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MINIMIZATION OF BOOLEAN FUNCTIONS IN THE CLASS OF ORTHOGONAL DISJUNCTIVE NORMAL FORMS

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The orthogonal disjunctive normal forms (DNFs) of Boolean functions have wide applications in the logical design of discrete devices [1–4]. The problem of DNF orthogonalization is to get DNF for a given function such that any two its terms would be orthogonal, i. e. the conjunction of them would be equal identically to zero. Orthogonalization of a system of Boolean functions is a more general problem that is set as follows. Let a system of completely specified Boolean functions $F = \{f_1, f_2, \ldots, f_m\}$ be given, and a family of mutually orthogonal functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ must be get such that any $f_i \in F$ can be expressed by a disjunction of some of the functions φ_j, $j = 1, 2, \ldots, N$, and N must be minimum. Here, orthogonality means that $\varphi_j \wedge \varphi_k = 0$ at each values of the arguments. Such a statement of the problem was in [5]. The orthogonalization of a DNF can be related as a particular case of the orthogonalization of a system of Boolean functions where every function is represented by an elementary conjunction.

The problem of DNF orthogonalization is of non-polynomial complexity [5]. A number of heuristic methods were described [3, 6–10]. Here, an approach to the problem of minimization of an orthogonal DNF is suggested that provides an exact solution using the means of graph theory.

The perfect DNF as a specification of a given function must be provided; i. e. the representation of the DNF is a binary matrix whose rows represent sets of values of the arguments where the function has value 1. Using the operation of partial merging on binary vector-rows all the intervals of the space of arguments are obtained where the function has value 1. The process of obtaining intervals is in the classical Quine – McCluskey method for minimization of DNF [11]. That method considers only maximal intervals, i. e. those ones, which are not proper subsets of other intervals. The problem of DNF minimization is reduced to finding a shortest cover of elements of Boolean space with the maximal intervals. In the case of orthogonal DNF minimization, all the intervals should be taken into consideration, and a cover whose elements do not intersect must be found.

The problem can be set in terms of graph theory as follows. The intersection relation between intervals including one-element intervals is represented by undirected graph whose vertices correspond to intervals. Its two vertices are connected with an edge if and only if the corresponding intervals intersect. This graph is called *graph of interval intersection*.

A *dominating independent set* in the graph corresponds to an orthogonal DNF. A set of vertices in a graph is independent if no two vertices in the set are connected with an edge. A dominating set in a graph is a set of vertices such that if any vertex does not belong to the set, it is adjacent to a vertex in the set [12]. So, a dominating independent set is a dominating set with the property of independency. A minimal dominating independent set of graph of interval intersection correspond to a minimal orthogonal DNF.

Any independent set in a graph of interval intersection corresponds to an orthogonal DNF. An independent set is maximal in a graph if it is not a proper subset of another independent set. A minimal orthogonal DNF corresponds to an independent set with minimal cardinality in the graph of interval intersection. Such a set is called least maximal independent set. Note that any maximal independent set is a dominating set. Indeed, suppose that a vertex in a graph is not in some its maximal independent set S. Then this vertex is adjacent to at least one vertex of S. So, a least dominating independent set is a least maximal independent set as well. Methods for finding both dominating and independent sets can be used in minimization of orthogonal DNF.

A dominating set of a graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E is obtained by solving the shortest cover problem. The set V is covered with its subsets $N[v_i] = \{v_i\} \cup N(v_i)$ where $N(v_i)$ is the neighborhood of vertex v_i. When a dominating independent set is being obtained, the demand of non-intersecting subsets is added. A method for shortest covering a set with its non-intersecting subsets is suggested.

A method for finding a least maximal independent set in a graph of interval intersection is suggested based on lexicographic enumeration of independent sets. An important part of this method is a special ordering of the vertices. The vertices are ordered relative to non-increasing of cardinalities of the intervals corresponding to the vertices. The vertices with equal values of this parameter are ordered relative to non-decreasing of their degrees. A heuristic method can be formed where the process of lexicographic enumeration comes to the end after obtaining the first maximal independent set in that order.

The described approach is extended to incompletely specified (partial) Boolean functions. Minimization of an incompletely specified Boolean function means obtaining minimal DNF realizing the given function, i. e.

DNF taking the same values as the function does everywhere they are defined. Usually, a partial Boolean function is given by two sets, M^1 and M^0, domains where the function has values 1 and 0, respectively. A method for minimization of partial Boolean functions in the class of orthogonal DNF is suggested. The method uses the concept of interval-covered set [6]. An interval-covered set is a subset of M^1 such that there is an interval that contains all the elements of the subset and does not intersect M^0.

The intervals related to interval-covered sets and the graph of intersection of those intervals are considered. Obtaining a minimal orthogonal DNF is reduced to finding a least dominating independent set or a least maximal independent set in the graph. If an interval-covered set M_j^1 is a proper subset of an interval-covered set M_i^1 and the intervals related to them coincide, then M_j^1 is excluded from the consideration. The obtained DNF must be simplified by widening the intervals if it is possible. At that, naturally, each interval must not intersect M^0 and other intervals in the solutions.

References

1. Synthesis of Asynchronous Automata in a Computer / A. D. Zakrevskij [et al.]. – Minsk : Navuka i tekhnika, 1975. – 184 p.

2. Bibilo, P. N. Synthesis of Combinational PLA-structures for VLSI / P. N. Bibilo. – Minsk : Navuka i tekhnika, 1992. – 323 p.

3. Zakrevskij, A. D. Calculation of Engineering System Reliability when the Critical Set of Events is Given / A. D. Zakrevskij, Yu. V. Pottosin // Logical Design. – Minsk : Inst. of Engineering Cybernetics AS of Belarus, 1996. – P. 123–131.

4. Matrosova, A. Yu. On Probabilistic Simulation of Discrete Devices / A. Yu. Matrosova // Avtomatika i telemekhanika. – 1995. – No. 3. – P. 156–154.

5. Kuznetsov, O. P. Orthogonal Systems of Boolean Functions and Their Applications in Analysis and Synthesis of Logical Networks / O. P. Kuznetsov // Avtomatika i telemekhanika. – 1970. – No. 10. – P. 117–128.

6. Zakrevskij, A. D. Optimization in Boolean Space / A. D. Zakrevskij, Yu. V. Pottosin, L. D. Cheremisinova; ed. A. Keevallik. – Tallinn , 2009. – 241 p.

7. Gavrilov, M. A. Logical Design of Discrete Automata (Languages, Methods, Algorithms) / M. A. Gavrilov, V. V. Devjatkov, E. I. Pupyrev. – M. : Nauka, 1977. – 352 p.

8. Bibilo, P. N. Synthesis of Combinational Circuits by the Method of Functional Decomposition / P. N. Bibilo, S. V. Enin. – Minsk : Navuka i tekhnika, 1987. – 189 p.

9. Kardash, S. N. Orthogonalization of a DNF System of Boolean Functions / S. N. Kardash // Informational Technologies and Systems 2020 (ITS 2020) : Proceedings of Intern. Scientific Conf., Republic of Belarus, Minsk, November 18, 2020. – Minsk : BSUIR, 2020. – P. 41–42.

10. Pottosin, Yu. V. Orthogonalization of a System of Completely Specified Boolean Functions / Yu. V. Pottosin, E. A. Shestakov // Logical Design. – Minsk : Inst. of Engineering Cybernetics AS of Belarus, 2000. –P. 107–115.

11. McCluskey, E. Minimization of Boolean Functions / E. McCluskey // Bell System Technical Journal. – 1956. – Vol. 35. – No. 6. – P. 1417–1444.

12. Lectures on Graph Theory / V. A. Emelichev [et al.]. – M. : Nauka, 1990. – 384 p.