Eigen Transformations of Symmetric Matrices in Information Processing Problems

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Abstract. **The mathematical justification of the algorithm for synthesis of proper transformation and the finding the eigenvalue of a symmetric matrix of dimension $N \times N$, $N = 4$ based on orthogonal rotation operators is given. Analytical relations for calculating the eigenvalues of symmetric matrix is optained. It is shown that the proper transformation has factorized structure in the form of a product of rotation operators. Each operator is a direct sum of elementary rotation matrices.**

Keywords: **symmetric matrix, eigenvalues, rotation operators**

I. Introduction

The analysis and processing of large amounts of data involves compression using fast algorithms, since it is important to have a high speed of information flow to the appropriate data analysis and processing systems. In statistics, the principal component method is used to compress information without significant loss of its informativeness [1]. It consists in a linear orthogonal transformation of the input vector X dimension n in the output vector of Y dimension p, where $p < n$. In this case, the components of the vector Y are uncorrelated and the total variance after the transformation remains unchanged.

The covariance matrix of the input data X is defined as

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix},$$

where r_{ij} - covariance between the i -th and j -th components of the input images.

The eigenvalues λ_k of the matrix R characterize the variance of the principal components. In this case, the sum of the variances in the space of the initial signs is equal to the sum of the variances in the space of the output signs:

$$\sum_{i=1}^{n} r_{ii}(x) = \sum_{i=1}^{n} \lambda_i .$$

II. Methodology

To solve the problem of eigenvalues of symmetric matrices, different approaches are used, in particular, in [2], a QL algorithm is proposed, which requires a preliminary reduction of the original matrix to a tridiagonal form. However, the QL algorithm uses similarity transformations of a rather complex structure, which do not allow to effectively organize the computational process when implementing its own basis. The method for rotations [2] allows to find all the eigenvalues and eigenvectors of a symmetric matrix without using the characteristic equation. It is known [4] that for a symmetric matrix A an orthogonal matrix Ψ exists when

$$\Psi^T A \Psi = \Lambda ,$$

where $\Psi^T = \Psi^{-1}$ (orthogonality condition), Λ is the diagonal matrix. In the method of rotations, the matrix Ψ is constructed as the limit of the sequence of products of matrices of simple rotations. Rotation operators [3], in contrast to rotation matrices, have invariant properties (Lemmas 1–4), that let solve this problem in an optimal way.

In this paper, we propose an approach for solving the symmetric eigenvalue problem based on the use of orthogonal rotation operators. The rotation operator is a matrix of dimension $n \times n,\ n = 2^i$ ($i = 2, 4, \ldots, N$), which is the direct sum of the elementary Givens and Jacobi rotation matrices. In this case, the symmetric matrix is preliminarily reduced to the persymmetric one (where the symmetry is less than and relative to the side diagonal), and then the synthesis of the proper transformation of the resulting matrix is carried out [3].

Pattern Recognition and Information Processing (PRIP'2021) : Proceedings of the 15th International Conference, 21–24 Sept. 2021, Minsk, Belarus. – Minsk : UIIP NASB, 2021. – 246 p. – ISBN 978-985-7198-07-8.

Suppose that a symmetric matrix $A=\left[a_{ij}\right]_{4\times 4}$ is given, as well as the rotation operators:

$$\bar{T}_2^{(4)}=\begin{bmatrix} c_1 & s_1 & 0 & \\ s_1 & -c_{11} & & 0 \\ 0 & & -c_2 & s_2 \\ & 0 & s_2 & c_2 \end{bmatrix},$$

$$T_0^{(4)}=\begin{bmatrix} c_3 & 0 & 0 & s_3 \\ 0 & c_4 & s_4 & 0 \\ 0 & s_4 & -c_4 & 0 \\ s_3 & 0 & 0 & -c_3 \end{bmatrix},$$

$$G_1^{(4)}=\begin{bmatrix} c_5 & 0 & s_5 & 0 \\ 0 & c_5 & 0 & s_5 \\ s_5 & 0 & -c_5 & 0 \\ 0 & s_5 & 0 & -c_5 \end{bmatrix},$$

$$T_1^{(4)}=\begin{bmatrix} c_6 & 0 & 0 & s_6 \\ 0 & c_6 & s_6 & 0 \\ 0 & s_6 & -c_6 & 0 \\ s_6 & 0 & 0 & -c_6 \end{bmatrix},$$

$$c_k=\cos\alpha_k\ ,\quad s_k=\sin\alpha_k$$

The following basic properties of the rotation operators are proved:

Lemma 1.

Let $\left[b_{ij}\right]_{4\times 4}=\bar{T}_2^{(4)}A\bar{T}_2^{(4)}=B_1$, $b_{11}=b_{22}$, $b_{33}=b_{44}$,

$\left[d_{ij}\right]_{4\times 4}=T_0^{(4)}B_1T_0^{(4)}=B_2$, $d_{11}=d_{44}$, $d_{22}=d_{33}$.

Then the diagnal elments of the matrix B_2 will be equal, i.e. $d_{11}=d_{22}=d_{33}=d_{44}=\frac{1}{2}L_1$, where

$$L_1=\frac{1}{2}\sum_{i=1}^{4}a_{ii}$$

Lemma 2. Let $\left[l_{ij}\right]_{4\times 4}=G_1^{(4)}B_2G_1^{(4)}$. Then for all α_5 the relation is valid:

$$l_{11}-l_{44}=l_{22}-l_{33}$$

Lemma 3. Suppose that $X=\left[x_{ij}\right]_{4\times 4}$ is a symmetric matrix, and $\left[y_{ij}\right]_{4\times 4}=G_1^{(4)}XG_1^{(4)}$ and $\left[\bar{y}_{ij}\right]_{4\times 4}=T_1^{(4)}XT_1^{(4)}$. Then equality $x_{14}=x_{23}$ follows that:

$$y_{14}=y_{23},\ \bar{y}_{14}=\bar{y}_{23}.$$

Lemma 4. Let $X=\left[x_{ij}\right]_{4\times 4}$, $\left[y_{ij}\right]_{4\times 4}=\bar{T}_2^{(4)}X\bar{T}_2^{(4)}$. If the conditions are met $x_{13}=x_{24}=0$, $x_{14}=x_{23}$, that $y_{14}=y_{23}$.

The invariant properties of the operators allow us to transform the symmetric matrix $A=\left[a_{ij}\right]_{4\times 4}$ into a persymmetric form and then use the diagonalization algorithm proposed in [3]. It should be noted, since the rotation operators are a direct sum of elementary rotation matrices that do not commute among themselves, and also, taking into account the invariant properties of the operators, the system of nonlinear equations for determining their parameters is a system with separated variables.

Thus, the eigenvalue of a symmetric matrix has a factorized structure in the form of a product of rotation operators, which implies its use in information processing problems.

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