DELAUNAY VARIABLES IN MODEL PROBLEMS OF CELESTIAL MECHANICS AND COSMODYNAMICS

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I. INTRODUCTION

One of the most important model problems of celestial mechanics and cosmodynamics is the two-body problem, which describes the interaction of two-point mass, moving under the action of mutual gravitational attraction according to Newton's law of universal gravitation.

Integration of differential equations of motion in the two-body problem is reduced to quadratures. The trajectory of a point with mass m relative to a point with mass M is a conical section. From the point of view of applications, the most important case is the case of the Keplerian elliptic orbit. Whereas in unperturbed problem a point with mass m moves along an ellipse, to derive the equations of perturbed motion, the canonical Delaunay variables are used [1, 2], which in unperturbed problem are the "action-angle" variables.

The equations of motion in Delaunay variables are convenient to apply numerical and asymptotic research methods. This paper gives an example of using Delaunay variables in the model problem of motion of a satellite in the Earth's gravitational field, modeled by an axisymmetric rigid body compressed along the axis of rotation.

II. DISTURBED SATELLITE MOTION

Consider the problem of the motion of a satellite (point particle P) with mass m in the Earth's gravitational field. Introduce a geocentric inertial coordinate system (ISC) $OXYZ$, which origin is coinciding with the center of the Earth O, and the OZ axis is directed along the axis of rotation of the planet. As an unperturbed problem, consider the motion of satellite in the gravitational field of a planet, modeled by homogeneous solid full-sphere, rotating evenly around its axis. In this case gravitational potential of the planet coincide with gravitational potential of particle point with mass equal to the mass of the planet, and the satellite's orbit is Keplerian. Lets consider the Keplerian elliptical orbit as the orbit of unperturbed motion.

Let $\mathbf{R}$ denote the satellite radius vector and let $R = |\mathbf{R}|$. Introduce the following notation for the parameters of the satellite orbit: a −semi-major axis, e − eccentricity i − inclination (angle between plane OXY and plane of the satellite's orbit), h − longitude of ascending node (angle between the axis OX and the line OX_1 intersection of the plane of the satellite's orbit and the plane OXY), g− longitude of the periapsis π from the ascending node OX_1, (π – the closest to the center of the Earth O point of the satellite's orbit), ϑ− true anomaly. In unperturbed motion, the parameters a, e, i, h, g are constant values, true anomaly ϑ is time function [1,2]:

$$\dot{\vartheta} = n(1 + e\cos\vartheta)^2/(1 - e^2)^{3/2}.$$

In this formula, n – is the mean motion of satellite on the orbit, which is related with semi-major axis a by the equality $n = \sqrt{f_0/a^3}$, where $f_0 = fM$, f – universal gravitational constant, M – Earth's mass.

To describe the orbital motion of the satellite, using canonical Delaunay variables L, G, H, l, g, h [1,2]. Here g, h −previously entered parameters pf the satellite orbit. Variable l is mean anomaly. In unperturbed problem, mean anomaly is linear function of time: $l = n(t - t_\pi)$, where t −current time moment, t_π−moment of time, when satellite passage through the periapsis. The Delaunay variables L, G, H are related to the orbit parameters a, e, i by the equalities:

$$L = \sqrt{f_0 m^2 a}, \qquad G = \sqrt{f_0 m^2 a(1 - e^2)}, \qquad H = \sqrt{f_0 m^2 a(1 - e^2)}\cos i.$$

The module of satellite's radius vector R and true anomaly ϑ are the functions of the Delaunay variables L, G, l. The dependence of R, ϑ on these variables is implicit and expressed through the relations:

$$R = \frac{G^2}{f_0 m^2(1 + e\cos\vartheta)}, \qquad \cos w = \frac{e + \cos\vartheta}{1 + e\cos\vartheta}, \qquad l = w - e\sin w.$$

In these formulas, w – eccentric anomaly.

The Hamiltonian of the unperturbed problem depends on only one Delaunay variable L:

$$\mathcal{H}_0 = -\frac{f_0^2 m^3}{2L^2}.$$

As a perturbing force function, consider the function [3,4]:

$$U_1 = -\frac{f_0 m J_2 r_0^2}{2R^3}(3\sin^2\varphi - 1),$$

where r_0 − average radius of the Earth, φ – geographic latitude of the satellite, $\sin\varphi = \sin i \sin(g + \vartheta)$, J_2 – second zonal harmonic (dimensionless coefficient, that is characterizing the contribution of nonspherical components, $J_2 = 1{,}0825 \cdot 10^{-3}$). The Hamiltonian of the problem, considering the indicated perturbation will take the form:

$$\mathcal{H} = \mathcal{H}_0(L) + \varepsilon\mathcal{H}_1(L, G, H, l, g, h),$$

$$\varepsilon\mathcal{H}_1(L, G, H, l, g, h) = \frac{f_0 m J_2 r_0^2}{2R^3}(3\sin^2 i \sin^2(g + \vartheta) - 1)$$

And the equations of the disturbed motion will be written in the form:

$$\dot{L} = -\varepsilon\frac{\partial\mathcal{H}_1}{\partial l}, \dot{G} = -\varepsilon\frac{\partial\mathcal{H}_1}{\partial g}, \dot{H} = -\varepsilon\frac{\partial\mathcal{H}_1}{\partial h}, \dot{l} = n + \varepsilon\frac{\partial\mathcal{H}_1}{\partial L}, \dot{g} = \varepsilon\frac{\partial\mathcal{H}_1}{\partial G}, \dot{h} = \varepsilon\frac{\partial\mathcal{H}_1}{\partial H}, \qquad n = \frac{f_0^2 m^3}{L^3} \quad (1)$$

Based on system (1), can get the system of ordinary differential equations of the 6th order in dimensionless variables $n_0, e, i, \beta, \vartheta, h$, where $\beta = g + \vartheta$, $n_0 = n/\omega$, ω − module of the angular velocity of the Earth's rotation. This system of equations has the form:

$$\dot{n}_0 = -\frac{3\varepsilon_2 n_0^{10/3} p^4}{q^9}\{\Phi e\sin\vartheta - p\sin^2 i \sin 2\beta\},$$

$$\dot{e} = \frac{\varepsilon_2 n_0^{7/3} p^3}{q^7}\{p\Phi\sin\vartheta - \sin^2 i \sin 2\beta\,(2\cos\vartheta + e + e\cos^2\vartheta)\},$$

$$\frac{di}{dt} = -\frac{\varepsilon_2 n_0^{7/3} p^3}{2q^7}\sin 2i \sin 2\beta,$$

$$\dot{\beta} = \frac{p^2 n_0 \omega}{q^3} + \frac{2\varepsilon_2 n_0^{7/3} p^3}{q^7}\cos^2 i \sin^2\beta,$$

$$\dot{\vartheta} = \frac{p^2 n_0 \omega}{q^3} + \frac{\varepsilon_2 n_0^{7/3} p^3}{q^7 e}\{p\Phi\cos\vartheta + \sin^2 i \sin 2\beta\,(2 + e\cos\vartheta)\sin\vartheta\},$$

$$\dot{h} = -\frac{2\varepsilon_2 n_0^{7/3} p^3}{q^7}\cos i \sin^2 \beta.$$

Here $\varepsilon_2 = 1.5 J_2 r_0^2 \omega^{7/3} f_0^{-2/3}$, $\Phi = 3\sin^2 i \sin^2 \beta - 1, \beta = g + \vartheta$, $p = 1 + e\cos\vartheta, q = \sqrt{1 - e^2}$.

III. CONCLUSIONS

Based on the system of equations of the perturbed satellite motion in the canonical Delaunay variables, a system of ordinary differential equations of the 6th order is obtained, which described the satellite's orbital motion in the Earth's gravitational field, considering its compression along the axis of rotation. The specified system of equations can be used in mathematical modeling in problems related to remote sensing of the Earth.

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