ISSN 1561-8323 (Print)
ISSN 2524-2431 (Online)

UDC 539.12
https://doi.org/10.29235/1561-8323-2019-63-3-282-290

Received 22.04.2019
Поступило в редакцию 22.04.2019

Alina V. Ivashkevich[1], Elena M. Ovsiyuk[1], Vasily V. Kisel[2], Viktor M. Red'kov[3]

[1]Mozyr State Pedagogical University named after I. P. Shamyakin, Mozyr, Republic of Belarus
[2]Belarusian State University of Informatics and Radioelectronics, Minsk, Republic of Belarus
[3]B. I. Stepanov Institute of Physics of the National Academy of Sciences of Belarus, Minsk, Republic of Belarus

SPHERICAL SOLUTIONS OF THE WAVE EQUATION FOR A SPIN 3/2 PARTICLE

(Communicated by Corresponding Member Lev M. Tomilchik)

Abstract. The wave equation for a spin 3/2 particle, described by 16-component vector-bispinor, is investigated in spherical coordinates. In the frame of the Pauli–Fierz approach, the complete equation is split into the main equation and two additional constraints, algebraic and differential. The solutions are constructed, on which 4 operators are diagonalized: energy, square and third projection of the total angular momentum, and spatial reflection, these correspond to quantum numbers $\{\varepsilon, j, m, P\}$. After separating the variables, we have derived the radial system of 8 first-order equations and 4 additional constraints. Solutions of the radial equations are constructed as linear combinations of the Bessel functions. With the use of the known properties of the Bessel functions, the system of differential equations is transformed to the form of purely algebraic equations with respect to three quantities a_1, a_2, a_3. Its solutions may be chosen in various ways by solving the simple linear equation $A_1a_1 + A_2a_2 + A_3a_3 = 0$, where the coefficients A_i are expressed trough the quantum numbers ε, j. Two most simple and symmetric solutions have been chosen. Thus, at fixed quantum numbers $\{\varepsilon, j, m, P\}$ there exists double-degeneration of the quantum states.

Keywords: spin 3/2 particle, degrees of freedom, spherical symmetry, exact solutions, Bessel functions, degeneration of quantum states

For citation: Ivashkevich A. V., Ovsiyuk E. M., Kisel V. V., Red'kov V. M. Spherical solutions of the wave equation for a spin 3/2 particle. *Doklady Natsional'noi akademii nauk Belarusi = Doklady of the National Academy of Sciences of Belarus*, 2019, vol. 63, no. 3, pp. 282–290 (in Russian). https://doi.org/10.29235/1561-8323-2019-63-3-282-290

А. В. Ивашкевич[1], Е. М. Овсиюк[1], В. В. Кисель[2], В. М. Редьков[3]

[1]Мозырский государственный педагогический университет имени И. П. Шамякина, Мозырь, Республика Беларусь
[2]Белорусский государственный университет информатики и радиоэлектроники, Минск, Республика Беларусь
[3]Институт физики имени Б. И. Степанова Национальной академии наук Беларуси, Минск, Республика Беларусь

СФЕРИЧЕСКИЕ РЕШЕНИЯ УРАВНЕНИЯ ДЛЯ ЧАСТИЦЫ СО СПИНОМ 3/2

(Представлено членом-корреспондентом Л. М. Томильчиком)

Аннотация. Волновое уравнение для частицы со спином 3/2, описываемой 16-компонентным вектор-биспинором, исследовано в сферической системе координат. В рамках подхода Паули–Фирца уравнение разбивается на основное и два дополнительных, алгебраическое и дифференциальное. Строятся решения, на которых диагонализируются четыре оператора: энергии, квадрата и третьей проекции полного момента, пространственного отражения, им соответствуют квантовые числа $\{\varepsilon, j, m, P\}$. После проведения разделения переменных выведена основная система из 8 зацепляющихся радиальных дифференциальных уравнений 1-го порядка и 4 условия связи: 2 алгебраических и 2 дифференциальных. Основная система приводится к виду 4 раздельных уравнений 2-го порядка, решения которых строятся в функциях Бесселя. С использованием свойств функций Бесселя вся система радиальных уравнений для частицы со спином 3/2 приведена к одному алгебраическому линейному уравнению $A_1a_1 + A_2a_2 + A_3a_3 = 0$ относительно величин a_1, a_2, a_3, в котором коэффициенты A_i выражаются через квантовые числа ε, j. Выбраны наиболее симметричные решения, которые определяют два решения при фиксированных квантовых числах $\{\varepsilon, j, m, P\}$.

Ключевые слова: частица со спином 3/2, степени свободы, сферическая симметрия, точные решения, функции Бесселя, вырождение квантовых состояний

Для цитирования. Сферические решения уравнения для частицы со спином 3/2 / А. В. Ивашкевич [и др.] // Докл. Нац. акад. наук Беларуси. – 2019. – Т. 63, № 3. – С. 282–290. https://doi.org/10.29235/1561-8323-2019-63-3-282-290

The basic wave equation and spherical symmetry. The theory of spin 3/2 particle is attracted steady interest after the seminal investigation by Pauli and Fierz – see [1–15]. Let us recall the most significant aspect of spin 3/2 particle theory. First of all, it is the problem of choosing an initial system of equations. The most consistent is an approach based on Lagrangian formalism and a correct first order equation for multi-component wave function which are based on the general theory of 1st order relativistic wave equations. However investigations are based on the use of 2nd order equations. Such a choice is of prime importance when we take into account the presence of external electromagnetic (or gravitational) fields. Applying the first order approach ensures correct solving the problem of independent degrees of freedom in presence of external fields; for instance see in [16]. The great attention was given to existence in this theory of solutions which correspond to a particle moving with velocity greater than the light velocity. Finally a separate interest has a massless case for spin 3/2 field, when – as shown by Pauli and Fierz – there exists specific gauge symmetry: the 4-gradient of arbitrary bispinor function provides us with solution for the massless field equation, for instance, see in [16].

In the present paper we examine the problem of degree of freedom for a massive spin 3/2 particle specified for solutions with spherical symmetry. For simplicity we restrict ourselves to Minkowski space-time model. The wave equation for such a particle in absence of external fields may be presented as a set of a main equation and two additional constraints (we assume the use of the tetrad formalism, see [16])

$$[\, i\gamma^{\beta}(x)(\nabla_{\beta} + \Gamma_{\beta}(x)) - m\,]\, \Psi_{\alpha}(x) = 0,$$

$$\gamma^{\alpha}(x)\Psi_{\alpha}(x) = 0, \;\; (\nabla_{\alpha} + \Gamma_{\alpha}(x))\Psi^{\alpha}(x) = 0, \tag{1}$$

the wave function $\Psi_{\alpha}(x)$ behaves as a bispinor with respect to tetrad transformations, and as a generally covariant vector with respect to coordinate transformations; we use the notations [16]:

$$m = \frac{Mc}{\hbar}, \;\; \gamma^{\beta}(x) = e^{\beta}_{(a)}(x)\gamma^{a}, \;\; \Gamma_{\beta}(x) = \frac{1}{2}(\sigma^{ab})_{k}^{\;\;l}\, e^{\beta}_{(a)}(\nabla_{\alpha} e_{(b)\beta}), \;\; \sigma^{ab} = \frac{\gamma^{a}\gamma^{b} - \gamma^{b}\gamma^{a}}{4}.$$

Below, it will be convenient to use the wave function with tetrad-vector index, $\Psi_{l}(x)$, it relates to the previous function $\Psi_{\alpha}(x)$ in accordance with the rule

$$\Psi_{l}(x) = e^{\beta}_{(l)}(x)\Psi_{\beta}(x), \quad \Psi_{\beta}(x) = e^{(l)}_{\beta}(x)\Psi_{l}(x).$$

Correspondingly, the system of equations (1) takes the form (for shortness, in the main equation we omit vector indices in the wave function)

$$[i\gamma^{\alpha}(x)(\partial_{\alpha} + B_{\alpha}(x)) - m]\Psi(x) = 0,$$

$$\gamma^{l}\Psi_{l}(x) = 0, \quad [e^{(l)\alpha}\partial_{\alpha} + e^{(l)\alpha}(x) + e^{(l)\alpha}(x)\Gamma_{\alpha}(x)]\Psi_{l}(x) = 0, \tag{2}$$

where

$$(j^{ab})_{k}^{\;\;l} = \delta^{a}_{k}\, g^{bl} - \delta^{b}_{k} g^{al}, \quad L_{\alpha}(x) = \frac{1}{2}(j^{ab})_{k}^{\;\;l}\, e^{\beta}_{(a)}(\nabla_{\alpha} e_{(b)\beta}),$$

$$J^{ab} = \sigma^{ab} \otimes I + I \otimes j^{ab}, \quad B_{\alpha}(x) = \Gamma_{\alpha}(x) \otimes I + I \otimes L_{\alpha}(x).$$

We will specify equations (2) in spherical coordinates

$$x^{\alpha} = (t, r, \theta, \phi), \quad dS^{2} = dt^{2} - dr^{2} - r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}),$$

the main equation takes the form

$$\left\{ i\gamma^{0}\partial_{0} + i\gamma^{3}\partial_{r} + \frac{\gamma^{1} \otimes T^{2} - \gamma^{2} \otimes T^{1}}{r} + \frac{1}{r}\Sigma_{\theta,\phi} - m \right\} \Psi(t, r, \theta, \phi) = 0,$$

where the angular operator $\Sigma_{\theta,\phi}$ is determined by the formula

$$\Sigma_{\theta,\phi} = i\gamma^{1}\partial_{\theta} + \gamma^{2}\,\frac{i\partial_{\phi} + (i\sigma^{12} \otimes I + I \otimes ij^{12})\cos\theta}{\sin\theta}.$$

Searching solutions in the form of spherical waves, we are to diagonalize operators of the square and third projection of the total angular momentum. In Cartesian basis, the components of the total angular momentum are defined as follows

$$J_i^{Cart} = l_i + S_i, \quad S_1 = iJ^{23}, \quad S_2 = iJ^{31}, \quad S_3 = iJ^{12},$$

$$S_i = \frac{1}{2}\Sigma_i \otimes I + I \otimes T_i, \quad T_i = \begin{vmatrix} 0 & 0 \\ 0 & \tau_i \end{vmatrix},$$

where symbol $\otimes$ stands for a direct product of the matrices. Operators J_i in spherical tetrad basis are given by the formulas [17]

$$J_1 = l_1 + S_3 \frac{\sin\phi}{\sin\theta}, \quad J_2 = l_2 + S_3 \frac{\cos\phi}{\sin\theta}, \quad J_3 = l_3 = -i\frac{\partial}{\partial\phi}.$$

The components of the wave function may be listed with the help of bispinor index A and 4-vector index (l):

$$\Psi_{A(l)} = \begin{vmatrix} \Psi_{1(0)} & \Psi_{1(1)} & \Psi_{1(2)} & \Psi_{1(3)} \\ \Psi_{2(0)} & \Psi_{2(1)} & \Psi_{2(2)} & \Psi_{2(3)} \\ \Psi_{3(0)} & \Psi_{3(1)} & \Psi_{3(2)} & \Psi_{3(3)} \\ \Psi_{4(0)} & \Psi_{4(1)} & \Psi_{4(2)} & \Psi_{4(3)} \end{vmatrix}.$$

It is convenient to use so called cyclic basis in which the generator ij^{12} becomes diagonal. The needed transformation $\tilde{\Psi} = U\Psi$ is given by the formulas

$$\begin{vmatrix} \tilde{\Psi}_{(0)} \\ \tilde{\Psi}_{(1)} \\ \tilde{\Psi}_{(2)} \\ \tilde{\Psi}_{(3)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & +1/\sqrt{2} & i/\sqrt{2} & 0 \end{vmatrix} \begin{vmatrix} \Psi_{(0)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{vmatrix}, \quad \begin{vmatrix} \Psi_{(0)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} \tilde{\Psi}_{(0)} \\ \tilde{\Psi}_{(1)} \\ \tilde{\Psi}_{(2)} \\ \tilde{\Psi}_{(3)} \end{vmatrix}.$$

In cyclic basis we have the diagonal form for $\tilde{S}_3$:

$$\tilde{S}_3 = \frac{1}{2}\begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \otimes I + I \otimes \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

In cyclic basis, the above equations (2) read

$$[i\gamma^{\alpha}(x)(\partial_{\alpha} + \Gamma_{\alpha}(x) \otimes I + I \otimes \tilde{L}_{\alpha}(x)) - m]\tilde{\Psi} = 0,$$

$$\gamma^{l}\Psi_{l}(x) = \gamma^{l}(U^{-1})_{lk}\tilde{\Psi}_{k}(x) = 0,$$

$$[e^{(l)\alpha}(x)\partial_{\alpha} + e^{(l)\alpha}(x) + e^{(l)\alpha}(x)\Gamma_{\alpha}(x)](U^{-1})_{lk}\Psi_{k}(x) = 0.$$

In cyclic basis the main equation takes the form (let $\tilde{\Psi} = \frac{1}{r}\tilde{\Phi}$)

$$\left[\gamma^0\varepsilon + i\gamma^3\partial_r + \frac{\gamma^1 \otimes \tilde{T}_2 - \gamma^2 \otimes \tilde{T}_1}{r} + \frac{1}{r}\tilde{\Sigma}_{\theta,\phi} - m\right]\tilde{\Phi}(r, \theta, \phi) = 0,$$

$$\tilde{\Sigma}_{\theta,\phi} = i\gamma^1\partial_{\theta} + \gamma^2\frac{i\partial_{\phi} + i\tilde{S}_3\cos\theta}{\sin\theta}.$$

Separating the variables. The general substitution for spherical solutions with quantum numbers j, m has the structure (for more details see [17])

$$\tilde{\xi}_l = \begin{vmatrix} f_0\, \delta_i^0\, D_{-1/2} + f_1\, \delta_i^1\, D_{-3/2} + f_2\, \delta_i^2\, D_{-1/2} + f_3\, \delta_i^3\, D_{+1/2} \\ g_0\, \delta_i^0\, D_{+1/2} + g_1\, \delta_i^1\, D_{-1/2} + g_2\, \delta_i^2\, D_{+1/2} + g_3\, \delta_i^3\, D_{+3/2} \end{vmatrix},$$

we take into account that vector-bispinor consists of two vector-spinors, $\tilde{\xi}_l, \tilde{\eta}_l,$ the second vector-spinor $\tilde{\eta}_i$ has the similar structure but with other radial functions

$$f_i(r) \Rightarrow h_i(r), \qquad g_i(r) \Rightarrow \nu_i(r).$$

Symbol D_σ stands for Wigner D-functions: $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0), \quad \sigma = -3/2, -1/2, +1/2, +3/2.$ Diagonalization of the spatial reflection operator leads to additional restrictions

$$\nu_0 = \delta f_0,\ \nu_1 = \delta f_3,\ \nu_2 = \delta f_2,\ \nu_3 = \delta f_1,$$

$$h_0 = \delta g_0,\ h_1 = \delta g_3,\ h_2 = \delta g_2,\ h_3 = \delta g_1,$$

where $\delta = +1$ corresponds to the parity $P = (-1)^{j+1}$ and $\delta = -1$ corresponds to $P = (-1)^j$.

After rather laborious calculation on separating the variables, we arrive at 8 first-order equations for states with parity $(-1)^{j+1}$:

$$\left(\varepsilon - i\frac{d}{dr}\right) g_0 = \left(m + i\frac{a}{r}\right) f_0, \quad \left(\varepsilon + i\frac{d}{dr}\right) f_0 = \left(m - i\frac{a}{r}\right) g_0,$$

$$\left(\varepsilon - i\frac{d}{dr}\right) g_3 - i\frac{b}{r} f_3 = m f_1, \quad \left(\varepsilon + i\frac{d}{dr}\right) f_1 + i\frac{b}{r} g_1 = m g_3,$$

$$\left(\varepsilon + i\frac{d}{dr}\right) f_3 + i\frac{\sqrt{2}}{r} g_2 + i\frac{b}{r} g_3 = m g_1, \quad \left(\varepsilon - i\frac{d}{dr}\right) g_1 - i\frac{\sqrt{2}}{r} f_2 - i\frac{b}{r} f_1 = m f_3,$$

$$\left(\varepsilon - i\frac{d}{dr}\right) g_2 = +i\frac{\sqrt{2}}{r} f_3 + \left(m + i\frac{a}{r}\right) f_2, \quad \left(\varepsilon + i\frac{d}{dr}\right) f_2 = -i\frac{\sqrt{2}}{r} g_1 + \left(m - i\frac{a}{r}\right) g_2; \tag{3}$$

where $a = j + 1/2, b = \sqrt{(j - 1/2)(j + 3/2)}, j = 1/2, 3/2, 5/2, \ldots;$ to obtain similar equations for states with parity $(-1)^j$ it suffices to make formal change m to $-m$.

The above additional constraints lead to equations (they are the same for both values of parity)

$$g_1 = \frac{1}{\sqrt{2}}(f_2 + f_0), \quad f_3 = \frac{1}{\sqrt{2}}(g_2 - g_0);$$

$$-i\varepsilon f_0 - \left(\frac{d}{dr} + \frac{1}{r}\right) f_2 = \frac{1}{\sqrt{2}\, r}(g_1 + b f_1 + a f_3),$$

$$-i\varepsilon g_0 - \left(\frac{d}{dr} + \frac{1}{r}\right) g_2 = \frac{1}{\sqrt{2}\, r}(f_3 + b g_3 + a g_1).$$

Equations for functions f_0, g_0. First we are to solve equations for functions f_0, g_0 (see two first equations in (3)). Summing and subtracting these two equations

$$\left(i\varepsilon + \frac{d}{dr}\right) g_0 = \left(im - \frac{a}{r}\right) f_0, \quad \left(i\varepsilon - \frac{d}{dr}\right) f_0 = \left(im + \frac{a}{r}\right) g_0, \tag{4}$$

we produce

$$\left(\frac{d}{dr} + \frac{a}{r}\right) F_0 = i(\varepsilon + m) G_0, \quad \left(\frac{d}{dr} - \frac{a}{r}\right) G_0 = i(\varepsilon - m) F_0,$$

where the notations $F_0 = f_0 + g_0$, $G_0 = f_0 - g_0$ are used. From (4), we derive two 2nd order equations

$$\left(\frac{d^2}{dr^2}-\frac{a^2+a}{r^2}+\varepsilon^2-m^2\right)F_0=0, \quad l=a;$$

$$\left(\frac{d^2}{dr^2}-\frac{(a-1)a}{r^2}+\varepsilon^2-m^2\right)G_0=0, \quad l'=a-1. \tag{5}$$

In the variable $x=\sqrt{\varepsilon^2-m^2}\,r$ they read

$$\left(\frac{d^2}{dx^2}+1-\frac{l(l+1)}{x^2}\right)F_0(x)=0, \qquad \left(\frac{d^2}{dx^2}+1-\frac{l'('l+1)}{x^2}\right)G_0(x)=0.$$

Let it be

$$F_0(x)=a_0\sqrt{x}\,Z_l(x), \quad G_0(x)=b_0\sqrt{x}\,Z_{l'}(x),$$

then eqs. (5) give

$$Z_l+\frac{1}{x}Z'_l+1-\frac{(l+1/2)^2}{x^2}Z_l=0, \quad l+1/2=j+1=p,$$

$$Z_{l'}+\frac{1}{x}Z'_{l'}+1-\frac{(l-1/2)^2}{x^2}Z_{l'}=0, \quad l'+1/2=j=p-1.$$

They have the Bessel's form

$$Z''+\frac{1}{x}Z'+\left(1-\frac{p^2}{x^2}\right)Z=0.$$

Thus, functions F_0 and G_0 relate to Bessel functions.

The first-order equations (4) for functions F_0, G_0, after transforming them to the variable x will take the form

$$\left(\frac{d}{dx}+\frac{p}{x}\right)Z_p=\sqrt{\frac{\varepsilon+m}{\varepsilon-m}}\left(i\frac{b_0}{a_0}\right)Z_{p-1}, \quad \left(\frac{d}{dx}-\frac{p-1}{x}\right)Z_{p-1}=\sqrt{\frac{\varepsilon-m}{\varepsilon+m}}\left(i\frac{a_0}{b_0}\right)Z_p. \tag{6}$$

Due to the known equations

$$\left(\frac{d}{dx}+\frac{p}{z}\right)Z_p=Z_{p-1}, \quad \left(\frac{d}{dx}-\frac{p}{z}\right)Z_p=-Z_{p+1}=0,$$

from (6) we find a relative coefficient between the quantities a_0 and b_0:

$$\sqrt{\varepsilon+m}\,b_0=-i\sqrt{\varepsilon-m}\,a_0.$$

Equations for functions f_i and g_i. Now, we turn to 6 equations from (3)

$$\left(\varepsilon+i\frac{d}{dr}\right)f_1+i\frac{b}{r}g_1=m\,g_3, \quad \left(\varepsilon-i\frac{d}{dr}\right)g_3-i\frac{b}{r}f_3=m\,f_1,$$

$$\left(\varepsilon+i\frac{d}{dr}\right)f_2+i\frac{\sqrt{2}}{r}g_1=\left(m-i\frac{a}{r}\right)g_2, \quad \left(\varepsilon-i\frac{d}{dr}\right)g_2-i\frac{\sqrt{2}}{r}f_3=\left(m+i\frac{a}{r}\right)f_2,$$

$$\left(\varepsilon+i\frac{d}{dr}\right)f_3+i\frac{\sqrt{2}}{r}g_2+i\frac{b}{r}g_3=m\,g_1, \quad \left(\varepsilon-i\frac{d}{dr}\right)g_1-i\frac{\sqrt{2}}{r}f_2-i\frac{b}{r}f_1=m\,f_3. \tag{7}$$

It is convenient to employ the following variables

$$f_1+g_3=F_1,\ f_1-g_3=G_1, \quad f_2+g_2=F_2,\ f_2-g_2=G_2, \quad f_3+g_1=F_3,\ f_3-g_1=G_3.$$

Summing and subtracting equations (7) within each pair, we produce

$$\frac{d}{dr}G_1 + i(m-\varepsilon)F_1 = +\frac{b}{r}G_3, \qquad \frac{d}{dr}F_1 - i(m+\varepsilon)G_1 = -\frac{b}{r}F_3;$$

$$\left(\frac{d}{dr}-\frac{a}{r}\right)G_2 + i(m-\varepsilon)F_2 = +\frac{\sqrt{2}}{r}G_3, \quad \left(\frac{d}{dr}+\frac{a}{r}\right)F_2 - i(m+\varepsilon)G_2 = -\frac{\sqrt{2}}{r}F_3;$$

$$\frac{d}{dr}G_3 - \frac{\sqrt{2}}{r}G_2 + i(m-\varepsilon)F_3 = +\frac{b}{r}G_1, \quad \frac{d}{dr}F_3 + \frac{\sqrt{2}}{r}F_2 - i(m+\varepsilon)G_3 = -\frac{b}{r}F_1.$$

This system may be presented in a matrix form

$$\frac{d}{dr}\begin{vmatrix} F_1 \\ F_2 \\ F_3 \\ G_1 \\ G_2 \\ G_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -b/r & i(m+\varepsilon) & 0 & 0 \\ 0 & -a/r & -\sqrt{2}/r & 0 & i(m+\varepsilon) & 0 \\ -b/r & -\sqrt{2}/r & 0 & 0 & 0 & i(m+\varepsilon) \\ -i(m-\varepsilon) & 0 & 0 & 0 & 0 & \frac{b}{r} \\ 0 & -i(m-\varepsilon) & 0 & 0 & a/r & \sqrt{2}/r \\ 0 & 0 & -i(m-\varepsilon) & b/r & \sqrt{2}/r & 0 \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \\ F_3 \\ G_1 \\ G_2 \\ G_3 \end{vmatrix},$$

or more shortly

$$\left(\frac{d}{dr}+\frac{A}{r}\right)F = i(m+\varepsilon)G, \quad \left(\frac{d}{dr}-\frac{A}{r}\right)G = -i(m-\varepsilon)F, \quad A = \begin{vmatrix} 0 & 0 & b \\ 0 & a & \sqrt{2} \\ b & \sqrt{2} & 0 \end{vmatrix}.$$

Applying the exclusion method, we derive equations for $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$

$$\Delta F = (A^2 + A)F, \quad \Delta G = (A^2 - A)G, \quad \Delta = r^2\left(\frac{d^2}{dr^2} + \varepsilon^2 - m^2\right).$$

We have found transformations S and S' which diagonalize two mixing matrices:

$$\bar{F} = SF, \quad \Delta\bar{F} = (STS^{-1})\bar{F}, \quad STS^{-1} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix};$$

$$\bar{G} = SG, \quad \Delta\bar{G} = (S'T'S'^{-1})\,\bar{G}, \quad S'T'S'^{-1} = \begin{vmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & \lambda'_3 \end{vmatrix}.$$

In this way, we get 8 separated equations (also we write down equations for F_0 and G_0)

$$\left(\frac{d^2}{dr^2} + \varepsilon^2 - m^2 - \frac{j'(j'+1)}{r^2}\right)F_0 = 0, \quad j' = j + 1/2, \quad F_0 = a_0 f_{j+1/2};$$

$$\left(\frac{d^2}{dr^2} + \varepsilon^2 - m^2 - \frac{j'(j'+1)}{r^2}\right)G_0 = 0, \quad j' = j - 1/2, \quad G_0 = b_0 f_{j-1/2};$$

$$\left(\frac{d^2}{dr^2} + \varepsilon^2 - m^2 - \frac{j'(j'+1)}{r^2}\right)\bar{F}_1 = 0, \quad j' = j - 1/2, \quad \bar{F}_1 = a_1 f_{j-1/2};$$

$$\left(\frac{d^2}{dr^2}+\varepsilon^2-m^2-\frac{j'(j'+1)}{r^2}\right)\bar{G}_1=0,\quad j'=j+1/2,\quad \bar{G}_1=b_1 f_{j+1/2};$$

$$\left(\frac{d^2}{dr^2}+\varepsilon^2-m^2-\frac{j'(j'+1)}{r^2}\right)\bar{F}_2=0,\quad j'=j+3/2,\quad \bar{F}_2=a_2 f_{j+3/2};$$

$$\left(\frac{d^2}{dr^2}+\varepsilon^2-m^2-\frac{j'(j'+1)}{r^2}\right)\bar{G}_2=0,\quad j'=j-3/2,\quad \bar{G}_2=b_2 f_{j-3/2};$$

$$\left(\frac{d^2}{dr^2}+\varepsilon^2-m^2-\frac{j'(j'+1)}{r^2}\right)\bar{F}_3=0,\quad j'=j-1/2,\quad \bar{F}_3=a_3 f_{j-1/2};$$

$$\left(\frac{d^2}{dr^2}+\varepsilon^2-m^2-\frac{j'(j'+1)}{r^2}\right)\bar{G}_3=0,\quad j'=j+1/2,\quad \bar{G}_3=b_3 f_{j+1/2}.$$

They have similar mathematical structure, and all may be reduced to Bessel's type.

It should be emphasized that parameters a_1, a_2, a_3 and b_1, b_2, b_3 cannot be considered as independent, because there exists the first-order differential equation which relates F and G:

$$\left(\frac{d}{dr}+\frac{A}{r}\right)F=i(m+\varepsilon)G;$$

whence with the formulas $F=S^{-1}\bar{F}$ and $G=S'^{-1}\bar{G}$ in mind, we derive

$$\left(\frac{d}{dr}+\frac{SAS^{-1}}{r}\right)\bar{F}=i(m+\varepsilon)SS'^{-1}\bar{G}.$$

We have found the matrices SAS^{-1} and SS'^{-1}, this makes possible to obtain linear constraints

$$b_1=i\sqrt{\frac{\varepsilon-m}{\varepsilon+m}}\frac{1}{4(j+1)}\left\{\sqrt{\frac{j+3/2}{j-1/2}}\left((2j+1)\sqrt{2}a_1-2a_3\right)+\sqrt{2}a_2\right\},$$

$$b_3=i\sqrt{\frac{\varepsilon-m}{\varepsilon+m}}\frac{1}{4(j+1)}\left\{\sqrt{\frac{j+3/2}{j-1/2}}\left((2j+1)a_1-\sqrt{2}a_3\right)-(2j+1)a_2\right\},$$

$$b_2=i\sqrt{\frac{\varepsilon-m}{\varepsilon+m}}\left\{\frac{1}{\sqrt{2}}a_1-a_3\right\}. \tag{8}$$

Two independent solutions. With the use of the properties of Bessel functions, all differential equations have been transformed to algebraic form, so in total we have 4 independent algebraic constraints

$$\sqrt{2}\,a_1-a_3=-\sqrt{j-1/2}\,\sqrt{2}b_0,\qquad \sqrt{2}\,b_1-b_3=+\sqrt{j+3/2}\,a_0,$$

$$i\frac{\Gamma}{2\alpha}a_0-(j+3/2)\sqrt{2}a_1+\sqrt{(j+3/2)(j-1/2)}\sqrt{2}a_2+(2j+1)a_3=0,$$

$$i\frac{\Gamma}{\beta}b_0+(j-1/2)\sqrt{2}b_1-\sqrt{(j+3/2)(j-1/2)}\sqrt{2}b_2-(2j+1)b_3=0; \tag{9}$$

$$\sqrt{\varepsilon+m}\,b_0=-i\sqrt{\varepsilon-m}\,a_0,\quad \frac{1}{2\alpha}=4(j+1)\sqrt{2j-1},\quad \frac{1}{\beta}=-4j\sqrt{j+3/2}.$$

Two first relations in (9) permit us to exclude the parameters a_0 and b_0:

$$a_0=\frac{\sqrt{2}}{\sqrt{j+3/2}}b_1-\frac{1}{\sqrt{j+3/2}}b_3,\quad b_0=-\frac{a_1}{\sqrt{j-1/2}}+\frac{1}{\sqrt{2}\sqrt{j-1/2}}a_3, \tag{10}$$

so from (9) we get

$$i\frac{\Gamma}{2\alpha}\frac{1}{\sqrt{j+3/2}}(\sqrt{2}b_1-b_3)-(j+3/2)\sqrt{2}a_1+\sqrt{(j+3/2)(j-1/2)}\sqrt{2}a_2+(2j+1)a_3=0,$$

$$i\frac{\Gamma}{\beta}\frac{1}{\sqrt{j-1/2}}\left(-a_1+\frac{a_3}{\sqrt{2}}\right)+(j-1/2)\sqrt{2}b_1-\sqrt{(j+3/2)(j-1/2)}\sqrt{2}b_2-(2j+1)b_3=0. \qquad (11)$$

In turn, taking into account $\sqrt{\varepsilon-m}a_0=i\sqrt{\varepsilon+m}b_0$ from (10) we derive

$$\frac{\sqrt{\varepsilon-m}}{\sqrt{j+3/2}}(\sqrt{2}b_1-b_3)=\frac{i\sqrt{\varepsilon+m}}{\sqrt{j-1/2}}\left(-a_1+\frac{a_3}{\sqrt{2}}\right).$$

This identity permits us to transform equations (11) to the form, when the first equation includes only a_i, and the second one contains only b_i:

$$\frac{4\varepsilon(j+1)}{\varepsilon-m}\left(a_1-\frac{a_3}{\sqrt{2}}\right)-\left(j+\frac{3}{2}\right)a_1+(2j+1)\frac{a_3}{\sqrt{2}}+\sqrt{\left(j+\frac{3}{2}\right)\left(j-\frac{1}{2}\right)}a_2=0,$$

$$\frac{4\varepsilon j}{\varepsilon+m}\left(b_1-\frac{b_3}{\sqrt{2}}\right)-\left(j-\frac{1}{2}\right)b_1+(2j+1)\frac{b_3}{\sqrt{2}}+\sqrt{\left(j+\frac{3}{2}\right)\left(j-\frac{1}{2}\right)}b_2=0; \qquad (12)$$

note that a_0, b_0 may be found from (10). The general structure of two equations (12) is

$$A_1a_1+A_2a_2+A_3a_3=0, \quad B_1b_1+B_2b_2+B_3b_3=0.$$

We are to fix two independent solutions of the main linear constraint for coefficients a_i. In this way obtain two independent solutions of the equations for spin 3/2 particle, evidently there exist many possibilities for that choosing. By simplicity reason, let us take the following solutions:

$$(1), \quad a_1^{(1)}=0 \;\Rightarrow\; A_2a_2^{(1)}+A_3a_3^{(1)}=0 \;\Rightarrow$$

$$a_3^{(1)}=-\frac{A_2}{A_3}a_2^{(1)}=-\frac{(\varepsilon-m)\sqrt{(j-1/2)(j+3/2)}}{4(j+1)\varepsilon+(2j+1)(\varepsilon-m)}\sqrt{2}a_2^{(1)};$$

$$(2), \quad a_3^{(2)}=0 \;\Rightarrow\; A_2a_2^{(2)}+A_1a_1^{(2)}=0 \;\Rightarrow$$

$$a_1^{(2)}=-\frac{A_2}{A_1}a_2^{(2)}=-\frac{(\varepsilon-m)\sqrt{(j-1/2)(j+3/2)}}{4(j+1)\varepsilon-(j+3/2)(\varepsilon-m)}a_2^{(2)}.$$

To simplify the above expression, we may set

$$a_2^{(1)}=\frac{1}{(\varepsilon-m)\sqrt{(j-1/2)(j+3/2)}}, \quad a_2^{(2)}=\frac{1}{(\varepsilon-m)\sqrt{(j-1/2)(j+3/2)}}.$$

Corresponding sets of parameters $b_i^{(1)}$ and $b_i^{(2)}$ may be found with the help of relations (8). It is readily verified that the second constraint $B_1b_1+B_2b_2+B_3b_3=0$ turns to be identity 0 = 0 for both solution (1) and (2).

Conclusions. The wave equation for a spin 3/2 particle is solved in spherical coordinates.. Solutions of the radial equations are constructed in the form of linear combinations of Bessel functions. With the use of the known properties of Bessel function, the system of differential equations is transformed to the purely algebraic equations with respect to three quantities a_1, a_2, a_3. Its solutions may be chosen in various ways as solutions of the simple linear constraint $A_1a_1+A_2a_2+A_3a_3=0$, where coefficients A_i are expressed trough the quantum numbers. Two most simple and symmetric sets are found, $a_i^{(1)}$ and $a_i^{(2)}$. Thus, at fixed quantum numbers $\{\varepsilon, j, m, P\}$ there exists double-degeneration of the quantum states. Explicit form of the operator associated with such degeneration has been not found.

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Information about the authors

Ivashkevich Alina Valentinovna – Student. Mozyr State Pedagogical University named after I. P. Shamyakin (28, Studencheskaya Str., 247760, Mozyr, Gomel region, Republic of Belarus). E-mail: ivashkevich.alina@yandex.by.

Ovsiyuk Elena Mikhailovna – Ph. D. (Physics and Mathematics), Assistant professor. Mozyr State Pedagogical University named after I. P. Shamyakin (28, Studencheskaya Str., 247760, Mozyr, Gomel region, Republic of Belarus). E-mail: e.ovsiyuk@mail.ru.

Kisel Vasily Vasilyevich – Ph. D. (Physics and Mathematics), Assistant professor. Belarusian State University of Informatics and Radioelectronics (6, P. Brovka Str., 220013, Minsk, Republic of Belarus). E-mail: vasiliy_bspu@mail.ru.

Red'kov Viktor Mikhailovich – Ph. D. (Physics and Mathematics), Chief researcher. B. I. Stepanov Institute of Physics of the National Academy of Sciences of Belarus (68-2, Nezavisimosti Ave., 220072, Minsk, Republic of Belarus). E-mail: v.redkov@dragon.bas-net.by.

Информация об авторах

Ивашкевич Алина Валентиновна – студент. Мозырский государственный педагогический университет им. И. П. Шамякина (ул. Студенческая, 28, 247760, Мозырь, Гомельская область, Республика Беларусь). E-mail: ivashkevich.alina@yandex.by.

Овсиюк Елена Михайловна – канд. физ.-мат. наук, доцент. Мозырский государственный педагогический университет им. И. П. Шамякина (ул. Студенческая, 28, 247760, Мозырь, Гомельская область, Республика Беларусь). E-mail: e.ovsiyuk@mail.ru.

Кисель Василий Васильевич – канд. физ.-мат. наук, доцент. Белорусский государственный университет информатики и радиоэлектроники (ул. П. Бровки, 6, 220013, Минск, Республика Беларусь). E-mail: vasiliy_bspu@mail.ru.

Редьков Виктор Михайлович – д-р физ.-мат. наук, гл. науч. сотрудник. Институт физики имени Б. И. Степанова Национальной академии наук Беларуси (пр-т Независимости, 68-2, 220072, Минск, Республика Беларусь). E-mail: v.redkov@dragon.bas-net.by.