals (2), (3), (4) were received in [3], but proof was given only for the equality (2). Now we give more detailed proof of the equality (2) and prove the equalities (3), (4).

P r o o f  o f  t h e  e q u a l i t y  (2). We will use the Gauss elimination method [4] that lets bring the matrix $A$ to the diagonal form and reduce the calculation of the multiple integral to the calculation of the repeated integral. The conditions for use the Gauss elimination method in this case hold, because the matrix $A$ is positive-definite.

We denote $F(\xi)$ the integrand function in (2):

$$F(\xi) = \exp\left( -\frac{1}{2} \xi^T A \xi + B^T \xi \right). \tag{5}$$

By using the Gauss elimination method [4] to the matrix $A = (a_{i,j})$ in (5), we receive the upper triangular matrix

$$G=\begin{pmatrix} a_{1,1}^{(0)} & a_{1,2}^{(0)} & \cdots & a_{1,k}^{(0)} \\ 0 & a_{2,2}^{(1)} & \cdots & a_{2,k}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{k,k}^{(k-1)} \end{pmatrix}=(a_{i,j}^{(i-1)}),\ i,j=\overline{1,k}\ .$$

Determinant of the upper triangular matrix $G$ is equal to the product of its diagonal elements and is equal to the determinant of the matrix $A$ [4]:

$$|G|=a_{1,1}^{(0)}a_{2,2}^{(1)}\cdots a_{k,k}^{(k-1)}=|A|=\prod_{i=1}^{k}a_{i,i}^{(i-1)}\ .$$

The mine minors of the matrices $A$ and $G$ is equal, $A_m=G_m$, $m=1,2,...,k$ [4], where $A_m=\begin{pmatrix}1,2,...,m\\1,2,...,m\end{pmatrix}$ is the mine minor of the order $m$ of the matrix $A$ and $G_m=\begin{pmatrix}1,2,...,m\\1,2,...,m\end{pmatrix}$ is the mine minor of the order $m$ of the matrix $G$. Moreover [4],

$$a_{1,1}^{(0)}=A_1,\ a_{2,2}^{(1)}=\frac{A_2}{A_1},\ a_{3,3}^{(2)}=\frac{A_3}{A_2},\dots,\ a_{k,k}^{(k-1)}=\frac{A_k}{A_{k-1}}\ . \tag{6}$$

The matrix $A$, as shown in [4], can be represented in the form

$$A=G_l^T\widehat{D}_1G\ , \tag{7}$$

where $G$ and $G_l$ are matrices received by the Gauss elimination method from the matrices $A$ and $A^T$ respectively, and $\widehat{D}_1$ is diagonal matrix of the form

$$\widehat{D}_1=diag\left\{\frac{1}{A_1},\frac{A_1}{A_2},\frac{A_2}{A_3},...,\frac{A_{k-1}}{A_k}\right\}.$$

Since the matrix $A$ is symmetrical, i. e. $A^T=A$, then $G_l=G$, and the equation (7) can be written in the form:

$$A=G^T\widehat{D}_1G\ .$$

Provided the expressions (6) we receive that the matrix $A$ can be represented in the following form:

$$A=G^T\widehat{D}G\ , \tag{8}$$

where $\widehat{D}$ is diagonal matrix of the form

$$\widehat{D}=diag\{(a_{1,1}^{(0)})^{-1},(a_{2,2}^{(1)})^{-1},\cdots,(a_{k,k}^{(k-1)})^{-1}\}\ . \tag{9}$$

From (8) it follows that

$$\widehat{D}^{-1}=GA^{-1}G^T\ . \tag{10}$$

We denote $\widehat{d}_{i,i}=(a_{i,i}^{(i-1)})^{-1}$, $i=\overline{1,k}$, the diagonal elements of the diagonal matrix $\widehat{D}$ (9) as  Obviously, the following equalities hold:

$$|\widehat{D}^{-1}|=|G|=|A|=\prod_{i=1}^{k}a_{i,i}^{(i-1)}\ . \tag{11}$$

Let us introduce in (5) the linear replacement of the variables

$$\xi = G^{-1}z, \tag{12}$$

with which the integrand function (5) is converted to the following function of the argument $z$:

$$F(z) = \exp\left(-\frac{1}{2}z^T Pz + Dz\right),$$

where

$$P = (G^{-1})^T AG^{-1},$$

$$D = B^T G^{-1} = (d_i),\ i = \overline{1,k}. \tag{13}$$

Since, provided (8),

$$P = (G^{-1})^T AG^{-1} = (G^{-1})^T G^T \widehat{D} GG^{-1} = \widehat{D},$$

then we have the following function of the argument $z$:

$$F(z) = \exp\left(-\frac{1}{2}z^T \widehat{D}z + Dz\right). \tag{14}$$

The following equality is holds when the replacing of the variables in the integral [5]:

$$\int_{E^k} F(\xi)d\xi = \int_{E^k} F(z)\,|\,J\,|\,dz, \tag{15}$$

where $|\,J\,|$ is absolute value of the Jacobian of the transformation (12):

$$|\,J\,| = |\,G^{-1}\,| = |\,\widehat{D}\,| = \prod_{i=1}^{k}(a_{i,i}^{(i-1)})^{-1}.$$

Let us rewrite the function $F(z)$ (14) as a function of the elements of the matrices $\widehat{D}$ and $D$, taking into account the notations above:

$$F(z) = \exp\left(-\frac{1}{2}z^T \widehat{D}z + Dz\right) = \exp\left(-\frac{1}{2}\sum_{i=1}^{k}(a_{i,i}^{(i-1)})^{-1}z_i^2 + \sum_{i=1}^{k}d_i z_i\right) = \prod_{i=1}^{k}\exp\left(-\frac{1}{2}(a_{i,i}^{(i-1)})^{-1}z_i^2 + d_i z_i\right). \tag{16}$$

Substituting (16) into (15) gives the following equality:

$$\int_{E^k} F(\xi)d\xi = |\,J\,|\prod_{i=1}^{k}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(a_{i,i}^{(i-1)})^{-1}z_i^2 + d_i z_i\right)dz_i. \tag{17}$$

The integral in the right part of the expression (17) is the table integral of the following form [6]:

$$\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\alpha x^2 + \beta x\right)dx = \sqrt{\frac{2\pi}{\alpha}}\exp\left(\frac{\beta^2}{2\alpha}\right). \tag{18}$$

If we take into account the integral (18), we will receive instead of (17):

$$\int_{E^k} F(\xi)d\xi = |\,J\,|\prod_{i=1}^{k}\sqrt{2\pi a_{i,i}^{(i-1)}}\exp\left(-\frac{1}{2}\sum_{i=1}^{k}a_{i,i}^{(i-1)}d_i^2\right),$$

or in matrix form

$$\int_{E^k} F(\xi)d\xi = \sqrt{(2\pi)^k\,|\,\widehat{D}\,|}\exp\left(-\frac{1}{2}D\widehat{D}^{-1}D^T\right). \tag{19}$$

Let us turn back from matrices $D$, $\hat{D}$ to matrices $A$, $B$. Since $|\hat{D}|=|A^{-1}|$ (formula (11)), $\hat{D}^{-1}=GA^{-1}G^T$ (formula (10)), $D=B^TG^{-1}$ (formula (13)), then

$$D\hat{D}^{-1}D^T=B^TG^{-1}GA^{-1}G^T(G^{-1})^TB=B^TA^{-1}B,$$

and instead (19) we will receive (2). Equality (2) is proved.

P r o o f  o f  t h e  e q u a l i t y  ( 3 ). We will calculate the integral

$$I_1=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}C^T\xi\exp\left(-\frac{1}{2}\xi^TA\xi+B^T\xi\right)d\xi. \qquad (20)$$

Let us complete the square of the expression $-\xi^TA\xi/2+B^T\xi$ in the integral (20). We will receive the expression

$$-\frac{1}{2}\xi^TA\xi+B^T\xi=-\frac{1}{2}(\xi-A^{-1}B)^TA(\xi-A^{-1}B)+\frac{1}{2}B^TA^{-1}B, \qquad (21)$$

and instead of the integral (20) we will receive the following integral:

$$I_1=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}C^T\xi\exp\left(-\frac{1}{2}(\xi-A^{-1}B)^TA(\xi-A^{-1}B)\right)\exp\left(\frac{1}{2}B^TA^{-1}B)\right)d\xi. \qquad (22)$$

Since the function

$$f(\xi)=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\exp\left(-\frac{1}{2}(\xi-A^{-1}B)^TA(\xi-A^{-1}B)\right) \qquad (23)$$

in (22) is probability density of the vector Gaussian distribution with mean value $A^{-1}B$ and dispersion matrix $A^{-1}$, then integral $I_1$ (22), (20) is equal

$$I_1=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}C^T\xi\exp\left(-\frac{1}{2}\xi^TA\xi+B^T\xi\right)d\xi=\exp\left(\frac{1}{2}B^TA^{-1}B)\right)C^TA^{-1}B.$$

As a result, we have the equality (3).

P r o o f  o f  t h e  e q u a l i t y  ( 4 ). We will perform it in a similar way to the proof of the equality (3). We will calculate the integral

$$I_2=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}\xi^TU\xi\exp\left(-\frac{1}{2}\xi^TA\xi+B^T\xi\right)d\xi. \qquad (24)$$

Completing the square of the expression $-\xi^TA\xi/2+B^T\xi$ in the integral (24) gives expression (21) and gives instead of the integral (21) the following integral:

$$I_2=\frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}\xi^TU\xi\exp\left(-\frac{1}{2}(\xi-A^{-1}B)^TA(\xi-A^{-1}B)\right)\exp\left(\frac{1}{2}B^TA^{-1}B)\right)d\xi. \qquad (25)$$

Since the function in (25) of the form (23) is probability density of the vector Gaussian distribution with mean value $A^{-1}B$ and dispersion matrix $A^{-1}$, then the value of the integral $I_2$ (25), (24) is determined by the expression

$$I_2=E(\Xi^TU\Xi)\exp\left(\frac{1}{2}B^TA^{-1}B)\right), \qquad (26)$$

where $E(\Xi^T U\Xi)$ is the mathematical expectation of the quadratic form $\Xi^T U\Xi$ of the random vector $\Xi$ with the Gaussian distribution (23). It is known the equality [2]

$$\Xi^T U\Xi = tr(U\Xi\Xi^T)\,. \tag{27}$$

Then

$$E(\Xi^T U\Xi) = E(tr(U\Xi\Xi^T)) = tr(UE(\Xi\Xi^T))\,.$$

Further, since for the Gaussian distribution (23) $E(\Xi\Xi^T) = A^{-1} + B^T A^{-1}A^{-1}B$, then

$$E(\Xi^T U\Xi) = tr(U(A^{-1} + B^T A^{-1}A^{-1}B)) = tr(UA^{-1}) + tr(UB^T A^{-1}A^{-1}B)\,.$$

If one takes into account the equality of the type (27) leading to the equality $tr(UB^T A^{-1}A^{-1}B) = B^T A^{-1}UA^{-1}B$, then one receives

$$E(\Xi^T U\Xi) = tr(UA^{-1}) + B^T A^{-1}UA^{-1}B\,,$$

and the value of the integral $I_2$ (26), (25), (24):

$$I_2 = \frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\int_{E^k}\xi^T U\xi\exp\left(-\frac{1}{2}\xi^T A\xi + B^T\xi\right)d\xi = (tr(UA^{-1}) + B^T A^{-1}UA^{-1}B)\exp\left(\frac{1}{2}B^T A^{-1}B)\right).$$

As the result we have received the equality (4).

Let us note that the equality (4) is more general then the according equality of the work [3], since the matrix $U$ in the work [3] is supposed diagonal.

**2. The total probability formula for vector Gaussian distributions.**

T h e o r e m 1 (The total probability formula for vector Gaussian distributions). *Let* $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_k)$ *be a row random vector with* $k$ *components,* $X^T = (X_1, X_2, ..., X_n)$ *be a row random vector with* $n$ *components,* $f(\xi)$ *be the probability density of the vector* $\Xi$, $f(x/\xi)$ *be a condition probability density of the vector* $X$, $E^k$ *be the* $k$*-dimensional Euclidean space. If in the total probability formula*

$$f(x) = \int_{E^k} f(x/\xi)\,f(\xi)d\xi \tag{28}$$

*the probability density* $f(x/\xi)$ *is represented in the form*

$$f(x/\xi) = \frac{1}{\sqrt{(2\pi)^n\,|d_X|}}\exp\left(-\frac{1}{2}\xi^T S\xi + V^T\xi - \frac{1}{2}W\right),$$

*and probability density* $f(\xi)$ *is represented in the form*

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^k\,|d_\Xi|}}\exp\left(-\frac{1}{2}\xi^T d_\Xi^{-1}\xi + \nu_\Xi^T d_\Xi^{-1}\xi - \frac{1}{2}\nu_\Xi^T d_\Xi^{-1}\nu_\Xi\right),$$

*then the integral* (28) *(the total probability formula) is defined by the following expression*

$$f(x) = \int_{E^k} f(x/\xi)\,f(\xi)d\xi = \frac{1}{\sqrt{(2\pi)^n\,|d_\Xi A d_X|}}\exp\left(\frac{1}{2}B^T A^{-1}B - \frac{1}{2}C\right), \tag{29}$$

*where*

$$A = d_{\Xi}^{-1} + S\,, \tag{30}$$

$$B = d_{\Xi}^{-1}\nu_{\Xi} + V\,, \tag{31}$$

$$C = \nu_{\Xi}^{T} d_{\Xi}^{-1}\nu_{\Xi} + W\,. \tag{32}$$

P r o o f. Performing the multiplication under the integral in (28), we receive

$$f(x/\xi)\,f(\xi) = \frac{1}{\sqrt{(2\pi)^{n+k}\,|d_X\|\,d_{\Xi}|}}\exp\left(-\frac{1}{2}\xi^T A\xi + B^T\xi - \frac{1}{2}C\right), \tag{33}$$

where $A$, $B$, $C$ are defined by the formulas (30), (31), (32). Integration of the expression (33) using the equality (2) gives the expression (29). This completes the proof of the theorem 1.

**3. The Bayes formula for vector Gaussian distributions.**

T h e o r e m 2 (Bayes formula for vector Gaussian distributions). *Let* $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_k)$ *be a random vector-row with* $k$ *components,* $X^T = (X_1, X_2, ..., X_n)$ *be a random vector-row with* $n$ *components,* $f(\xi)$ *be the probability density of the vector* $\Xi$, $f(x/\xi)$ *be the condition probability density of the vector* $X$, $E^k$ *be the* $k$*-dimensional Euclidean space. If in the Bayes formula*

$$f(\xi/x) = \frac{f(\xi)\,f(x/\xi)}{\int\limits_{E^k} f(\xi)\,f(x/\xi)d\xi} \tag{34}$$

*the probability density* $f(x/\xi)$ *is represented in the form*

$$f(x/\xi) = \frac{1}{\sqrt{(2\pi)^n\,|d_X|}}\exp\left(-\frac{1}{2}\xi^T S\xi + V^T\xi - \frac{1}{2}W\right),$$

*and the probability density* $f(\xi)$ *is represented in the form*

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^k\,|d_{\Xi}|}}\exp\left(-\frac{1}{2}\xi^T d_{\Xi}^{-1}\xi + \nu_{\Xi}^T d_{\Xi}^{-1}\xi - \frac{1}{2}\nu_{\Xi}^T d_{\Xi}^{-1}\nu_{\Xi}\right),$$

*then the posteriori probability density* $f(\xi/x)$ *of the random vector* $\xi$ *defined by the Bayes formula* (34) *haves the following form*

$$f(\xi/x) = \frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\exp\left(-\frac{1}{2}(\xi - A^{-1}B)^T A(\xi - A^{-1}B)\right), \tag{35}$$

*where* $A = d_{\Xi}^{-1} + S$, $B = d_{\Xi}^{-1}\nu_{\Xi} + V$.

P r o o f. We note that this theorem is formulated under the same conditions and designations as the theorem 1. In this case, the numerator of the Bayes formula (34) is defined by the expressions (33), and the denominator is defined by the formula (29). Dividing (33) on (29) we receive the formula

$$f(\xi/x) = \frac{1}{\sqrt{(2\pi)^k\,|A^{-1}|}}\exp\left(-\frac{1}{2}\xi^T A\xi + B^T\xi - \frac{1}{2}B^T A^{-1}B\right), \tag{36}$$

which can be written in the form (35). The equality of the expression (36) and (35) is easy verified by performing the multiplying in the expression (35). This completes the proof of the theorem 2.

Obviously, that expression $A^{-1}B$ in the Bayes formula (34) is the posteriori mathematical expectation of the random vector $\Xi$, i.e. $A^{-1}B = E(\Xi / x)$, and the matrix $A^{-1}$ is the posteriori dispersion matrix of the random vector $\Xi$, i.e. $A^{-1} = E((\Xi - A^{-1}B)(\Xi - A^{-1}B)^T / x)$.

**4. Example.** As an example we consider the problem of calculation of the Bayesian estimations of the coefficients of the multiple regression function.

Let $U$ and $Y$ be the input and output vectors of the controlled object respectively and the object is described by the conditional probability density $f(\bar{y}/\bar{\theta},\bar{u})$. As a rule it is the Gaussian (normal) probability density:

$$f(\bar{y}/\bar{\theta},\bar{u})) \sim N(\varphi(\bar{\theta},\bar{u}), d_Y), \tag{37}$$

where $\varphi(\bar{\theta},\bar{u}) = \bar{y}$ is the regression function of $Y$ on $U$, $\bar{u}$ and $\bar{y}$ are input and output vectors of the regression function respectively, $\bar{\theta}$ is the vector of the coefficients of the regression function, $d_Y$ is the constant dispersion matrix of the internal noise of the object. The description (37) could be represented in the form

$$Y = \varphi(\bar{\theta},\bar{u}) + E,$$

where $E$ is the random vector with Gaussian distribution $N(0, d_E)$.

The multiple regression function is considered most often, when $\bar{y}$ is scalar (we will denote it $y$) and $\bar{u}$ is vector.

The class of the functions represented in the form

$$y = \varphi(\bar{\theta},\bar{u}) = \sum_{j=1}^{m} h_j(\bar{u})\theta_j = \bar{h}^T\bar{\theta},$$

where $h_j(\bar{u})$, $j = 1,2,...,m$, are some functions called basis functions, $\bar{h}^T = \bar{h}^T(\bar{u}) = (h_1(\bar{u}), h_2(\bar{u}),..., h_m(\bar{u}))$ is the vector of the basis functions, $\bar{\theta}^T = (\theta_1, \theta_2,..., \theta_m)$ is the vector of the coefficients of the regression function, is usually used to describe of the multiple regression function. For example, if we want to write in the vector form the function of the two variables $u_1, u_2$ having the form

$$y = \alpha + \beta u_1 + \gamma u_2 + \tau u_1^2,$$

then we have to choose $\bar{h}^T = (1, u_1, u_2, u_1^2)$, $\bar{\theta}^T = (\alpha, \beta, \gamma, \tau)$.

Let $y_{o,i} = \bar{h}_i^T\Theta + \varepsilon_i$, $\Theta^T = (\Theta_1, \Theta_2,..., \Theta_m)$, be the $i$-th observed value of the output variable $Y$ of the object on the observed value $\bar{h}_i^T$ of the input vector of the basis functions, $i = 1,2,..., n$, and the vector $\Theta$ has the normal priory probability density $N(a_\Theta, d_\Theta)$. The problem consists of finding of estimation $\hat{\bar{\theta}}$ of the vector $\bar{\theta}$ on the base of observers $(\bar{h}_1, y_{o,1})$, $(\bar{h}_2, y_{o,2})$,…, $(\bar{h}_n, y_{o,n})$ provided $\Theta = \bar{\theta}$.

In our case we have the following probability density functions:

$$f(\bar{\theta}) = \frac{1}{\sqrt{(2\pi)^m d_\Theta}} \exp\left(-\frac{1}{2}(\bar{\theta} - a_\Theta)^T d_\Theta^{-1}(\bar{\theta} - a_\Theta)\right),$$

$$f(y/\bar{\theta}) = \frac{1}{\sqrt{2\pi d_E}} \exp\left(-\frac{1}{2}(y - \bar{h}^T\bar{\theta})^T d_E^{-1}(y - \bar{h}^T\bar{\theta})\right).$$

The vector of observations $\vec{y} = (y_{o,1}, y_{o,2},..., y_{o,n})$ will be having the following probability density function:

$$f(\vec{y}_o/\bar{\theta}) = \prod_{i=1}^{n} f(y_{o,i}/\bar{\theta}) \sim \frac{1}{\sqrt{(2\pi)^n d_E^n}} \exp\left(-\sum_{i=1}^{n}\frac{1}{2}(y_{o,i} - \bar{h}_i^T\bar{\theta})^T d_E^{-1}(y_{o,i} - \bar{h}_i^T\bar{\theta})\right).$$

We find now the posterior probability density functions $f(\overline{\theta}/\vec{y}_o)$ of the vector coefficient $\overline{\theta}$ by the Bayes formula:

$$f(\overline{\theta}/\vec{y}_o)=\frac{f(\overline{\theta})f(\vec{y}_o/\overline{\theta})}{\int\limits_{E^m} f(\overline{\theta})f(\vec{y}_o/\overline{\theta})d\overline{\theta}}.$$

We will use for this the theorem 2. Because

$$f(\overline{\theta})=\frac{1}{\sqrt{(2\pi)^m d_\Theta}}\exp\left(-\frac{1}{2}\overline{\theta}^T d_\Theta^{-1}\overline{\theta}+a_\Theta^T d_\Theta^{-1}\overline{\theta}-\frac{1}{2}a_\Theta^T d_\Theta^{-1}a_\Theta\right),$$

$$f(\vec{y}_o/\overline{\theta})=\frac{1}{\sqrt{(2\pi)^n d_{\mathrm{E}}^n}}\exp\left(-\frac{1}{2}\sum_{i=1}^n y_{o,i}^T d_{\mathrm{E}}^{-1}y_{o,i}+\sum_{i=1}^n y_{o,i}^T d_{\mathrm{E}}^{-1}\overline{h}_i^T\overline{\theta}-\frac{1}{2}\sum_{i=1}^n\overline{\theta}^T\overline{h}_i d_{\mathrm{E}}^{-1}\overline{h}_i^T\overline{\theta}\right),$$

then, in accordance with the theorem 2, we have

$$f(\overline{\theta}/\vec{y}_o)=\frac{1}{\sqrt{(2\pi)^m|A^{-1}|}}\exp\left(-\frac{1}{2}(\overline{\theta}-A^{-1}B)^T A(\overline{\theta}-A^{-1}B)\right),$$

where $A=d_\Theta^{-1}+\sum_{i=1}^n\overline{h}_i d_{\mathrm{E}}^{-1}\overline{h}_i^T$, $B=d_\Theta^{-1}a_\Theta+\sum_{i=1}^n h_i d_{\mathrm{E}}^{-1}y_{o,i}$.

Provided the loss function is quadratic, $W(\hat{\overline{\theta}},\Theta)=(\hat{\overline{\theta}}-\Theta)^T(\hat{\overline{\theta}}-\Theta)$, we get the Bayesian estimation $\hat{\overline{\theta}}$ of the vector $\overline{\theta}$: $\hat{\overline{\theta}}=A^{-1}B$.

**Conclusion.** The results represented in the article provide the same basis to obtain the theoretical solutions of the vector problems formulated in the framework of the statistical decision theory. The integrals can be used also yourself as the table integrals. The possible generalizations of the obtained results for solving more complicated problems in the framework of the statistical decision theory are of interest.

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