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ON THE MATRIX EQUATION FOR A SPIN 2 PARTICLE IN PSEUDO-RIEMANNIAN SPACE-TIME

After the study by Pauli and Fierz [1, 2], the theory of massive and massless fields with spin 2 has always attracted much attention [3–13]. Most of the studies were performed in the framework of 2-nd order differential equations. It is known that many specific difficulties may be avoided if from the very beginning we start with 1-st order systems. Apparently, the first systematic study of the theory of spin 2 fields within the first order formalism was done by F. I. Fedorov [4]. It turns out that this description requires a field function with 3 independent components. This theory was re-discovered and improved by Regee [5]. In the present paper we develop the theory of the spin 2 field, in both massive and massless variants, starting from the matrix equation in Minkowski space-time and extending it to the generally covariant theory within the Tetrode-Weyl-Fock-Ivanenko tetrad method.

We start with the known system of the first order equations for a massive spin 2 particle:

$$\partial^a \Phi_a = m\Phi, \quad \frac{1}{2}\partial_a \Phi - \frac{1}{3}\partial^b \Phi_{(ab)} = m\Phi_a,$$
$$\frac{1}{2}\left(\partial^k \Phi_{[ka]b} + \partial^k \Phi_{[kb]a} - \frac{1}{2} g_{ab}\partial^k \Phi_{[kn]}{}^{n}\right) + \left(\partial_a \Phi_b + \partial_b \Phi_a - \frac{1}{2} g_{ab}\partial^k \Phi_k\right) = m\Phi_{(ab)},$$
$$\partial_a \Phi_{(bc)} - \partial_b \Phi_{(ac)} + \frac{1}{3}\left(g_{bc}\partial^k \Phi_{(ak)} - g_{ac}\partial^k \Phi_{(bk)}\right) = m\Phi_{[ab]c}, \qquad (1)$$

where the field variables are scalar, vector, symmetric 2-rank tensor, and 3-rank skew-symmetric in two first indices tensor, $m = \mathrm{i}M$. By excluding the vector and the 3-rank tensor, we obtain the 2-nd order equations with respect to the scalar and symmetric tensor:

$$\Phi = 0, \quad (\Box + M^2)\Phi_{(ab)} = 0, \quad \Phi_{(ab)} = \Phi_{(ba)}, \quad \Phi^a{}_a = 0, \quad \partial^k \Phi_{(ka)} = 0. \qquad (2)$$

In massless case, the first order system reads

$$\partial^a \Phi_a = 0, \frac{1}{2}\partial_a \Phi - \frac{1}{3}\partial^b \Phi_{(ab)} = \Phi_a,$$

$$\frac{1}{2}\left(\partial^k \Phi_{[ka]b} + \partial^k \Phi_{[kb]a} - \frac{1}{2} g_{ab} \partial^k \Phi_{[kn]}{}^{n}\right) + \left(\partial_a \Phi_b + \partial_b \Phi_a - \frac{1}{2} g_{ab} \partial^k \Phi_k\right) = 0,$$

$$\partial_a \Phi_{(bc)} - \partial_b \Phi_{(ac)} + \frac{1}{3}\left(g_{bc} \partial^k \Phi_{(ak)} - g_{ac} \partial^k \Phi_{(bk)}\right) = \Phi_{[ab]c}. \quad (3)$$

From (3) we derive the 2-nd order equations for the massless field:

$$\frac{1}{2} \Box \Phi - \frac{1}{3} \partial^k \partial^l \Phi_{(kl)} = 0,$$

$$(\partial_a \partial_b + \frac{1}{2} g_{ab} \Box)\Phi - \frac{1}{4} g_{ab} \Box \Phi_c^c + \Box \Phi_{(ab)} - \partial_a \partial^l \Phi_{(bl)} - \partial_b \partial^l \Phi_{(al)} = 0. \quad (4)$$

Massless equations have a class of gauge solutions:

$$\bar{\Phi} = \partial^l L_l, \quad \bar{\Phi}_{(ab)} = \partial_a L_b + \partial_b L_a - \frac{1}{2} g_{ab} \partial^l L_l, \quad (5)$$

where $L_l(x)$ stands for an arbitrary 4-vector. These special states do not contribute to physically observable quantities, like the energy-momentum tensor. The concomitant gauge components are as follows:

$$\bar{\Phi}_a = \frac{1}{3}\partial_a \partial^l L_l - \frac{1}{3} L_a, \bar{\Phi}_{[ab]c} = \partial_c (\partial_a L_b - \partial_b L_a) - \frac{g_{cb}\partial_a - g_{ca}\partial_b}{3} \partial^l L_l + \frac{g_{cb} L_a - g_{ca} L_b}{3}. \quad (6)$$

The system (1) can be re-written in equivalent block form

$$\partial_a (G^a)_{(0)}{}^{k} \Phi_k = m\Phi_{(0)}, \quad \partial_a \left\{ \frac{1}{2} (\Delta^a)_k{}^{(0)} \Phi_{(0)} - \frac{1}{3} (K^a)_k{}^{(mn)} \Phi_{mn} \right\} = m\, \Phi_k \ ,$$

$$\partial_a \left\{ \frac{1}{2} (B^a)_{(cd)}{}^{[mn]l} \Phi_{mnl} + (\Lambda^a)_{(dc)}{}^{k} \Phi_k \right\} = m\Phi_{dc} \ , \partial_a \left\{ (F^a)_{[kb]c}{}^{(mn)} \Phi_{mn} \right\} = m\, \Phi_{kbc}, \quad (7)$$

The corresponding matrix equation

$$(\Gamma^a \frac{\partial}{\partial x^a} - m) \Psi(x) = 0, \quad \Psi = \{H; H_1; H_2; H_3\} \quad (8)$$

is extended to the Riemannian space-time in accordance with the tetrad method. In a space-time with given metric, we fix a tetrad:

$$dS^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta, \quad g_{\alpha\beta}(x) \to e_{(a)\alpha}(x), g_{\alpha\beta}(x) = \eta^{ab} e_{(a)\alpha}(x) e_{(b)\beta}(x), \quad (9)$$

and then the generalized form gets written as follows

$$\left[\Gamma^\alpha(x)\left(\frac{\partial}{\partial x^\alpha} + \Sigma_\alpha(x)\right) - m\right]\Psi(x) = 0, \quad (10)$$

where the local matrices $\Gamma^\alpha(x)$ are determined with the use of the tetrad

$$\Gamma^\alpha(x) = e^\alpha_{(a)}(x)\Gamma^a = \begin{pmatrix} 0 & G^\alpha(x) & 0 & 0 \\ \frac{1}{2}\Delta^\alpha(x) & 0 & -\frac{1}{3}K^\alpha(x) & 0 \\ 0 & \Lambda^\alpha(x) & 0 & \frac{1}{2}B^\alpha(x) \\ 0 & 0 & F^\alpha(x) & 0 \end{pmatrix}, \quad (11)$$

and connection $\Sigma_\alpha(x)$ is defined by relations

$$\Sigma_\alpha(x) = J^{ab} e^\beta_{(a)}(x)\, e_{(b)\beta;\alpha}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\Sigma_1)_\alpha & 0 & 0 \\ 0 & 0 & (\Sigma_2)_\alpha & 0 \\ 0 & 0 & 0 & (\Sigma_3)_\alpha \end{pmatrix}, \quad (12)$$

where $\Sigma_i(x) = J_i^{ab}\, e^\beta_{(a)}(x)\, e_{(b)\beta;\alpha}(x), \quad i = 1,2,3;$ and $J_1^{ab}, J_2^{ab}, J_3^{ab}$ stand for the generators for the tensors $\Phi_k, \Phi_{(mn)}, \Phi_{[mn]l}$. The equation (10) can be presented by using the Ricci rotation coefficients

$$\left[\Gamma^c\left(e^\alpha_{(c)}(x)\frac{\partial}{\partial x^\alpha} + \frac{1}{2}J^{ab}\gamma_{abc}\right) - m\right]\Psi(x) = 0. \quad (13)$$

In block form, eq. (13) reads

$$G^{\alpha}(x)[\partial_{\alpha}+(\Sigma_1)_{\alpha}]H_1 = mH, \quad \frac{1}{2}\Delta^{\alpha}(x)\partial_{\alpha}H - \frac{1}{3}K^{\alpha}(x)[\partial_{\alpha}+(\Sigma_2)_{\alpha}]H_2 = mH_1,$$
$$\Lambda^{\alpha}(x)[\partial_{\alpha}+(\Sigma_1)_{\alpha}]H_1 + \frac{1}{2}[\partial_{\alpha}+(\Sigma_3)_{\alpha}]H_3 = mH_2, \quad F^{\alpha}(x)[\partial_{\alpha}+(\Sigma_2)_{\alpha}]H_2 = mH_3.$$

In the massless case, the system slightly changes:

$$G^{\alpha}(x)[\partial_{\alpha}+(\Sigma_1)_{\alpha}]H_1 = 0, \quad \frac{1}{2}\Delta^{\alpha}(x)\partial_{\alpha}H - \frac{1}{3}K^{\alpha}(x)[\partial_{\alpha}+(\Sigma_2)_{\alpha}]H_2 = H_1,$$
$$\Lambda^{\alpha}(x)[\partial_{\alpha}+(\Sigma_1)_{\alpha}]H_1 + \frac{1}{2}[\partial_{\alpha}+(\Sigma_3)_{\alpha}]H_3 = 0, \quad F^{\alpha}(x)[\partial_{\alpha}+(\Sigma_2)_{\alpha}]H_2 = H_3,$$

but its physical content is completely different. In particular, let us detail tetrad representation for the gauge solutions:

$$\bar{\Phi} = \nabla_{\alpha}L^{\alpha}(x) \quad \Rightarrow \quad \bar{\Phi} = e^{(c)\alpha}\partial_{\alpha}L_{(c)} + e^{\alpha}_{(c);\alpha}L^{(c)},$$
$$\bar{\Phi}_{(\alpha\beta)} = \nabla_{\alpha}L_{\beta} + \nabla_{\beta}L_{\alpha} - \frac{1}{2}g_{\alpha\beta}(x)\nabla_{\rho}\Lambda^{\rho} \quad \Rightarrow$$
$$\bar{\Phi}_{(ab)} = -\left(\gamma_{[ca]b} + \gamma_{[cb]a}\right)L^{(c)} + e^{\alpha}_{(a)}\partial_{\alpha}\Lambda_{(b)} + e^{\alpha}_{(b)}\partial_{\alpha}\Lambda_{(a)} - \frac{1}{2}g_{ab}\bar{\Phi}. \qquad (14)$$

The concomitant gauge components are determined by the formulas

$$\bar{H}_1 = \frac{1}{2}\Delta^{\alpha}(x)\partial_{\alpha}\bar{H} - \frac{1}{3}K^{\alpha}(x)[\partial_{\alpha}+(\Sigma_2)_{\alpha}]\bar{H}_2, \quad \bar{H}_3 = F^{\alpha}(x)\left[\partial_{\alpha}+(\Sigma_2)_{\alpha}\right]\bar{H}_2. \qquad (15)$$

The covariant equation is symmetric under the local Lorentz group, in accordance with the following relations

$$\Psi'(x) = S(x)\Psi(x), \quad S(x)\Gamma^{\alpha}(x)S^{-1}(x) = \Gamma'^{\alpha}(x),$$
$$S(x)\Sigma_{\alpha}(x)S^{-1}(x) + S(x)\frac{\partial}{\partial x^{\alpha}}S^{-1}(x) = \Sigma'_{\alpha}, \qquad (16)$$

where the prime indicates that quantities are determined with the use of the primed tetrad related to initial one by the local Lorentz transformation $e^{\sigma}_{(a')}(x) = L_a{}^b(x)\, e^{\sigma}_{(b)}(x)$. With respect to the coordinate transformation, the field function Ψ behaves as a scalar, $x^{\alpha} \to x'^{\alpha}$, $\Psi(x) = \Psi'(x')$.

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