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MASSLESS SPIN 2 PARTICLE: CYLINDRICAL SYMMETRY, PROJECTIVE OPERATORS, GAUGE DEGREES OF FREEDOM

(Communicated by Corresponding Member Dmitry S. Mogilevtsev)

Abstract. In the present paper, we have developed the theory of a massless spin 2 particle. We apply the matrix equation in Minkowski space-time, specifying it in cylindrical coordinates t, r, ϕ, z and tetrad. By diagonalizing energy operators, the third projection of total angular momentum, and the third projection of linear momentum, we derive the system of 39 differential equations in a polar coordinate *r*. In order to resolve this system, we apply the Fedorov–Gronskiy method based on the projective operator method. In accordance with this method, the dependence of all 39 functions is determined only by five different functions of the polar variable *r* that in the considered case are expressed in terms of Bessel functions. We find the explicit form of six independent solutions of the basic matrix equation. In order to eliminate gauge degrees of freedom, we use the general structure of gauge solutions according to the Pauli–Fierz approach, when the gauge solutions for the spin 2 field are constructed on the basis of the exact solution for a massless spin 1 field (in Bessel functions as well). In this way, we find the explicit form of two independent gauge solutions for the spin 2 field. In the end, we derive the explicit form of two gauge-free solutions for the massless spin 2 field, as should be expected by physical reason.

Keywords: massless particles, spin 2, matrix Fedorov–Regge equation, cylindrical symmetry, projective operators, exact solutions, Bessel functions, Pauli–Fierz theory, gauge degrees of fredom

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БЕЗМАССОВАЯ ЧАСТИЦА СО СПИНОМ 2: ЦИЛИНДРИЧЕСКАЯ СИММЕТРИЯ, ПРОЕКТИВНЫЕ ОПЕРАТОРЫ, КАЛИБРОВОЧНЫЕ СТЕПЕНИ СВОБОДЫ

(Представлено членом-корреспондентом Д. С. Могилевцевым)

Аннотация. Исследуется безмассовая частица со спином 2, при этом применяется матричное уравнение в цилиндрической тетраде пространства Минковского. На решениях диагонализируются операторы энергии, третьей проекции полного момента и третьей проекции импульса; после разделения переменных получена система из 39 уравнений по полярной координате *r*. Для нахождения решений этих уравнений используется метод Федорова–Гронского, основанный на теории проективных операторов. В соответствии с этим 39 функций выражаются через пять основных функций от переменной *r*, строящихся в терминах функций Бесселя. Найден явный вид шести независимых решений. Чтобы исключить калибровочные степени свободы, используется явный вид четырех калибровочных решений, строящихся согласно теории Паули–Фирца на основе точных решений уравнения для безмассовой частицы со спином 1 в цилиндрических координатах. После исключения из шести решений четырех калибровочных найдены явные выражения для двух независимых решений, не содержащих калибровочных степеней свободы.

Ключевые слова: безмассовая частица, спин 2, матричное уравнение Федорова–Редже, цилиндрическая симметрия, проективные операторы, точные решения, функции Бесселя, теория Паули–Фирца, калибровочные степени свободы

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Introduction. After the study by Pauli and Fierz [1–4] the theory of massive and massless fields with spin 2 has always attracted much attention [5–20]. Most of the studies were performed in the framework of 2-nd order differential equations. It is known that many specific difficulties may be avoided if from the very beginning we start with 1-st order systems. Apparently, the first systematic study of the theory of spin 2 fields within the first order formalism was done by F. I. Fedorov [5]. It turns out that this description requires a field function with 3 independent components. This theory was re-discovered and improved by Regee [6]. The first order approach is based from the very beginning on the general theory of relativistic wave equations by the Gel'fand–Yaglom [3] and the Lagrangian formalisms.

In the present paper we develop the theory of the massless spin 2 particle. We apply the matrix equation in Minkowski space-time, specifying it in cylindrical coordinates t, r, ϕ, z and tetrad. By diagonalizing the operators of the energy, the third projection of the total angular momentum, and the third projection of the linear momentum, we derive the system of 39 differential equations in polar coordinate r. In order to resolve this system, we apply the method by Fedorov–Gronskiy [7] based on the projective operator method. In accordance with this method, dependance of all 39 functions is determined by only five different functions of the polar variable r, which in the case under consideration are expressed in Bessel functions. We find in explicit form four independent solutions of the basic matrix equation. In order to eliminate the gauge degrees of freedom we use the general structure of gauge solutions according to Pauli–Fierz approach, when the gauge solutions for spin 2 field are constructed on the based of exact solution for massless spin 1 field (in Bessel functions as well). In this way, we find explicit form of two independent gauge solutions for spin 2 field. In the end, we derive explicit form of two gauge-free solutions for massless spin 2 field, as should be expected by physical reason.

The system of differential equations and projective operators. Let us start with the known system of differential equations which describes the spin 2 particle in cylindrical coordinates (we use results obtained in [20] for massive spin 2 particle, but now specified for the massless case). First we write down the first equation related to the scalar field:

$$\frac{1}{\sqrt{2}} a_{m-1} h_1 \varphi_2 - \frac{1}{\sqrt{2}} b_{m+1} h_3 \varphi_3 - i\varepsilon h_0 \varphi_1 - ikh_2 \varphi_1 = 0.$$

Remaining equations are collected in five groups (we preserve notations from [20])

V_1, S_1, P_1

$$-\sqrt{2} b_{m-1} d_1 \varphi_2 + \sqrt{2} a_{m+1} d_3 \varphi_3 - 3i\varepsilon h(r) + 2ikd_2 \varphi_1 + 2i\varepsilon f_0 \varphi_1 = 6h_0 \varphi_1,$$

$$\sqrt{2} a_{m+1} c_1 \varphi_3 - \sqrt{2} b_{m-1} c_3 \varphi_2 + 3ikh(r) + 2i\varepsilon d_2 \varphi_1 + 2ikf_2 \varphi_1 = 6h_2 \varphi_1,$$

$$\begin{gathered} 3\sqrt{2} a_{m+1} B_{12} \varphi_3 - \sqrt{2} b_{m-1} B_{21} \varphi_2 - \sqrt{2} a_{m+1} B_{23} \varphi_3 + 3\sqrt{2} b_{m-1} B_{32} \varphi_2 - \sqrt{2} b_{m-1} E_{10} \varphi_2 + \\ + \sqrt{2} a_{m+1} E_{30} \varphi_3 + 2\sqrt{2} b_{m-1} h_1 \varphi_2 - 2\sqrt{2} a_{m+1} h_3 \varphi_3 - \\ -2ikB_{11}\varphi_1 - 2i\varepsilon E_{13}\varphi_1 - 6i\varepsilon E_{22}\varphi_1 - 2i\varepsilon E_{31}\varphi_1 - 4i\varepsilon h_0\varphi_1 + 2ikB_{33}\varphi_1 + 2ikE_{20}\varphi_1 + 12ikh_2\varphi_1 = 0, \end{gathered}$$

$$\begin{gathered} \sqrt{2} a_{m+1} B_{12} \varphi_3 - \sqrt{2} b_{m-1} B_{21} \varphi_2 - \sqrt{2} a_{m+1} B_{23} \varphi_3 + \\ + \sqrt{2} b_{m-1} B_{32} \varphi_2 + \sqrt{2} b_{m-1} E_{10} \varphi_2 - \sqrt{2} a_{m+1} E_{30} \varphi_3 + \\ + 2\sqrt{2} b_{m-1} h_1 \varphi_2 - 2\sqrt{2} a_{m+1} h_3 \varphi_3 - \end{gathered}$$

$$-2i\varepsilon E_{13}\varphi_1 - 2i\varepsilon E_{22}\varphi_1 - 2i\varepsilon E_{31}\varphi_1 + 4i\varepsilon h_0\varphi_1 - 2ikB_{11}\varphi_1 + 2ikB_{33}\varphi_1 - 2ikE_{20}\varphi_1 + 4ikh_2\varphi_1 = 0,$$

$$\sqrt{2}a_{m+1}B_{10}\varphi_3 + \sqrt{2}b_{m-1}B_{30}\varphi_2 + \sqrt{2}a_{m+1}E_{32}\varphi_3 - \sqrt{2}b_{m-1}E_{12}\varphi_2 +$$
$$+ 2ikE_{22}\varphi_1 + 4ikh_0\varphi_1 - 2i\varepsilon E_{20}\varphi_1 - 4i\varepsilon h_2\varphi_1 = 0,$$

$$\sqrt{2}a_{m+1}B_{12}\varphi_3 + \sqrt{2}b_{m-1}B_{21}\varphi_2 + \sqrt{2}a_{m+1}B_{23}\varphi_3 +$$
$$+\sqrt{2}b_{m-1}B_{32}\varphi_2 - 3\sqrt{2}b_{m-1}E_{10}\varphi_2 + 3\sqrt{2}a_{m+1}E_{30}\varphi_3 -$$
$$-2\sqrt{2}b_{m-1}h_1\varphi_2 + 2\sqrt{2}a_{m+1}h_3\varphi_3 +$$
$$+2i\varepsilon E_{13}\varphi_1 - 2i\varepsilon E_{22}\varphi_1 + 2i\varepsilon E_{31}\varphi_1 - 12i\varepsilon h_0\varphi_1 + 2ikB_{11}\varphi_1 - 2ikB_{33}\varphi_1 + 6ikE_{20}\varphi_1 + 4ikh_2\varphi_1 = 0,$$

$$\frac{1}{3\sqrt{2}}a_{m+1}c_1\varphi_3 - \frac{1}{3\sqrt{2}}b_{m-1}c_3\varphi_2 - \frac{2}{3}i\varepsilon d_2\varphi_1 - ikf_0\varphi_1 + \frac{1}{3}ikf_2\varphi_1 - E_{20}\varphi_1 = 0,$$

$$\frac{1}{3\sqrt{2}}a_{m+1}c_1\varphi_3 + \frac{1}{3}\sqrt{2}b_{m-1}c_3\varphi_2 + \frac{1}{3}i\varepsilon d_2\varphi_1 + \frac{1}{3}ikf_2\varphi_1 - ikc_2\varphi_1 + B_{11}\varphi_1 = 0,$$

$$\frac{1}{3}\sqrt{2}b_{m-1}d_1\varphi_2 + \frac{1}{3\sqrt{2}}a_{m+1}d_3\varphi_3 + \frac{1}{3}i\varepsilon f_0\varphi_1 + \frac{1}{3}ikd_2\varphi_1 + i\varepsilon c_2\varphi_1 + E_{31}\varphi_1 = 0,$$

$$\frac{1}{3\sqrt{2}}b_{m-1}d_1\varphi_2 + \frac{1}{3}\sqrt{2}a_{m+1}d_3\varphi_3 - \frac{1}{3}i\varepsilon f_0\varphi_1 - \frac{1}{3}ikd_2\varphi_1 - i\varepsilon c_2\varphi_1 - E_{13}\varphi_1 = 0,$$

$$\frac{1}{3\sqrt{2}}b_{m-1}d_1\varphi_2 - \frac{1}{3\sqrt{2}}a_{m+1}d_3\varphi_3 - \frac{1}{3}i\varepsilon f_0\varphi_1 + i\varepsilon f_2\varphi_1 + \frac{2}{3}ikd_2\varphi_1 + E_{22}\varphi_1 = 0,$$

$$\frac{1}{3}\sqrt{2}a_{m+1}c_1\varphi_3 + \frac{1}{3\sqrt{2}}b_{m-1}c_3\varphi_2 - \frac{1}{3}i\varepsilon d_2\varphi_1 - \frac{1}{3}ikf_2\varphi_1 + ikc_2\varphi_1 + B_{33}\varphi_1 = 0,$$

$$\frac{1}{\sqrt{2}}a_{m+1}c_1\varphi_3 + \frac{1}{\sqrt{2}}b_{m-1}c_3\varphi_2 - B_{22}\varphi_1 = 0, \quad \frac{1}{\sqrt{2}}b_{m-1}d_1\varphi_2 + \frac{1}{\sqrt{2}}a_{m+1}d_3\varphi_3 - B_{20}\varphi_1 = 0;$$

V_2, S_2, P_2

$$\sqrt{2}a_m c_2\varphi_1 - \sqrt{2}b_{m-2}f_1\varphi_4 - \frac{3}{2}\sqrt{2}a_m h(r) + 2ikc_3\varphi_2 + 2i\varepsilon d_1\varphi_2 = 6h_1\varphi_2,$$

$$\sqrt{2}a_m B_{11}\varphi_1 - \sqrt{2}a_m B_{22}\varphi_1 + \sqrt{2}b_{m-2}B_{31}\varphi_4 - 2\sqrt{2}a_m h_2\varphi_1 -$$
$$-2i\varepsilon E_{12}\varphi_2 - 2i\varepsilon E_{21}\varphi_2 + 2ikB_{32}\varphi_2 + 4ikh_1\varphi_2 = 0,$$

$$-\sqrt{2}a_m B_{20}\varphi_1 - \sqrt{2}b_{m-2}E_{11}\varphi_4 + \sqrt{2}a_m E_{31}\varphi_1 - 2\sqrt{2}a_m h_0\varphi_1 + 2ikB_{30}\varphi_2 +$$
$$+2ikE_{21}\varphi_2 - 2i\varepsilon E_{10}\varphi_2 - 4i\varepsilon h_1\varphi_2 = 0,$$

$$\frac{1}{3\sqrt{2}}b_{m-2}f_1\varphi_4 - \frac{1}{3\sqrt{2}}a_m c_2\varphi_1 - \frac{1}{\sqrt{2}}a_m f_0\varphi_1 - \frac{1}{3}ikc_3\varphi_2 + \frac{2}{3}i\varepsilon d_1\varphi_2 + E_{10}\varphi_2 = 0,$$

$$\frac{1}{3\sqrt{2}}b_{m-2}f_1\varphi_4 - \frac{1}{3\sqrt{2}}a_m c_2\varphi_1 + \frac{1}{\sqrt{2}}a_m f_2\varphi_1 + \frac{2}{3}ikc_3\varphi_2 - \frac{1}{3}i\varepsilon d_1\varphi_2 + B_{32}\varphi_2 = 0,$$

$$\frac{2}{3}\sqrt{2}a_m c_2\varphi_1 + \frac{1}{3}\sqrt{2}b_{m-2}f_1\varphi_4 + \frac{1}{3}ikc_3\varphi_2 + \frac{1}{3}i\varepsilon d_1\varphi_2 - B_{21}\varphi_2 = 0,$$

$$\frac{1}{\sqrt{2}}a_m d_2\varphi_1 - i\varepsilon c_3\varphi_2 - ME_{12}\varphi_2 = 0, \quad \frac{1}{\sqrt{2}}a_m d_2\varphi_1 + ikd_1\varphi_2 + B_{30}\varphi_2 = 0,$$
$$i\varepsilon c_3\varphi_2 + ikd_1\varphi_2 + E_{21}\varphi_2 = 0;$$

$$\frac{3}{2}\sqrt{2}b_m h(r) - \sqrt{2}b_m c_2\varphi_1 + \sqrt{2}a_{m+2}f_3\varphi_5 + 2ikc_1\varphi_3 + 2i\varepsilon d_3\varphi_3 = 6h_3\varphi_3,$$

$$\sqrt{2}a_{m+2}B_{13}\varphi_5 - \sqrt{2}b_m B_{22}\varphi_1 + \sqrt{2}b_m B_{33}\varphi_1 + 2\sqrt{2}b_m h_2\varphi_1 -$$
$$-2i\varepsilon E_{23}\varphi_3 - 2i\varepsilon E_{32}\varphi_3 - 2ikB_{12}\varphi_3 + 4ikh_3\varphi_3 = 0,$$

$$-\sqrt{2}b_m B_{20}\varphi_1 - \sqrt{2}b_m E_{13}\varphi_1 + \sqrt{2}a_{m+2}E_{33}\varphi_5 + 2\sqrt{2}b_m h_0\varphi_1 -$$
$$-2ikB_{10}\varphi_3 + 2ikE_{23}\varphi_3 - 2i\varepsilon E_{30}\varphi_3 - 4i\varepsilon h_3\varphi_3 = 0;$$

V_3, S_3, P_3

$$\frac{1}{3\sqrt{2}}b_m c_2\varphi_1 - \frac{1}{3\sqrt{2}}a_{m+2}f_3\varphi_5 + \frac{1}{\sqrt{2}}b_m f_0\varphi_1 - \frac{1}{3}ikc_1\varphi_3 + \frac{2}{3}i\varepsilon d_3\varphi_3 + E_{30}\varphi_3 = 0,$$
$$\frac{1}{3\sqrt{2}}b_m c_2\varphi_1 - \frac{1}{3\sqrt{2}}a_{m+2}f_3\varphi_5 - \frac{1}{\sqrt{2}}b_m f_2\varphi_1 + \frac{2}{3}ikc_1\varphi_3 - \frac{1}{3}i\varepsilon d_3\varphi_3 - B_{12}\varphi_3 = 0,$$
$$\frac{2}{3}\sqrt{2}b_m c_2\varphi_1 + \frac{1}{3}\sqrt{2}a_{m+2}f_3\varphi_5 - \frac{1}{3}ikc_1\varphi_3 - \frac{1}{3}i\varepsilon d_3\varphi_3 - B_{23}\varphi_3 = 0,$$
$$\frac{1}{\sqrt{2}}b_m d_2\varphi_1 - ikd_3\varphi_3 + B_{10}\varphi_3 = 0, \quad \frac{1}{\sqrt{2}}b_m d_2\varphi_1 + i\varepsilon c_1\varphi_3 + E_{32}\varphi_3 = 0,$$
$$i\varepsilon c_1\varphi_3 + ikd_3\varphi_3 + E_{23}\varphi_3 = 0;$$

V_4, S_4, P_4

$$a_{m-1}B_{21}\varphi_2 + 2a_{m-1}h_1\varphi_2 - \sqrt{2}ikB_{31}\varphi_4 + \sqrt{2}i\varepsilon E_{11}\varphi_4 = 0,$$
$$\frac{1}{\sqrt{2}}a_{m-1}d_1\varphi_2 - i\varepsilon f_1\varphi_4 - E_{11}\varphi_4 = 0, \quad \frac{1}{\sqrt{2}}a_{m-1}c_3\varphi_2 + ikf_1\varphi_4 + B_{31}\varphi_4 = 0;$$

V_5, S_5, P_5

$$-b_{m+1}B_{23}\varphi_3 + 2b_{m+1}h_3\varphi_3 - \sqrt{2}ikB_{13}\varphi_5 - \sqrt{2}i\varepsilon E_{33}\varphi_5 = 0,$$
$$\frac{1}{\sqrt{2}}b_{m+1}d_3\varphi_3 + i\varepsilon f_3\varphi_5 + E_{33}\varphi_5 = 0, \quad \frac{1}{\sqrt{2}}b_{m+1}c_1\varphi_3 - ikf_3\varphi_5 + B_{13}\varphi_5 = 0,$$

where short notations for differential operators are used

$$a_m = \frac{d}{dr} + \frac{m}{r},\ a_{m+2} = \frac{d}{dr} + \frac{m+2}{r},\ a_{m+1} = \frac{d}{dr} + \frac{m+1}{r},\ a_{m-1} = \frac{d}{dr} + \frac{m-1}{r},$$
$$b_m = \frac{d}{dr} - \frac{m}{r},\ b_{m-2} = \frac{d}{dr} - \frac{m-2}{r},\ b_{m+1} = \frac{d}{dr} - \frac{m+1}{r},\ b_{m-1} = \frac{d}{dr} - \frac{m-1}{r}.$$

According to Fedorov–Gronskiy method [7], the above system should be consistent with the following differential constraints (they permit us to transform the above system to algebraic form)

$$b_{m-1}\varphi_2 = C_1\varphi_1,\ a_m\varphi_1 = C_3\varphi_2; \quad a_{m+1}\varphi_3 = C_2\varphi_1,\ b_m\varphi_1 = C_5\varphi_3;$$
$$b_{m-2}\varphi_4 = C_4\varphi_2,\ a_{m-1}\varphi_2 = C_7\varphi_4; \quad a_{m+2}\varphi_5 = C_6\varphi_3,\ b_{m+1}\varphi_3 = C_8\varphi_5;$$

we can assume identities $C_3 = C_1, C_5 = C_2, C_7 = C_4, C_8 = C_6$, they lead to equations for separate functions:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - X\right]\varphi_1 = 0,$$
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - X\right]\varphi_2 = 0, \quad \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - X\right]\varphi_3 = 0, \qquad (1)$$
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-2)^2}{r^2} - X\right]\varphi_4 = 0, \quad \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+2)^2}{r^2} - X\right]\varphi_5 = 0,$$

where $X = C_3^2 = C_5^2 = C_7^2 = C_8^2$. Solutions of these equations should be in agree with the gauge solutions. Because, the gauge solution are expressed in terms of Bessel functions of the argument $z = \sqrt{\varepsilon^2 - k^2}\ r$

(see below), in equations (1) we should set $X=-(\varepsilon^2-k^2)$. In this way, instead of (1) we get equations of Bessel type in the variable $z=\sqrt{\varepsilon^2-k^2}\, r$:

$$\varphi_1 = L_1 J_m(z),\ \varphi_2 = L_2 J_{m-1}(z),\ \varphi_3 = L_3 J_{m+1}(z),\ \varphi_4 = L_4 J_{m-2}(z),\ \varphi_5(z) = L_5 J_{m+2}(z). \qquad (2)$$

Eliminating the variables referring to the tensors of first and third ranks, we should get 13 equations for the variables $h, f_1, f_2, f_3,\ c_1, c_2, c_3,\ d_1, d_2, d_3, f_0$. After performing calculations in the resulting algebraic system, we find 6 independent solutions (they are presented in Bessel form):

$$h(r)=0, \qquad \Psi_1 = \left| \begin{array}{c} 0 \\ -J_{m-2} \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}J_m \\ 0 \\ -\frac{\sqrt{\varepsilon^2-k^2}}{2\sqrt{2}\,\varepsilon} iJ_{m-1} \\ 0 \\ \frac{\sqrt{\varepsilon^2-k^2}}{2\sqrt{2}\varepsilon} iJ_{m+1} \\ 0 \end{array} \right|, \quad \Psi_2 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ -J_{m+2} \\ 0 \\ \frac{1}{2}J_m \\ 0 \\ +\frac{\sqrt{\varepsilon^2-k^2}}{2\sqrt{2}\varepsilon} iJ_{m-1} \\ 0 \\ -\frac{\sqrt{\varepsilon^2-k^2}}{2\sqrt{2}\varepsilon} iJ_{m+1} \\ 0 \end{array} \right|, \quad \Psi_3 = \left| \begin{array}{c} 0 \\ 0 \\ -\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}k} J_m \\ 0 \\ iJ_{m+1} \\ \frac{k}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} J_m \\ 0 \\ +\frac{k}{2\varepsilon} iJ_{m-1} \\ 0 \\ -\frac{k}{2\varepsilon} iJ_{m+1} \\ 0 \end{array} \right|,$$

$$\Psi_4 = \left| \begin{array}{c} 0 \\ 0 \\ \frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}k} J_m \\ 0 \\ 0 \\ -\frac{k}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} J_m \\ -iJ_{m-1} \\ +\frac{k}{2\varepsilon} iJ_{m-1} \\ 0 \\ -\frac{k}{2\varepsilon} iJ_{m+1} \\ 0 \end{array} \right|, \quad \Psi_5 = \left| \begin{array}{c} 0 \\ 0 \\ -\frac{\varepsilon}{k} J_m \\ 0 \\ 0 \\ \frac{k\varepsilon}{k^2-\varepsilon^2} J_m \\ 0 \\ -\frac{k}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} iJ_{m-1} \\ J_m \\ -\frac{k}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} iJ_{m+1} \\ 0 \end{array} \right|, \quad \Psi_6 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\varepsilon^2}{k^2-\varepsilon^2} J_m \\ 0 \\ -\frac{\varepsilon}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} iJ_{m-1} \\ 0 \\ -\frac{\varepsilon}{\sqrt{2}\sqrt{\varepsilon^2-k^2}} iJ_{m+1} \\ J_m \end{array} \right|.$$

Gauge type solutions for massless spin 2 field. For massless spin 2 field there exist gauge solutions, which are defined through an arbitrary vector $L_c^C(x), c=0,1,2,3$; the symbol $C=(\text{Cart})$ designates Cartesian basis. In this way, the gauge scaler field $\overline{h}(x)$ is given by the formula

$$\bar{h}(x) = \bar{\Phi} = \partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right) L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C;$$

in turn, 10 components of the gauge symmetric tensor are given by the formula

$$\bar{f}_1 = \bar{\Phi}_{11} = 2\partial_r L_1^C + \frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right) L_1^C - \frac{}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

$$\bar{f}_2 = \bar{\Phi}_{22} = \frac{2}{r} L_1^C + \frac{2}{r}\partial_\phi L_2^C + \frac{}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right) L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

$$\bar{f}_3 = \bar{\Phi}_{33} = 2\partial_z L_3^C + \frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right) L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

$$\bar{c}_1 = \bar{\Phi}_{23} = \frac{1}{r}\partial_\phi L_3^C + \partial_z L_2^C, \quad \bar{c}_2 = \bar{\Phi}_{31} = \partial_z L_1^C + \partial_r L_3^C, \quad \bar{c}_3 = \bar{\Phi}_{12} = -\frac{1}{r} L_2^C + \partial_r L_2^C + \frac{1}{r}\partial_\phi L_1^C,$$

$$\bar{d}_1 = \bar{\Phi}_{01} = \partial_t L_1^C + \partial_r L_0^C, \quad \bar{d}_2 = \bar{\Phi}_{02} = \partial_t L_2^C + \frac{1}{r}\partial_\phi L_0^C, \quad \bar{d}_3 = \bar{\Phi}_{03} = \partial_t L_3^C + \partial_z L_0^C;$$

$$\bar{f}_0 = \bar{\Phi}_{00} = 2\partial_t L_0^C - \frac{1}{2}\left[\partial_t L_0^C - \left(\partial_r + \frac{1}{r}\right) L_1^C - \frac{1}{r}\partial_\phi L_2^C - \partial_z L_3^C\right],$$

these relations are referred to Cartesian basis. Taking into account expressions for the gauge vector $L_c^C(x)$

$$L^{\text{Cart}}(t, r, \varphi, z) = e^{-i\varepsilon t} e^{im\varphi} e^{ikz} \begin{vmatrix} L_0^C \\ L_1^C \\ L_2^C \\ L_3^C \end{vmatrix} = e^{-i\varepsilon t} e^{im\varphi} e^{ikz} \begin{vmatrix} L_0 \\ \frac{1}{\sqrt{2}}(L_3 - L_1) \\ \frac{-i}{\sqrt{2}}(L_3 + L_1) \\ L_2 \end{vmatrix},$$

the previous formulas transform to the following form (the total multiplier $e^{-i\varepsilon t} e^{im\phi} e^{ikz}$ is omitted):

$$\bar{h}(r) = -\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{1}{r}\right)(L_3 - L_1) - \frac{m}{r\sqrt{2}}(L_3 + L_1) - i\varepsilon L_0 - ikL_2;$$

$$\bar{f}_1(r) = +\sqrt{2}\frac{d}{dr}(L_3 - L_1) - \frac{1}{2\sqrt{2}}\left(\frac{d}{dr} + \frac{1}{r}\right)(L_3 - L_1) - \frac{m}{2\sqrt{2}r}(L_3 + L_1) - \frac{i\varepsilon}{2} L_0 - \frac{ik}{2} L_2,$$

$$\bar{f}_2(r) = -\frac{1}{2\sqrt{2}}\left(\frac{d}{dr} + \frac{1}{r}\right)(L_3 - L_1) + \frac{\sqrt{2}}{r}(L_3 - L_1) + \frac{\sqrt{2}m}{r}(L_3 + L_1) - \frac{m}{2\sqrt{2}r}(L_3 + L_1) - \frac{i\varepsilon}{2} L_0 - \frac{ik}{2} L_2,$$

$$\bar{f}_3(r) = -\frac{1}{2\sqrt{2}}\left(\frac{d}{dr} + \frac{1}{r}\right)(L_3 - L_1) - \frac{m}{2\sqrt{2}r}(L_3 + L_1) + 2ikL_2 - \frac{i\varepsilon}{2} L_0 - \frac{ik}{2} L_2;$$

$$\bar{c}_1(r) = \frac{im}{r} L_2 + \frac{k}{\sqrt{2}}(L_3 + L_1), \quad \bar{c}_2(r) = \frac{d}{dr} L_2 + \frac{ik}{\sqrt{2}}(L_3 - L_1),$$

$$\bar{c}_3(r) = -\frac{i}{\sqrt{2}}\frac{d}{dr}(L_3 + L_1) + \frac{i}{r\sqrt{2}}(L_3 + L_1) + \frac{im}{r\sqrt{2}}(L_3 - L_1);$$

$$\overline{d}_1(r)=+\frac{d}{dr}L_0-\frac{i\varepsilon}{\sqrt{2}}(L_3-L_1),\quad \overline{d}_2(r)=+\frac{im}{r}L_0-\frac{\varepsilon}{\sqrt{2}}(L_3+L_1),\quad \overline{d}_3(r)=-i\varepsilon L_2+ikL_0;$$

$$\overline{f}_0(r)=+\frac{1}{2\sqrt{2}}\left(\frac{d}{dr}+\frac{1}{r}\right)(L_3-L_1)+\frac{m}{2\sqrt{2}r}(L_3+L_1)-2i\varepsilon L_0+\frac{i\varepsilon}{2}L_0+\frac{ik}{2}L_2.$$

For symmetric tensor, transition to cyclic basis is performed by means of the known linear transformation. After needed calculation, we obtain expressions for 10 components of the symmetric tensor in cyclic basis

$$\overline{f}_1(z)=-\sqrt{2}\lambda\left(\frac{d}{dz}+\frac{m-1}{z}\right)L_1(z),\quad \overline{f}_3(z)=\sqrt{2}\lambda\left(\frac{d}{dz}-\frac{m+1}{z}\right)L_3(z),$$

$$\overline{f}_2(z)=\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}-\frac{m-1}{z}\right)L_1(z)-\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}+\frac{m+1}{z}\right)L_3(z)-\frac{1}{2}i(\varepsilon L_0(z)-3kL_2)(z),$$

$$\overline{c}_1(z)=\frac{\sqrt{2}}{2}\lambda\left(\frac{d}{dz}-\frac{m}{z}\right)L_2(z)+ikL_3(z),\quad \overline{c}_3(z)=-\frac{\sqrt{2}}{2}\lambda\left(\frac{d}{dz}+\frac{m}{z}\right)L_2(z)+ikL_1(z),$$

$$\overline{c}_2(z)=\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}-\frac{m-1}{z}\right)L_1(z)-\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}+\frac{m+1}{z}\right)L_3(z)+\frac{1}{2}i(\varepsilon L_0+kL_2(z)),$$

$$\overline{d}_1(z)=-\frac{\sqrt{2}}{2}\lambda\left(\frac{d}{dz}+\frac{m}{z}\right)L_0(z)-i\varepsilon L_1(z),\quad \overline{d}_3(z)=\frac{\sqrt{2}}{2}\lambda\left(\frac{d}{dz}-\frac{m}{z}\right)L_0(z)-i\varepsilon L_3(z),$$

$$\overline{d}_2(z)=i(kL_0(z)-\varepsilon L_2(z)),$$

$$\overline{f}_0(z)=-\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}-\frac{m-1}{z}\right)L_1(z)+\frac{\sqrt{2}}{4}\lambda\left(\frac{d}{dz}+\frac{m+1}{z}\right)L_3(z)-\frac{1}{2}i(3\varepsilon L_0(z)-kL_2(z)).$$

Explicit expressions for four vectors $L_c^{(0,1,2,3)}$ in terms of Bessel functions are known; note notations $z=\sqrt{\varepsilon^2-k^2}\,r$; let $\sqrt{\varepsilon^2-k^2}=\lambda$ (to have the same dimension for all four solutions, we set $L=1/\lambda$):

$$\overline{L}_0^{(0)}=-i\varepsilon LJ_m,\overline{L}_2^{(0)}=ikLJ_m,\overline{L}_1^{(0)}=-\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}LJ_{m-1},\overline{L}_3^{(0)}=-\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}LJ_{m+1};$$

$$L_1^{(1)}=-i\,J_{m-1},\quad L_2^{(1)}=0,\quad L_3^{(1)}=0,\quad L_0^{(1)}=\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}\varepsilon}J_m;$$

$$L_1^{(2)}=0,\quad L_2^{(2)}=J_m,\quad L_3^{(2)}=0,\quad L_0^{(2)}=-\frac{k}{\varepsilon}J_m; \qquad (3)$$

$$L_1^{(3)}=0,\quad L_2^{(3)}=0,\quad L_3^{(3)}=+i\,J_{m+1},\quad L_0^{(3)}=-\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}\varepsilon}J_m.$$

We substitute these expressions (3) in the formula for components of the gauge tensor. This results in

$$L_c^{(1)},\quad \begin{array}{l}\overline{f}_1=i\sqrt{2}\lambda\,J_{m-2},\ \overline{f}_2=0,\ \overline{f}_3=0,\ \overline{c}_1=0,\ \overline{c}_2=\dfrac{i\lambda}{\sqrt{2}}J_m,\ \overline{c}_3=kJ_{m-1},\\ \overline{d}_1=-\left(\dfrac{\lambda^2}{2\varepsilon}+\varepsilon\right)J_{m-1},\ \overline{d}_2=ik\dfrac{\lambda}{\sqrt{2}\varepsilon}J_m,\ \overline{d}_3=-\dfrac{\lambda^2}{2\varepsilon}J_{m+1},\ \overline{f}_0=-i\sqrt{2}\lambda J_m;\end{array}$$

$$L_c^{(2)},\quad \begin{array}{l}\overline{f}_1=0,\ \overline{f}_2=2ikJ_m,\ \overline{f}_3=0,\ \overline{c}_1=-\dfrac{\lambda}{\sqrt{2}}J_{m+1},\ \overline{c}_2=0,\ \overline{c}_3=-\dfrac{\lambda}{\sqrt{2}}J_{m-1},\\ \overline{d}_1=\dfrac{k\lambda}{\varepsilon\sqrt{2}}J_{m-1},\ \overline{d}_2=-i\varepsilon\left(\dfrac{k^2}{\varepsilon^2}+1\right)J_m,\ \overline{d}_3=\dfrac{k\lambda}{\varepsilon\sqrt{2}}J_{m+1},\ \overline{f}_0=2ikJ_m;\end{array}$$

$$L_c^{(3)},\quad \begin{array}{l} \overline{f}_1 = 0,\ \overline{f}_2 = 0,\ \overline{f}_3 = -\sqrt{2}i\lambda J_{m+2},\ \overline{c}_1 = -kJ_{m+1},\ \overline{c}_2 = -\frac{\sqrt{2}}{2}i\lambda J_m,\ \overline{c}_3 = 0, \\ \overline{d}_1 = \frac{\lambda^2}{2\varepsilon}J_{m-1},\ \overline{d}_2 = -ik\frac{\lambda}{\sqrt{2\varepsilon}}J_m,\ \overline{d}_3 = \left(\frac{\lambda^2}{2\varepsilon}+\varepsilon\right)J_{m+1},\ \overline{f}_0 = \sqrt{2}i\lambda J_m. \end{array}$$

Verifying the gauge solutions. We should remind expressions for all 6 solutions from Section 2. With 4 vectors, $L_c^{(1)}, L_c^{(2)}, L_c^{(3)}, L_c^{(0)}$ we may associate corresponding 10-dimensional columns Q, R, S, T with the structure $\{f_i, c_i, d_i, f_0\}$:

$$L_c^{(1)} \Rightarrow Q,\quad \begin{array}{l} \overline{f}_1 = i\sqrt{2}\lambda\, J_{m-2},\ \overline{f}_2 = 0,\ \overline{f}_3 = 0,\quad \overline{c}_1 = 0, \overline{c}_2 = \frac{i\lambda}{\sqrt{2}}J_m, \overline{c}_3 = kJ_{m-1}, \\ \overline{d}_1 = -\left(\frac{\lambda^2}{2\varepsilon}+\varepsilon\right)J_{m-1}, \overline{d}_2 = ik\frac{\lambda}{\sqrt{2\varepsilon}}J_m, \overline{d}_3 = -\frac{\lambda^2}{2\varepsilon}J_{m+1}, \overline{f}_0 = -i\sqrt{2}\lambda J_m; \end{array}$$

$$L_c^{(2)} \Rightarrow R,\quad \begin{array}{l} \overline{f}_1 = 0,\ \overline{f}_2 = 2ikJ_m,\ \overline{f}_3 = 0, \overline{c}_1 = -\frac{\lambda}{\sqrt{2}}J_{m+1}, \overline{c}_2 = 0, \overline{c}_3 = -\frac{\lambda}{\sqrt{2}}J_{m-1}, \\ \overline{d}_1 = \frac{k\lambda}{\varepsilon\sqrt{2}}J_{m-1}, \overline{d}_2 = -i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right)J_m, \overline{d}_3 = \frac{k\lambda}{\varepsilon\sqrt{2}}J_{m+1}, \overline{f}_0 = 2ikJ_m; \end{array}$$

$$L_c^{(3)} \Rightarrow S,\quad \begin{array}{l} \overline{f}_1 = 0,\ \overline{f}_2 = 0,\ \overline{f}_3 = -\sqrt{2}i\lambda J_{m+2}, \overline{c}_1 = -kJ_{m+1}, \overline{c}_2 = -\frac{\sqrt{2}}{2}i\lambda J_m, \overline{c}_3 = 0, \\ \overline{d}_1 = \frac{\lambda^2}{2\varepsilon}J_{m-1}, \overline{d}_2 = -ik\frac{\lambda}{\sqrt{2\varepsilon}}J_m, \overline{d}_3 = \left(\frac{\lambda^2}{2\varepsilon}+\varepsilon\right)J_{m+1}, \overline{f}_0 = \sqrt{2}i\lambda J_m; \end{array}$$

$$L_c^{(0)} \Rightarrow T,\quad \begin{array}{l} \overline{f}_1 = \lambda J_{m-2},\ \overline{f}_2 = -2\frac{k^2}{\lambda}J_m,\ \overline{f}_3 = \lambda J_{m+2}, \\ \overline{c}_1 = -\sqrt{2}ikJ_{m+1}, \overline{c}_2 = \lambda J_m, \overline{c}_3 = -\sqrt{2}ikJ_{m-1}, \\ \overline{d}_1 = \sqrt{2}i\varepsilon J_{m-1}, \overline{d}_2 = 2\frac{\varepsilon k}{\lambda}J_m, \overline{d}_3 = \sqrt{2}i\varepsilon J_{m+1}, \overline{f}_0 = -\frac{2\varepsilon^2}{\lambda}J_m. \end{array}$$

These 4 columns Q, R, S, T should be decomposed in linear combinations of solutions $\Psi_i, i = 1, ..., 6$:

$$\begin{array}{l} Q = q_1\Psi_1 + q_2\Psi_2 + q_3\Psi_3 + q_4\Psi_4 + q_5\Psi_5 + q_6\Psi_6, \\ R = r_1\Psi_1 + r_2\Psi_2 + r_3\Psi_3 + r_4\Psi_4 + r_5\Psi_5 + r_6\Psi_6, \\ S = s_1\Psi_1 + s_2\Psi_2 + s_3\Psi_3 + s_4\Psi_4 + s_5\Psi_5 + s_6\Psi_6, \\ T = t_1\Psi_1 + t_2\Psi_2 + t_3\Psi_3 + t_4\Psi_4 + t_5\Psi_5 + t_6\Psi_6; \end{array}$$

whence we derive 4 algebraic systems. The first system has the form

$$Q,\quad \begin{array}{l} -i\sqrt{2}\lambda = q_1,\quad 0 = \sqrt{2}(q_4 - q_3)\lambda - 2q_5\varepsilon,\quad 0 = -q_2,\quad 0 = iq_3, \\ \frac{i\lambda}{\sqrt{2}} = q_1\lambda^2 + q_2\lambda^2 + \sqrt{2}q_3k\lambda - \sqrt{2}q_4k\lambda - 2q_5k\varepsilon - 2q_6\varepsilon^2, \\ k = -iq_4,\quad -\left(\frac{\lambda^2}{2\varepsilon}+\varepsilon\right) = -i\frac{\sqrt{2}(q_1 - q_2)\lambda^2 - 2(q_3+q_4)k\lambda + 2\sqrt{2}\varepsilon(q_5k + q_6\varepsilon)}{4\lambda\varepsilon}, \\ ik\frac{\lambda}{\sqrt{2\varepsilon}} = q_5,\quad -\frac{\lambda^2}{2\varepsilon} = i\frac{\sqrt{2}(q_1 - q_2)\lambda^2 - 2(q_3+q_4)k\lambda - 2\sqrt{2}\varepsilon(q_5k + q_6\varepsilon)}{4\lambda\varepsilon},\quad -i\sqrt{2}\lambda = q_6. \end{array}$$

Its solution reads

$$Q, \quad q_1 = -i\sqrt{2}\lambda,\ q_2 = 0,\ q_3 = 0,\ q_4 = ik,\ q_5 = ik\frac{\lambda}{\sqrt{2}\varepsilon},\ q_6 = -i\sqrt{2}\lambda.$$

The second system reads

$$R,\ 0 = -r_1,\ 2ik = \frac{\sqrt{2}(r_4 - r_3)\lambda - 2r_5\varepsilon}{2k},\ 0 = -r_2,\ -\frac{\lambda}{\sqrt{2}} = ir_3,$$

$$0 = r_1\lambda^2 + r_2\lambda^2 + \sqrt{2}r_3k\lambda - \sqrt{2}r_4k\lambda - 2r_5k\varepsilon - 2r_6\varepsilon^2, \quad -\frac{\lambda}{\sqrt{2}} = -ir_4,$$

$$\frac{k\lambda}{\varepsilon\sqrt{2}} = -i\frac{\sqrt{2}(r_1 - r_2)\lambda^2 - 2(r_3 + r_4)k\lambda + 2\sqrt{2}\varepsilon(r_5k + r_6\varepsilon)}{4\lambda\varepsilon},\ -i\varepsilon\left(\frac{k^2}{\varepsilon^2} + 1\right) = r_5,$$

$$\frac{k\lambda}{\varepsilon\sqrt{2}} = i\frac{\sqrt{2}(r_1 - r_2)\lambda^2 - 2(r_3 + r_4)k\lambda - 2\sqrt{2}\varepsilon(r_5k + r_6\varepsilon)}{4\lambda\varepsilon},\ 2ik = r_6;$$

its solution has a structure

$$R,\ r_1 = 0,\ r_2 = 0,\ r_3 = i\frac{\lambda}{\sqrt{2}},\ r_4 = -i\frac{\lambda}{\sqrt{2}},\ r_5 = -i\varepsilon\left(\frac{k^2}{\varepsilon^2} + 1\right),\ r_6 = 2ik.$$

The third system reads

$$S,\ 0 = -s_1J_{m-2},\ 0 = \frac{\sqrt{2}(s_4 - s_3)\lambda - 2s_5\varepsilon}{2k}J_m,\ -i\sqrt{2}\lambda J_{m+2} = -s_2J_{m+2},\ -kJ_{m+1} = is_3J_{m+1},$$

$$\frac{1}{2}(-\sqrt{2})i\lambda J_m = \frac{s_1\lambda^2 + s_2\lambda^2 + \sqrt{2}s_3k\lambda - \sqrt{2}s_4k\lambda - 2s_5k\varepsilon - 2s_6\varepsilon^2}{2\lambda^2}J_m,\ 0 = -is_4J_{m-1},$$

$$\frac{\lambda^2J_{m-1}}{2\varepsilon} = -i\frac{\sqrt{2}(s_1 - s_2)\lambda^2 - 2(s_3 + s_4)k\lambda + 2\sqrt{2}\varepsilon(s_5k + s_6\varepsilon)}{4\lambda\varepsilon}J_{m-1},\ -\frac{ik\lambda J_m}{\sqrt{2}\varepsilon} = s_5J_m,$$

$$\varepsilon\left(1 + \frac{\lambda^2}{2\varepsilon^2}\right)J_{m+1} = i\frac{(\sqrt{2}(s_1 - s_2)\lambda^2 - 2(s_3 + s_4)k\lambda - 2\sqrt{2}\varepsilon(s_5k + s_6\varepsilon))}{4\lambda\varepsilon}J_{m+1},\ \sqrt{2}i\lambda J_m = s_6J_m;$$

its solution reads

$$S,\ s_1 = 0,\ s_2 = i\sqrt{2}\lambda,\ s_3 = -\frac{k}{i},\ s_4 = 0,\ s_5 = -\frac{ik\lambda}{\sqrt{2}\varepsilon},\ s_6 = \sqrt{2}i\lambda.$$

The fourth system reads

$$T,\ t_1 = -\lambda,\ t_2 = -\lambda,\ t_3 = -\sqrt{2}k,\ t_4 = \sqrt{2}k,\ t_5 = \frac{2k\varepsilon}{\lambda},\ t_6 = -\frac{2\varepsilon^2}{\lambda}.$$

So we have found 4 decompositions

$$\begin{aligned}
Q &= -i\sqrt{2}\lambda\Psi_1 + ik\Psi_4 + ik\frac{\lambda}{\sqrt{2}\varepsilon}\Psi_5 - i\sqrt{2}\lambda\Psi_6,\\
R &= i\frac{\lambda}{\sqrt{2}}\Psi_3 - i\frac{\lambda}{\sqrt{2}}\Psi_4 - i\varepsilon\left(\frac{k^2}{\varepsilon^2} + 1\right)\Psi_5 + 2ik\Psi_6,\\
S &= i\sqrt{2}\lambda\Psi_2 - \frac{k}{i}\Psi_3 - \frac{ik\lambda}{\sqrt{2}\varepsilon}\Psi_5 + \sqrt{2}i\lambda\Psi_6,\\
T &= -\lambda\Psi_1 - \lambda\Psi_2 - \sqrt{2}k\Psi_3 + \sqrt{2}k\Psi_4 + \frac{2k\varepsilon}{\lambda}\Psi_5 - \frac{2\varepsilon^2}{\lambda}\Psi_6.
\end{aligned} \tag{4}$$

Because when finding 6 independent solutions in massless case it was shown that corresponding scalar function is equal to zero, $\bar{h}(r)=0,$ we have to calculate the component $\bar{h}(r)$ for each gauge vector $L_c^{(i)}$, $i=1,2,3,0.$ First, we find the general expression for $\bar{h}(r)$:

$$\bar{h}(r)=-\frac{1}{\sqrt{2}}\left(\frac{d}{dr}+\frac{1}{r}\right)(L_3-L_1)-\frac{m}{r\sqrt{2}}(L_3+L_1)-i\varepsilon L_0-ikL_2;$$

in the variable $z=\sqrt{\varepsilon^2-k^2}\, r,$ $\sqrt{\varepsilon^2-k^2}=\lambda$ this equality reads

$$\bar{h}(z)=-\frac{\lambda}{\sqrt{2}}\left(\frac{d}{dz}+\frac{1}{z}\right)(L_3-L_1)-\lambda\frac{m}{z\sqrt{2}}(L_3+L_1)-i\varepsilon L_0-ikL_2,$$

after performing needed calculations, we obtain

$$L_c^{(1)},\quad \bar{h}(z)=i\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}J_m-i\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}J_m\equiv 0,$$
$$L_c^{(2)},\quad \bar{h}(z)=ikJ_m-ikJ_m\equiv 0,$$
$$L_c^{(3)},\quad \bar{h}(z)=-i\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}J_m+i\frac{\sqrt{\varepsilon^2-k^2}}{\sqrt{2}}J_m\equiv 0,$$
$$L_c^{(0)},\quad \bar{h}(z)=\sqrt{\varepsilon^2-k^2}J_m-\sqrt{\varepsilon^2-k^2}J_m\equiv 0.$$

We conclude that for all 4 cases the gauge component $\bar{h}(r)$ turns out to be vanishing.

Eliminating the gauge degrees of freedom. As shown above, the four gauge solutions are decomposed on 6 independent solutions (see (4)). The six t solutions Ψ_i can be considered as an orthogonal basis:

$$\Psi_i\Psi_j=\delta_{ij},\ i,j\in\{1,...,6\}.$$

Let us search two solutions, which are decomposed in combination of 6 vectors Ψ_i:

$$\Phi^{(1)}=\phi_1^{(1)}\Psi_1+\phi_2^{(1)}\Psi_2+\phi_3^{(1)}\Psi_3+\phi_4^{(1)}\Psi_4+\phi_5^{(1)}\Psi_5+\phi_6^{(1)}\Psi_6,$$

$$\Phi^{(2)}=\phi_1^{(2)}\Psi_1+\phi_2^{(2)}\Psi_2+\phi_3^{(2)}\Psi_3+\phi_4^{(2)}\Psi_4+\phi_5^{(2)}\Psi_5+\phi_6^{(2)}\Psi_6;$$

besides, these 2 solutions should be orthogonal to 4 gauge solutions

$$\Phi^{(1)}Q=0,\quad \Phi^{(1)}R=0,\quad \Phi^{(1)}S=0,\quad \Phi^{(1)}T=0;$$
$$\Phi^{(2)}Q=0,\quad \Phi^{(2)}R=0,\quad \Phi^{(2)}S=0,\quad \Phi^{(2)}T=0.$$

It suffices to study only one equation (the index in brackets is omitted)

$$\Phi Q=0,\ \Phi R=0,\ \Phi S=0,\ \Phi T=0. \qquad (5)$$

In detailed form we get equations

$$(\phi_1\Psi_1+\phi_2\Psi_2+\phi_3\Psi_3+\phi_4\Psi_4+\phi_5\Psi_5+\phi_6\Psi_6)\left(-i\sqrt{2}\lambda\Psi_1+ik\Psi_4+ik\frac{\lambda}{\sqrt{2}\varepsilon}\Psi_5-i\sqrt{2}\lambda\Psi_6\right)=0;$$

$$(\phi_1\Psi_1+\phi_2\Psi_2+\phi_3\Psi_3+\phi_4\Psi_4+\phi_5\Psi_5+\phi_6\Psi_6)\left(i\frac{\lambda}{\sqrt{2}}\Psi_3-i\frac{\lambda}{\sqrt{2}}\Psi_4-i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right)\Psi_5+2ik\Psi_6\right)=0;$$

$$(\phi_1\Psi_1+\phi_2\Psi_2+\phi_3\Psi_3+\phi_4\Psi_4+\phi_5\Psi_5+\phi_6\Psi_6)\left(i\sqrt{2}\lambda\Psi_2-\frac{k}{i}\Psi_3-\frac{ik\lambda}{\sqrt{2}\varepsilon}\Psi_5+\sqrt{2}i\lambda\Psi_6\right)=0;$$

$$(\phi_1\Psi_1+\phi_2\Psi_2+\phi_3\Psi_3+\phi_4\Psi_4+\phi_5\Psi_5+\phi_6\Psi_6)\times$$
$$\times\left(-\lambda\Psi_1-\lambda\Psi_2-\sqrt{2}k\Psi_3+\sqrt{2}k\Psi_4+\frac{2k\varepsilon}{\lambda}\Psi_5-\frac{2\varepsilon^2}{\lambda}\Psi_6\right)=0.$$

Therefore, from conditions (5) we derive algebraic system of 4 equations

$$\phi_1(-i\sqrt{2}\lambda)+\phi_4(ik)+\phi_5\left(ik\frac{\lambda}{\sqrt{2}\varepsilon}\right)+\phi_6(-i\sqrt{2}\lambda)=0,$$
$$\phi_3\left(i\frac{\lambda}{\sqrt{2}}\right)+\phi_4\left(-i\frac{\lambda}{\sqrt{2}}\right)+\phi_5\left(-i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right)\right)+\phi_6^{(1)}(2ik)=0,$$
$$\phi_2(i\sqrt{2}\lambda)+\phi_3\left(-\frac{k}{i}\right)+\phi_5\left(-\frac{ik\lambda}{\sqrt{2}\varepsilon}\right)+\phi_6(\sqrt{2}i\lambda)=0,$$
$$\phi_1(-\lambda)+\phi_2(-\lambda)+\phi_3(-\sqrt{2}k)+\phi_4(\sqrt{2}k)+\phi_5\left(\frac{2k\varepsilon}{\lambda}\right)+\phi_6\left(-\frac{2\varepsilon^2}{\lambda}\right)=0.$$

This system may presented in the matrix form $B^{(1)}_{4\times 6}\Phi^{(1)}=0$:

$$\begin{vmatrix} -i\sqrt{2}\lambda & . & . & ik & ik\frac{\lambda}{\sqrt{2}\varepsilon} & -i\sqrt{2}\lambda \\ . & . & i\frac{\lambda}{\sqrt{2}} & -i\frac{\lambda}{\sqrt{2}} & -i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right) & 2ik \\ . & i\sqrt{2}\lambda & -\frac{k}{i} & . & -\frac{ik\lambda}{\sqrt{2}\varepsilon} & \sqrt{2}i\lambda \\ -\lambda & -\lambda & -\sqrt{2}k & \sqrt{2}k & \frac{2k\varepsilon}{\lambda} & -\frac{2\varepsilon^2}{\lambda} \end{vmatrix}\begin{vmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{vmatrix}=0.$$

The rank of the matrix B equals 3. We readily verify that after eliminating the forth row we obtain matrix with the same rank 3. Thus, we have an equivalent system

$$\begin{vmatrix} -i\sqrt{2}\lambda & 0 & 0 \\ 0 & 0 & \frac{i\lambda}{\sqrt{2}} \\ 0 & i\sqrt{2}\lambda & -\frac{k}{i} \end{vmatrix}\begin{vmatrix}\phi_1\\ \phi_2\\ \phi_3\end{vmatrix}=-\phi_4\begin{vmatrix} ik \\ -\frac{i\lambda}{\sqrt{2}} \\ 0\end{vmatrix}-\phi_5\begin{vmatrix}\frac{ik\lambda}{\sqrt{2}\varepsilon} \\ -i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right) \\ -\frac{ik\lambda}{\sqrt{2}\varepsilon}\end{vmatrix}-\phi_6\begin{vmatrix}-i\sqrt{2}\lambda \\ 2ik \\ \sqrt{2}i\lambda\end{vmatrix}. \tag{6}$$

Because all parameters $\phi_i, i=1,...,6,$ are fixed up to an arbitrary total multiplier, we can set $\phi_3=1$. In this way, from (6) we derive the simple solution

$$-i\sqrt{2}\lambda\phi_1=-ik\phi_4-\frac{ik\lambda}{\sqrt{2}\varepsilon}\phi_5+i\sqrt{2}\lambda\phi_6,\qquad \frac{i\lambda}{\sqrt{2}}=\frac{i\lambda}{\sqrt{2}}\phi_4+i\varepsilon\left(\frac{k^2}{\varepsilon^2}+1\right)\phi_5-2ik\phi_6,$$
$$i\sqrt{2}\lambda\phi_2+ik=\frac{ik\lambda}{\sqrt{2}\varepsilon}\phi_5-i\sqrt{2}\lambda\phi_6.$$

The second equation permits us to eliminate ϕ_4:

$$\phi_3 = 1,\ \phi_4 = 1 - \sqrt{2}\frac{k^2+\varepsilon^2}{\varepsilon\lambda}\phi_5 + 2\sqrt{2}\frac{k}{\lambda}\phi_6,$$

$$\phi_1 = \frac{k}{\sqrt{2}\lambda} - \frac{k}{\varepsilon}\frac{\varepsilon^2+3k^2}{2\lambda^2}\phi_5 + 2\frac{2k^2-\varepsilon^2}{\lambda^2}\phi_6,\ \phi_2 = \frac{k}{2\varepsilon}\phi_5 - \phi_6 - \frac{k}{\sqrt{2}\lambda}.$$

Thus, we can fix 2 independent solutions as follows

$$\phi_3 = 1,\ \phi_5 = 1,\ \phi_6 = 0,\ \phi_4 = \frac{\lambda\varepsilon - \sqrt{2}(\varepsilon^2+k^2)}{\lambda\varepsilon},\ \phi_1 = \frac{(\sqrt{2}\lambda\varepsilon - \varepsilon^2 - 3k^2)k}{2\lambda^2\varepsilon},\ \phi_2 = \frac{(\lambda - \sqrt{2}\varepsilon)k}{2\varepsilon\lambda};$$

$$\phi_3 = 1,\ \phi_5 = 0,\ \phi_6 = 1,\ \phi_4 = 2\sqrt{2}\frac{k}{\lambda},\ \phi_1 = \frac{\lambda k + 2\sqrt{2}k^2 - \sqrt{2}\varepsilon^2 + \sqrt{2}k^2}{\sqrt{2}\lambda^2},\ \phi_2 = -\frac{k}{\sqrt{2}\lambda} - 1.$$

So, we have found two gauge-free solutions for massless spin 2 field, as expected by physical reason.

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