ELECTRODYNAMICS, COMPLEX ROTATION GROUP, MEDIA

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I. INTRODUCTION

In 1931, Majorana and Oppenheimer proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein , he showed the possibility to have formulated Maxwell equation in term of complex 3-vector entities.

Silberstein writes that the complex form of Maxwell equations has been known before; he refers this to the second volume of the lecture notes on the differential equations of mathematical physics by Riemann that were edited and published by H. Weber in 1901.

II. MATRIX FORM IN THE VACUUM

Let us introduce 3-dimensional complex vector $\psi^k = E^k + icB^k$, then equations can be joint into the one matrix equation (see notations in [1])

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = J\,, \quad \Psi = \begin{vmatrix} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + icB^3 \end{vmatrix}, \quad J = 1\epsilon_0 \begin{vmatrix} j^0 \\ i\,j^1 \\ i\,j^2 \\ i\,j^3 \end{vmatrix},$$

$$(\alpha^i)^2 = -I, \quad \alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3\,, \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1\,, \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2\,.$$

III. MINKOWSKI ELECTRODYNAMICS FOR THE UNIFORM MEDIUM

Let us introduce the quantities with simple transformation properties under the Lorentz group:

$$f = E + icB\,, \quad h = 1\epsilon_0\,(D + iH/c)\,;$$

where f and h are complex 3-vectors under the group $SO(3.C)$, the latter is isomorphic to the Lorentz group. Let us introduce the new quantities

$$M = (h + f)/2\,, N = (h^* - f^*)/2\,, \quad M' = O\,M\,, \quad N' = O^*\,N\,.$$

they are different 3-vectors under the group $SO(3.C)$. In terms of the variables M and N, the Maxwell equations in the uniform medium read in the matrix form

$$(-i\partial_0 + \alpha^i\partial_i)\,M + (-i\partial_0 + \beta^i\partial_i)\,N = J\,, \quad M = \begin{vmatrix} 0 \\ M \end{vmatrix}, \quad N = \begin{vmatrix} 0 \\ N \end{vmatrix}, J = 1\epsilon_0\,\begin{vmatrix} \rho \\ i\,j \end{vmatrix},$$

where three additional matrices β^i are used.

IV. MINKOWSKI CONSTITUTIVE RELATIONS IN THE COMPLEX FORM

Let us examine how the constitutive relations for the uniform medium behave under the Lorentz transformations. We start with these relations in the rest reference frame

$$D = \epsilon_0\epsilon\,E\,, \quad Hc = 1\mu_0\mu 1c^2\,cB = \epsilon_0\mu\,cB\,,$$

whence it follows their different representation in terms of complex vectors

$$2h = (\epsilon + 1\mu)\,f + (\epsilon - 1\mu)\,f^*\,, \quad 2h^* = (\epsilon + 1\mu)\,f^* + (\epsilon - 1\mu)\,f\,;$$

$$2f = (1\epsilon + \mu)\,h + (1\epsilon - \mu)\,h^*\,, \quad 2f^* = (1\epsilon + \mu)\,h^* + (1\epsilon - \mu)\,h\,.$$

Let us apply the Lorentz transformations

$$f' = O\,f\,, \quad f'^* = O^*\,f^*\,, \quad h' = O\,h\,, \quad h'^* = O^*\,h^*\,,$$

then we get

$$2h' = (\epsilon + 1\mu)\,f' + (\epsilon - 1\mu)\,O(O^{-1})^*\,f'^*\,, \quad 2h'^* = (\epsilon + 1\mu)\,f'^* + (\epsilon - 1\mu)\,O^*O^{-1}\,f'\,.$$

$$2f' = (1\epsilon + \mu)\,h' + (1\epsilon - \mu)\,O(O^{-1})^*\,h'^*\,, \quad 2f'^* = (1\epsilon + \mu)\,h'^* + (1\epsilon - \mu)\,O^*O^{-1}\,h'\,.$$

In general, we notice evident modifications of the constitutive relations in the moving reference frame. In this point one should distinguish between two cases: Euclidean rotations and Lorentzian boosts. Indeed, for any Euclidean rotation we have identities $O^* = O \implies O(O^{-1})^* = I,\ O^*O^{-1} = I$; therefore in this case the constitutive relations preserve their form. However, for any pseudo-Euclidean rotation we have other identities

$$O^* = O^{-1} \quad \implies \quad O(O^{-1})^* = O^2, \quad O^*O^{-1} = O^{*2}\,;$$

therefore the constitutive relations are modified.

Extensions. The previous results can be extended to more generale media, let us restrict ourselves to linear media. Arbitrary linear media is characterized by the following constitutive equations:

$$D = \epsilon_0\,\epsilon(x)\,E + \epsilon_0 c\,\alpha(x)\,B\,, \quad H = \epsilon_0 c\,\beta(x)\,E + 1\mu_0\,\mu(x)\,B\,,$$

where $\epsilon(x), \mu(x), \alpha(x), \beta(x)$ are (3×3)-dimensionless matrices. These equations may be re-written in terms of complex vectors f, h:

$$h = [\,(\epsilon + \mu) + i(\beta - \alpha)\,]\, f + [\,(\epsilon - \mu) + i(\beta + \alpha)\,]\, f^{*}\,,$$

$$h^{*} = [\,(\epsilon + \mu) - i(\beta - \alpha)\,]\, f^{*} + [\,(\epsilon - \mu) - i(\beta + \alpha)\,]\, f\,.$$

For Euclidean rotations, the constitutive relations preserve their form. For Lorentzian boosts, however these relations change their form in the moving reference frame in accordance with the rules

$$h' = [\,(\epsilon + \mu) + i\,(\beta - \alpha)\,] f' + [\,(\epsilon - \mu) + i\,(\beta + \alpha)\,] O^{2} f'^{*}\,,$$

$$h'^{*} = [\,(\epsilon + \mu) - i\,(\beta - \alpha)\,] f'^{*} + [\,(\epsilon - \mu) - i\,(\beta + \alpha)\,] O^{*2} f'.$$

V. CONCLUSIONS

The matrix form of the Maxwell theory in the form of Riemann – Silberstein – Majorana – Oppenheimerand, based on the theory of complex rotation group $SO(3.C)$, may be effectively used in practical calculation when studying electromagnetic problems. This representation is closely related to spinor formalism in Maxwell theory (see in [2]).

REFERENCES

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