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Spin 3/2 Particle in External Uniform Magnetic Field, the Method of Projective Operators

In the present paper, an algebraic method for solving the system of equations describing the spin 3/2 particle in the presence of uniform magnetic field has been elaborated. The method is based on decomposition of 16-components wave function with transformation properties of vector-bispinor in the sum of four constitutes, which are determined by four projective operators. With the use of formalism of elements of complete matrix algebra the system is transformed to the form, in which only projective constituents $\Psi_{\pm 1/2}(x), \Psi_{\pm 3/2}(x)$ *enter. This system of equations is transformed to cylindric coordinates. On the wave functions three operators are digitalized: of energy, third projection of linear momentum, third projection of the total angular momentum. After separating the variables, we derive 4 linked subsystems of equations for 16-component functions* $\Psi_{\pm 1/2}(r),\ \Psi_{\pm 3/2}(r)$. *After performing needed calculations, the problem reduces to four independent second order equations for four primary functions. These equations are solved in terms of confluent hypergeometric functions, four different energy spectra are found.*

Key words: *spin 3/2 particle, magnetic field, projective operators, exact solutions, energy spectra.*

ЧАСТИЦА СО СПИНОМ 3/2 ВО ВНЕШНЕМ МАГНИТНОМ ПОЛЕ, МЕТОД ПРОЕКТИВНЫХ ОПЕРАТОРОВ

В настоящей работе развит алгебраический метод анализа системы уравнений, описывающей частицу со спином 3/2 во внешнем однородном магнитном поле. Метод основан на разложении 16-компонентной волновой функции с трансформационными свойствами вектора биспинора в сумму 4-х составляющих, которые определяются действием 4-х проективных операторов на полную волновую функцию. С использованием формализма элементов полной матричной алгебры и свойств матриц Дирака система уравнений приведена к виду, когда в ней присутствует только 4 проективные составляющие $\Psi_{\pm 1/2}(x), \Psi_{\pm 3/2}(x)$. *Полученная система уравнений записывается в цилиндрической системе координат. На волновых функциях диагонализируются операторы энергии, третьей проекции импульса и третьей проекции полного углового момента. С учетом соответствующей подстановки для волновой функции из системы уравнений исключается зависимость от переменных* (t, z, ϕ); *в результате получены 4 связанные между собой подсистемы, в которые входят зависящие от полярной координаты r функции* $\Psi_{\pm 1/2}(r), \Psi_{\pm 3/2}(r)$. *Задача приводится к раздельным дифференциальным урав-нениям второго порядка для некоторых 4-х основных функций. Эти уравнения решаются в терминах вырож-денных гипергеометрических функций. Получены 4 различных спектра энергий.*

Ключевые слова: *частица со спином 3/2, магнитное поле, проективные операторы, точные решения, спектры энергий.*

1. Initial equation and projective operators

The basic equation for the 16-component wave function has the form ([1–3], also see [4–14])

$$(\Gamma_\mu \partial_\mu + M)\Psi = 0\,. \qquad (1)$$

The third projection of the spin operator is $\Sigma_3 = -iJ_{12} = Y;$ its explicit form is given by the formula

$$Y = -i\{\frac{1}{2}\gamma_1\gamma_2 \otimes I + I \otimes (e^{1,2} - e^{2,1})\}, \qquad (2)$$

where γ_μ designates the Dirac matrices, $e^{\mu,\nu}$ stands for the elements of the complete matrix algebra $(e^{\mu,\nu})_{\varrho\lambda} = \delta_{\mu\varrho}\delta_{\nu\lambda}$, obeying the following multiplication rule $e^{\mu,\nu}e^{\varrho,\lambda} = \delta_{\nu\varrho}e^{\mu,\lambda}$. We can verify that the minimal equation for the matrix Y has the form

$$(Y^2 - \frac{1}{4})(Y^2 - \frac{9}{4}) = 0. \qquad (3)$$

The last equation permits us to define four projective operators

$$P_{-1/2} = \frac{1}{2}(Y - 1/2)(Y^2 - 9/4), \quad P_{+1/2} = -\frac{1}{2}(Y + 1/2)(Y^2 - 9/2),$$
$$P_{+3/2} = \frac{1}{6}(Y^2 - 1/4)(Y - 3/2), \quad P_{-3/2} = -\frac{1}{6}(Y^2 - 1/4)(Y - 3/2). \qquad (4)$$

Correspondingly, the complete wave function can be decomposed into the sum of four constituents

$$P_{-1/2}\Psi = \Psi_{-1/2}, \quad P_{+1/2}\Psi = \Psi_{+1/2}, \quad P_{+3/2}\Psi = \Psi_{+3/2}, \quad P_{-3/2}\Psi = \Psi_{-3/2},$$
$$\Psi_{-1/2} + \Psi_{+1/2} + \Psi_{+3/2} + \Psi_{-3/2} = \Psi. \qquad (5)$$

We will study the basic equation (1) in presence of the uniform magnetic field

$$B = (0, 0, B), \quad A_1 = -\frac{1}{2}Bx_2, \quad A_2 = \frac{1}{2}Bx_1, \quad A_3 = 0, \quad A_4 = 0.$$

In the cylindric coordinates (r, ϕ, z) the basic equation reads

$$(\Gamma_3\partial_3 + \Gamma_4\partial_4 + M)\Psi + (\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi})\Gamma_1\Psi +$$
$$+(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi})\Gamma_2\Psi + \frac{ieB}{2}r\sin\phi\,\Gamma_1\Psi - \frac{ieB}{2}r\cos\phi\,\Gamma_2\Psi = 0. \qquad (6)$$

With the use of projective operators (5), we can transform eq. (6) to the form of four linked equations in which only projective constituents enter (for shortness we perform the change in notation $eB \to B$):

$$(\Gamma_3\partial_3 + \Gamma_4\partial_4 + M)\Psi_{+3/2} + 2e^{-i\phi}\{\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi}\}(\Gamma_1 + i\Gamma_2)\Psi_{+3/2} +$$
$$+\frac{1}{2}e^{-i\phi}\{\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi}\}(\Gamma_1 + i\Gamma_2)\Psi_{+1/2} - \frac{1}{2}e^{i\phi}\{\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\}(\Gamma_1 - i\Gamma_2)\Psi_{-3/2} -$$
$$-iBe^{-i\phi}(\Gamma_1 + i\Gamma_2)\Psi_{+3/2} - \frac{B}{4}re^{-i\phi}(\Gamma_1 + i\Gamma_2)\Psi_{+1/2} - \frac{B}{4}re^{i\phi}(\Gamma_1 - i\Gamma_2)\Psi_{-3/2} = 0, \qquad (7)$$

$$(\Gamma_3\partial_3+\Gamma_4\partial_4+M)\Psi_{+1/2}-$$
$$-3e^{-i\phi}\{\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi}\}(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+\frac{1}{2}e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{+3/2}+$$
$$+\frac{1}{2}e^{-i\phi}(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1+i\Gamma_2)\Psi_{-1/2}+2e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{-3/2}+$$
$$+\frac{3}{2}Bre^{-i\phi}(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+\frac{1}{4}Bre^{i\phi}(\Gamma_1-i\Gamma_2)\Psi_{+3/2}-$$
$$-\frac{1}{4}Bre^{-i\phi}(\Gamma_1+i\Gamma_2)\Psi_{-1/2}+Bre^{+i\phi}(\Gamma_1-i\Gamma_2)\Psi_{-3/2}=0, \qquad (8)$$

$$(\Gamma_3\partial_3+\Gamma_4\partial_4+M)\Psi_{-1/2}+$$
$$+2e^{-i\phi}(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+\frac{1}{2}e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{+1/2}-$$
$$-3e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{-3/2}+\frac{1}{2}e^{-i\phi}(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1+i\Gamma_2)\Psi_{-3/2}-$$
$$-Bre^{-i\phi}(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+\frac{Br}{4}e^{i\phi}(\Gamma_1-i\Gamma_2)\Psi_{+1/2}-$$
$$-\frac{3}{2}Bre^{i\phi}(\Gamma_1-i\Gamma_2)\Psi_{-3/2}-\frac{Br}{4}e^{-i\phi}(\Gamma_1+i\Gamma_2)\Psi_{-3/2}=0, \qquad (9)$$

$$(\Gamma_3\partial_3+\Gamma_\mu\partial_\mu+M)\Psi_{-3/2}-\frac{1}{2}e^{-i\phi}(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+$$
$$+\frac{1}{2}e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{-1/2}+2e^{i\phi}(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\phi})(\Gamma_1-i\Gamma_2)\Psi_{-3/2}+$$
$$+\frac{B}{4}re^{-i\phi}(\Gamma_1+i\Gamma_2)\Psi_{+3/2}+\frac{B}{2}re^{i\phi}(\Gamma_1-i\Gamma_2)\Psi_{-1/2}+Bre^{i\phi}(\Gamma_1-i\Gamma_2)\Psi_{-3/2}=0. \qquad (10)$$

We will use the following substitutions (they correspond to diagonalization of operators of energy, linear momentum along the axis x_3, and the third projection of the total angular momentum)

$$\Psi_{+3/2}=e^{ip_4x_4}e^{ip_3x_3}e^{i(m-3/2)\phi}f_{+3/2}(r), \quad \Psi_{+1/2}=e^{ip_4x_4}e^{ip_3x_3}e^{i(m-1/2)\phi}f_{+1/2}(r),$$
$$\Psi_{-1/2}=e^{ip_4x_4}e^{ip_3x_3}e^{i(m+1/2)\phi}f_{-1/2}(r), \quad \Psi_{-3/2}=e^{ip_4x_4}e^{ip_3x_3}e^{i(m+3/2)\phi}f_{-3/2}(r), \qquad (11)$$

where $f_{\pm3/2}(r), f_{\pm1/2}(r)$ stands for 16-component radial columns. After separating the variables we get 4 linked equations (for convenience, we mark them by symbols $S_3=\pm3/2,\ \pm1/2$)

$$\begin{array}{ll} S_3=+3/2, & (iP+M)f_{+3/2}+a_{m-1/2}\Gamma_+f_{+1/2}=0,\\ S_3=-3/2, & (iP+M)f_{-3/2}-b_{m+1/2}\Gamma_-f_{-1/2}=0,\\ S_3=+1/2, & (iP+M)f_{+1/2}-b_{m-3/2}\Gamma_-f_{+3/2}+a_{m+1/2}\Gamma_+f_{-1/2}=0,\\ S_3=-1/2, & (iP+M)f_{-1/2}-b_{m-1/2}\Gamma_-f_{+1/2}+a_{m+3/2}\Gamma_+f_{-3/2}=0, \end{array} \qquad (12)$$

where the special notations are used

$$a_{m-1/2}=\frac{1}{\sqrt{2}}(\frac{d}{dr}+\frac{1}{r}(m-\frac{1}{2})-\frac{Br}{2}), \quad b_{m+1/2}=\frac{1}{\sqrt{2}}(-\frac{d}{dr}+\frac{1}{r}(m+1/2)-\frac{Br}{2});$$

$$a_{m+3/2} = \frac{1}{\sqrt{2}}(\frac{d}{dr} + \frac{1}{r}(m+\frac{3}{2}) - \frac{Br}{2}), \quad b_{m-3/2} = \frac{1}{\sqrt{2}}(-\frac{d}{dr} + \frac{1}{r}(m-3/2) - \frac{Br}{2});$$

$$b_{m-1/2} = \frac{1}{\sqrt{2}}(-\frac{d}{dr} + \frac{m-1/2}{r} - \frac{Br}{2}); \; \Gamma_{\pm} = \frac{1}{\sqrt{2}}(\Gamma_1 \pm i\Gamma_2), \quad P = p_3\Gamma_3 + p_4\Gamma_4, \quad p = p_3\gamma_3 + p_4\gamma_4.$$

Let us detail the structure of 16-dimensional columns $f_{\pm 1/2}(r), f_{\pm 3/2}(r)$. For this we should take into account expressions for the projective operators:

$$P_{+3/2} = \frac{1}{4}(1 - i\gamma_1\gamma_2) \otimes [e^{1,1} + e^{2,2} - i(e^{1,2} - e^{2,1})], \quad P_{-3/2} = \frac{1}{4}(1 + i\gamma_1\gamma_2) \otimes [e^{1,1} + e^{2,2} + i(e^{1,2} - e^{2,1})],$$

$$P_{+1/2} = \frac{1}{4}\{2(1 - i\gamma_1\gamma_2) \otimes (e^{3,3} + e^{4,4}) + (1 + i\gamma_1\gamma_2) \otimes [e^{1,1} + e^{2,2} - i(e^{1,2} - e^{2,1})]\},$$

$$P_{-1/2} = \frac{1}{4}\{2(1 + i\gamma_1\gamma_2) \otimes (e^{3,3} + e^{4,4}) + (1 - i\gamma_1\gamma_2) \otimes [e^{1,1} + e^{2,2} + i(e^{1,2} - e^{2,1})]\};$$

after performing the needed calculations we get

$$f_{+3/2} = \frac{1}{4}\begin{vmatrix} (1 - i\gamma_1\gamma_2)(f_1 - if_2) \\ i(1 - i\gamma_1\gamma_2)(f_1 - if_2) \\ 0 \\ 0 \end{vmatrix}, \quad f_{-3/2} = \frac{1}{4}\begin{vmatrix} (1 + i\gamma_1\gamma_2)(f_1 + if_2) \\ -i(1 + i\gamma_1\gamma_2)(f_1 + if_2) \\ 0 \\ 0 \end{vmatrix},$$
$$f_{+1/2} = \frac{1}{4}\begin{vmatrix} (1 + i\gamma_1\gamma_2)(f_1 - if_2) \\ i(1 + i\gamma_1\gamma_2)(f_1 - if_2) \\ 2(1 - i\gamma_1\gamma_2) f_3 \\ 2(1 - i\gamma_1\gamma_2) f_4 \end{vmatrix}, \quad f_{-1/2} = \frac{1}{4}\begin{vmatrix} (1 - i\gamma_1\gamma_2)(f_1 + if_2) \\ -i(1 - i\gamma_1\gamma_2)(f_1 + if_2) \\ 2(1 + i\gamma_1\gamma_2) f_3 \\ 2(1 + i\gamma_1\gamma_2) f_4 \end{vmatrix}, \quad (13)$$

where f_1, f_2, f_3, f_4 designate 4-dimensional columns. Below we will apply the notations

$$\Psi_{A\mu} = \begin{vmatrix} \Psi_{A1} \\ \Psi_{A2} \\ \Psi_{A3} \\ \Psi_{A4} \end{vmatrix} = \begin{vmatrix} \text{function}(t,z,\phi)\, f_1(r) \\ \text{function}(t,z,\phi)\, f_2(r) \\ \text{function}(t,z,\phi)\, f_3(r) \\ \text{function}(t,z,\phi)\, f_4(r) \end{vmatrix}, \quad (14)$$

A is a bispinor index, and μ is a vector index.

2. Equation related to $S_3 = +3/2$

Let us find the 4-component structure of the first equation in (12), related to $S_3 = +3/2$. Starting with the identity

$$(iP + M)\Phi|_A = (ip_3\Gamma_3 + ip_4\Gamma_4 + M)\Phi|_A = iP\delta_{A,\rho}\Phi_\rho + \frac{i}{\sqrt{3}} p_\mu\gamma_\rho[\delta_{A,\rho}\Phi_\mu - \delta_{A,\mu}\Phi_\rho] + M\delta_{A,\rho}\Phi_\rho,$$

depending on the value of A we get

$$(iP + M)\Phi|_1 = (ip + M)\frac{1}{4}(1 - i\gamma_1\gamma_2)(f_1 - if_2), \; (iP + M)\Phi|_2 = (ip + M)\frac{1}{4}(1 - i\gamma_1\gamma_2)(f_1 - if_2),$$

$$(iP+M)\Phi|_3=\frac{i}{\sqrt{3}}\gamma_3 p_\mu \Phi_\mu-\frac{i}{\sqrt{3}}p_3\gamma_\mu\Phi_\mu=-\frac{i}{\sqrt{3}}p_3(\gamma_1+i\gamma_2)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1-if_2)=0,$$

$$(iP+M)\Phi|_4=\frac{i}{\sqrt{3}}\gamma_3 p_\mu \Phi_\mu-\frac{i}{\sqrt{3}}p_4\gamma_\mu\Phi_\mu=-\frac{i}{\sqrt{3}}p_4(\gamma_1+i\gamma_2)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1-if_2)=0.$$

Thus, we arrive at

$$(iP+M)f_{+3/2}=\begin{vmatrix}(ip+M)(1-i\gamma_1\gamma_2)(f_1-if_2)/4\\ i(ip+M)(1-i\gamma_1\gamma_2)(f_1-if_2)/4\\ 0\\ 0\end{vmatrix}.$$

Let us find expression for $\Gamma_+ f_{+1/2}$. Starting with

$$\Gamma_+\Phi|_A=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\delta_{A,\rho}\Phi_\rho+\frac{1}{\sqrt{3}}\gamma_\rho[\delta_{A,\rho}\Phi_1-\delta_{A,1}\Phi_\rho+i(\delta_{A,\rho}\Phi_2-\delta_{A,2}\Phi_\rho)]\},$$

we obtain

$$\Gamma_+\Phi|_1=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_1+\frac{1}{\sqrt{3}}[\gamma_1\Phi_1-\gamma_\rho\Phi_\rho+i\gamma_1\Phi_2)]\},$$

whence it follows

$$\Gamma_+\Phi|_1=\frac{1}{2\sqrt{6}}\{(\sqrt{3}-1)(\gamma_1+i\gamma_2)(f_1-if_2)-(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\}.$$

Similarly, we get

$$\Gamma_+\Phi|_2=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_2+\frac{1}{\sqrt{3}}[\gamma_2\Phi_1+i\gamma_2\Phi_2-i\gamma_p\Phi_p]\}=$$

$$=\frac{1}{2\sqrt{6}}\{(\sqrt{3}-1)(\gamma_1+i\gamma_2)(f_1-if_2)-(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\};$$

$$\Gamma_+\Psi|_3=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\frac{1}{2}(1-i\gamma_1\gamma_2)f_3+$$

$$+\frac{1}{\sqrt{3}}[\gamma_3\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)-\gamma_3\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)]\}=0;$$

$$\Gamma_+\Psi|_4=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\frac{1}{2}(1-i\gamma_1\gamma_2)f_4+$$

$$+\frac{1}{\sqrt{3}}[\gamma_4\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)-\gamma_4\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)]\}=0.$$

Thus, we arrive at the formula

$$\Gamma_+ f_{+1/2}=\frac{1}{2\sqrt{6}}\begin{vmatrix}(\sqrt{3}-1)(\partial_1+i\partial_2)(f_1-if_2)-(1-i\partial_1\partial_2)(\partial_3 f_3+\partial_4 f_4)\\ i[(\sqrt{3}-1)(\partial_1+i\partial_2)(f_1-if_2)-(1-i\partial_1\partial_2)(\partial_3 f_3+\partial_4 f_4)]\\ 0\\ 0\end{vmatrix}.$$

Therefore, the first 16-component equation in (12) gives two 4-component equations:

$$(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1-if_2)+$$
$$+\frac{1}{2\sqrt{6}}a_{m-1/2}[(\sqrt{3}-1)(\gamma_1+i\gamma_2)(f_1-if_2)-(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]=0, \quad (15)$$
$$i(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1-if_2)+$$
$$+\frac{i}{2\sqrt{6}}a_{m-1/2}[(\sqrt{3}-1)(\gamma_1+i\gamma_2)(f_1-if_2)-(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]=0,$$

where the second equation coincides with the first. Equation (15) may be transformed to other form. To this end, bearing in mind the identities

$$(\gamma_1+i\gamma_2)(f_1-if_2)=\gamma_1 f_1-i\gamma_1 f_2+i\gamma_2 f_1+\gamma_2 f_2,$$
$$(1-i\gamma_1\gamma_2)(\gamma_1 f_1+\gamma_2 f_2)=\gamma_1 f_1-i\gamma_1 f_2+i\gamma_2 f_1+\gamma_2 f_2,$$

we derive $(\gamma_1+i\gamma_2)(f_1-if_2)=(1-i\gamma_1\gamma_2)(\gamma_1 f_1+\gamma_2 f_2)$, whence it follows

$$(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)=$$
$$=(1-i\gamma_1\gamma_2)(\gamma_1 f_1+\gamma_2 f_2+\gamma_3 f_3+\gamma_4 f_4)=(1-i\gamma_1\gamma_2)(\gamma_\mu f_\mu). \quad (16)$$

Thus, eq. (15) is presented as follows

$$(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1-if_2)+\frac{1}{2\sqrt{6}}a_{m-1/2}[\sqrt{3}(\gamma_1+i\gamma_2)(f_1-if_2)-(1-i\gamma_1\gamma_2)(\gamma_\mu f_\mu)]=0. \quad (17)$$

3. Additional constraint

As known, in absence of external fields, from the initial system follows the constraint $\gamma_\mu\Psi_\mu=0$. Let us consider an analog of such a constraint in the presence of external fields. To this end, we turn to the equation (1) in tensor form

$$\hat{D}\Psi_\nu+\frac{1}{\sqrt{3}}\gamma_\nu(D_\mu\Psi_\mu)-\frac{1}{\sqrt{3}}D_\nu(\gamma_\mu\Psi_\mu)+M\Psi_\nu=0, \quad \hat{D}=\gamma_\mu D_\mu. \quad (18)$$

First, we act on this equation by the operator D_ν, and perform the convolution in the index ν:

$$D_\mu\hat{D}\Psi_\mu+\frac{1}{\sqrt{3}}\hat{D}(uD_\mu\Psi_\mu)-\frac{1}{\sqrt{3}}D^2(\gamma_\mu\Psi_\mu)+M(D_\mu\Psi_\mu)=0. \quad (19)$$

With the use of the identities

$$D_\mu D_\nu-D_\nu D_\mu=-ieF_{[\mu\nu]}, \quad \hat{D}\hat{D}=D^2-\frac{ie}{4}F_{[\varrho\lambda]}(\gamma_\varrho\gamma_\lambda-\gamma_\lambda\gamma_\varrho), \quad D^2=D_\mu D_\mu,$$

eq. (19) transforms to other form

$$\{\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M\}(D_\mu\Psi_\mu)-\frac{1}{\sqrt{3}}D^2(\gamma_\mu\Psi_\mu)-ieF_{[\mu\nu]}\gamma_\nu\Psi_\mu=0. \quad (20)$$

Similarly, let us multiply eq. (18) by γ_ν which after convolution in index ν leads to

$$\{-\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M\}(\gamma_\mu\Psi_\mu)+\frac{(1+\sqrt{3})^2}{\sqrt{3}}(D_\mu\Psi_\mu)=0. \tag{21}$$

Let us act by operator $(\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M)$ on eq. (21), this results in

$$[\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M][-\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M](\gamma_\mu\Psi_\mu)+\frac{(1+\sqrt{3})^2}{\sqrt{3}}[-\frac{1+\sqrt{3}}{\sqrt{3}}\hat{D}+M](D_\mu\Psi_\mu)=0, \tag{22}$$

whence bearing in mind (20) we obtain

$$\{1+\frac{(1+\sqrt{3})^2}{12}\frac{ie}{M^2}F_{[\varrho\lambda]}(\gamma_\varrho\gamma_\lambda-\gamma_\lambda\gamma_\varrho)\}(\gamma_\mu\Psi_\mu)=-\frac{(1+\sqrt{3})^2}{\sqrt{3}}\frac{ie}{M^2}F_{[\mu\nu]}\gamma_\nu\Psi_\mu. \tag{23}$$

We note that in absence of the external field, relation (23) reduces to the well-known constraint for the free particle, $\gamma_\mu\Psi_\mu=0$.

In what follows, we will take into account the presence of the uniform magnetic field, $F_{[12]}=B$, then expression (23) takes on the form

$$\{1+\frac{(1+\sqrt{3})^2}{3}\frac{B}{M^2}(i\gamma_1\gamma_2)\}(\gamma_\mu\Psi_\mu)=-\frac{(1+\sqrt{3})^2}{\sqrt{3}}\frac{iB}{M^2}(\gamma_2\Psi_1-\gamma_1\Psi_2). \tag{24}$$

Using the temporary notation

$$R=\frac{3M^2}{9M^4-(1+\sqrt{3})^4B^2}\{3M^2-(1+\sqrt{3})^2B\ \ (i\gamma_1\gamma_2)\}; \tag{25}$$

we readily verify that the following identity holds

$$R\{1+\frac{(1+\sqrt{3})^2}{3}\frac{B}{M^2}(i\gamma_1\gamma_2)\}=1.$$

Therefore, from (24) it follows the new expression for $\gamma_\mu\Psi_\mu$:

$$\gamma_\mu\Psi_\mu=\frac{\sqrt{3}(1+\sqrt{3})^2B}{9M^4-(1+\sqrt{3})^4B^2}\{3M^2(i\gamma_1\gamma_2)-(1+\sqrt{3})^2B\}(\gamma_1\Psi_1+\gamma_2\Psi_2), \tag{26}$$

whence bearing in mind the substitution for the wave function $\Psi_\mu=e^{i(p_3x_3+p_4x_4)}\Phi_\mu(x_1,x_2)$, we obtain

$$\gamma_\mu\Phi_\mu=\frac{\sqrt{3}(1+\sqrt{3})^2B}{9M^4-(1+\sqrt{3})^4B^2}\{3M^2(i\gamma_1\gamma_2)-(1+\sqrt{3})^2B\}(\gamma_1\Phi_1+\gamma_2\Phi_2). \tag{27}$$

After multiplying the last relation by $(1-i\gamma_1\gamma_2)$, we get

$$(1-i\gamma_1\gamma_2)(\gamma_\mu\Phi_\mu)=-\frac{\sqrt{3}(1+\sqrt{3})^2B}{3M^2-(1+\sqrt{3})^2B}(\gamma_1+i\gamma_2)(\Phi_1-i\Phi_2). \tag{28}$$

Allowing for the identity $(\gamma_1 + i\gamma_2)(\Phi_1 - i\Phi_2) = (1 - i\gamma_1\gamma_2)(\gamma_1\Phi_1 + \gamma_2\Phi_2)$, from (28) we arrive at the radial relation

$$(\gamma_1 + i\gamma_2)\frac{1}{4}(1 + i\gamma_1\gamma_2)(\Phi_1 - i\Phi_2) + \gamma_3\frac{1}{2}(1 - i\gamma_1\gamma_2)\Phi_3 + \gamma_4\frac{1}{2}(1 - i\gamma_1\gamma_2)\Phi_4 =$$

$$= -\frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2 - (1+\sqrt{3})^2 B}(\gamma_1 + i\gamma_2)\frac{1}{4}(1 + i\gamma_1\gamma_2)(\Phi_1 - i\Phi_2). \qquad (29)$$

Bearing in mind the substitution from (29), we derive

$$(1 - i\gamma_1\gamma_2)(\gamma_\mu f_\mu) = -\frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2 - (1+\sqrt{3})^2 B}(\gamma_1 + i\gamma_2)(f_1 - if_2). \qquad (30)$$

Taking into account relation (30), from (17) we can eliminate the combination $\gamma_\mu f_\mu$, so we get

$$S_3 = +3/2, \quad (ip + M)\frac{1}{2}(1 - i\gamma_1\gamma_2)(f_1 - if_2) + a_{m-1/2}\frac{3M^2}{3M^2 - (1+\sqrt{3})^2 B}\gamma_+(f_1 - if_2) = 0, \qquad (31)$$

where $\gamma_+ = 1/\sqrt{2}(\gamma_1 + i\gamma_2)$. We will consider the function $(f_1 - if_2)$ is the primary one

$$(1 - i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) = -\frac{3M^2 + 2(1+\sqrt{3})B}{3M^2 - (1+\sqrt{3})^2 B}(\gamma_1 + i\gamma_2)(f_1 - if_2). \qquad (32)$$

4. Equation related to $S = -3/2$

Let us consider the second equation in (12):

$$(iP + M)f_{-3/2} - b_{m+1/2}\Gamma_- f_{-1/2} = 0.$$

For the term

$$(iP + M)f_{-3/2} = ip\delta_{A,p}\Phi_p + \frac{i}{\sqrt{3}}p_\mu\gamma_p[\delta_{A,p}\Phi_\mu - \delta_{A,\mu}\Phi_p] + M\delta_{A,p}\Phi_p,$$

depending on the value of A we get:

$$A = 1, \quad (ip + M)\Phi_1 + \frac{i}{\sqrt{3}}[\gamma_1 p_\mu\Phi_\mu - p_1\gamma_p\Phi_p] = (ip + M)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2);$$

$$A = 2, \quad (ip + M)\Phi_2 + \frac{i}{\sqrt{3}}[\gamma_2 p_\mu\Phi_\mu - p_2\gamma_\mu\Phi_\mu] = (ip + M)(-\frac{i}{4})(1 + i\gamma_1\gamma_2)(f_1 + if_2);$$

$$A = 3, \quad (ip + M)\Phi_3 + \frac{i}{\sqrt{3}}[\gamma_3 p_\mu\Phi_\mu - p_3\gamma_\mu\Phi_\mu] = -\frac{i}{\sqrt{3}}p_3(\gamma_1 - i\gamma_2)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2) = 0;$$

$$A = 4, \quad (ip + M)\Phi_4 + \frac{i}{\sqrt{3}}[\gamma_4 p_\mu\Phi_\mu - p_4\gamma_\mu\Phi_\mu] = -\frac{i}{\sqrt{3}}p_4(\gamma_1 - i\gamma_2)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2) = 0.$$

Thus, we find

$$(iP + M)f_{-3/2} = \frac{1}{4}\begin{vmatrix} (ip + M)(1 + i\gamma_1\gamma_2)(f_1 + if_2) \\ -i(ip + M)(1 + i\gamma_1\gamma_2)(f_1 + if_2) \\ 0 \\ 0 \end{vmatrix}. \qquad (33)$$

Now consider the term $\Gamma_- f_{-1/2}$; starting with

$$\Gamma\_\Phi|_A = \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\delta_{A,p}\Phi_p + \frac{1}{\sqrt{3}}\gamma_p[\delta_{A,p}\Phi_1 - \delta_{A,1}\Phi_p - i(\delta_{a,p}\Phi_2 - \delta_{A,2}\Phi_p)]\},$$

depending on the value of A we find:

$$A = 1, \quad \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_1 + \frac{1}{\sqrt{3}}[\gamma_1\Phi_1 - \gamma_p\Phi_p - i\gamma_1\Phi_2]\} =$$
$$= \frac{1}{2\sqrt{6}}\{(\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4)\};$$
$$A = 2, \quad \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_2 + \frac{1}{\sqrt{3}}[\gamma_2\Phi_1 - i\gamma_2\Phi_2 + i\gamma_p\Phi_p]\} =$$
$$= -\frac{i}{2\sqrt{6}}\{(\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4)\};$$
$$A = 3, \quad \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_3 + \frac{1}{\sqrt{3}}[\gamma_3\Phi_1 - i\gamma_3\Phi_2]\} = \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\frac{1}{2}(1 + i\gamma_1\gamma_2) f_3\} = 0;$$
$$A = 4, \quad \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_4 + \frac{1}{\sqrt{3}}[\gamma_4\Phi_1 - i\gamma_4\Phi_2]\} = 0.$$

Thus, we derive the formula

$$\Gamma_- f_{-1/2} = \frac{1}{2\sqrt{6}}\begin{vmatrix} (\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) \\ -i[(\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4)] \\ 0 \\ 0 \end{vmatrix}. \tag{34}$$

Therefore, the radial equations related to $S_3 = -3/2$ are

$$(ip + M)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2) -$$
$$- \frac{1}{2\sqrt{6}} b_{m+1/2}\{(\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4)\} = 0, \tag{35}$$

$$-i(ip + M)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2) +$$
$$+ \frac{i}{2\sqrt{6}} b_{m+1/2}\{(\sqrt{3}-1)(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4)\} = 0; \tag{36}$$

the second equation coincides the first one. Bearing in mind the identities

$$(\gamma_1 - i\gamma_2)(f_1 + if_2) = (1 + i\gamma_1\gamma_2)(\gamma_1 f_1 + \gamma_2 f_2),$$
$$(\gamma_1 - i\gamma_2)(f_1 + if_2) + (1 + i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) = (1 + i\gamma_1\gamma_2)(\gamma_\mu f_\mu),$$

we can present eq. (35) differently

$$(ip + M)\frac{1}{4}(1 + i\gamma_1\gamma_2)(f_1 + if_2) - \frac{1}{2\sqrt{6}} b_{m+1/2}\{\sqrt{3}(\gamma_1 - i\gamma_2)(f_1 + if_2) - (1 + i\gamma_1\gamma_2)(\gamma_\mu f_\mu)\} = 0. \tag{37}$$

We are to eliminate the combination $\gamma_\mu f_\mu$. To this end, let us turn to eq. (27), acting on it by the matrix $(1+i\gamma_1\gamma_2)$, so we get

$$(1+i\gamma_1\gamma_2)(\gamma_\mu \Phi_\mu) = \frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2+(1+\sqrt{3})^2 B}(\gamma_1 - i\gamma_2)(\Phi_1 + i\Phi_2). \quad (38)$$

The last equation can be presented differently

$$(\gamma_1 - i\gamma_2)\frac{1}{4}(1-i\gamma_1\gamma_2)(\Phi_1 + i\Phi_2) + \gamma_3 \frac{1}{2}(1+i\gamma_1\gamma_2)\Phi_3 + \gamma_4 \frac{1}{2}(1+i\gamma_1\gamma_2)(\Phi_4) =$$

$$= \frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2+(1+\sqrt{3})^2 B}(\gamma_1 - i\gamma_2)\frac{1}{4}(1-i\gamma_1\gamma_2)(\Phi_1 + i\Phi_2). \quad (39)$$

Whence, bearing in mind the substitution for the wave function, we derive the radial relation

$$(\gamma_1 - i\gamma_2)(f_1 + if_2) + \gamma_3(1+i\gamma_1\gamma_2) f_3 + \gamma_4(1+i\gamma_1\gamma_2) f_4 = \frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2+(1+\sqrt{3})^2 B}(\gamma_1 - i\gamma_2)(f_1 + if_2).$$

Further, allowing for the identity

$$(\gamma_1 - i\gamma_2)(f_1 + if_2) + (1+i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) = (1+i\gamma_1\gamma_2)(\gamma_\mu f_\mu),$$

we get

$$(1+i\gamma_1\gamma_2)(\gamma_\mu f_\mu) = \frac{\sqrt{3}(1+\sqrt{3})^2 B}{3M^2+(1+\sqrt{3})^2 B}(\gamma_1 - i\gamma_2)(f_1 + if_2). \quad (40)$$

With the help of the last relation, we can eliminate from (37) the combination $\gamma_\mu f_\mu$, so we obtain

$$S_3 = -3/2, \quad (ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)(f_1 + if_2) - \frac{3M^2}{3M^2+(1+\sqrt{3})^2 B} b_{m+1/2}\gamma_-(f_1 + if_2) = 0, \quad (41)$$

where $\gamma_- = \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2)$. The function $(\gamma_3 f_3 + \gamma_4 f_4)$ will be considered as the secondary one; from (40) it follows

$$(1+i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) = \frac{-3M^2 + 2(1+\sqrt{3})B}{3M^2+(1+\sqrt{3})^2 B}(\gamma_1 - i\gamma_2)(f_1 + if_2). \quad (42)$$

5. Equation related to $S = +1/2$

Let us turn to the third equation in (12):

$$S = +1/2, \quad (iP+M) f_{+1/2} - b_{m-3/2}\Gamma_- f_{+3/2} + a_{m+1/2}\Gamma_+ f_{-1/2} = 0. \quad (43)$$

From relation

$$(iP+M)\Phi|_A = (ip+M)\delta_{A,p}\Phi_p + \frac{i}{\sqrt{3}} p_\mu \gamma_p [\delta_{A,p}\Phi_\mu - \delta_{A,\mu}\Phi_p],$$

depending on the value A we get:

$$A=1,\quad (ip+M)\Phi_1+\frac{i}{\sqrt{3}}\gamma_1 p_\mu\Phi_\mu-\frac{i}{\sqrt{3}}p_1\gamma_\mu\Phi_\mu=$$

$$=(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)+\frac{i}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4).$$

$$A=2,\quad (ip+M)\Phi_2+\frac{i}{\sqrt{3}}\gamma_2 p_\mu\Phi_\mu-\frac{i}{\sqrt{3}}p_2\gamma_\mu\Phi_\mu=$$

$$=i(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)-\frac{1}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4).$$

$$A=3,\quad (ip+M)\Phi_3+\frac{i}{\sqrt{3}}\gamma_3 p_\mu\Phi_\mu-\frac{i}{\sqrt{3}}p_3\gamma_\mu\Phi_\mu=$$

$$=(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3\{(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\}.$$

$$A=4,\quad (ip+M)\Phi_4+\frac{i}{\sqrt{3}}\gamma_4 p_\mu\Phi_\mu-\frac{i}{\sqrt{3}}p_4\gamma_\mu\Phi_\mu=$$

$$=(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_3(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4\{(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\}.$$

Thus, we obtain

$$(iP+M)f_{+1/2}=\left|\begin{array}{l}(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)+\frac{i}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4)\\ \hline i(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)-\frac{1}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4)\\ \hline (ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-\\ -\frac{i}{2\sqrt{3}}p_3\{(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\}\\ \hline (ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-\\ -\frac{i}{2\sqrt{3}}p_4\{(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)\}\end{array}\right| \qquad (44)$$

(four matrix rows are divided by lines). Now consider expression for $\Gamma_- f_{+3/2}$:

$$\Gamma_-\Phi\to\frac{1}{\sqrt{2}}\{(\gamma_1-i\gamma_2)\delta_{A,p}\Phi_p+\frac{1}{\sqrt{3}}\gamma_p[\delta_{A,p}\Phi_1-\delta_{A,1}\Phi_p-i(\delta_{A,p}\Phi_2-\delta_{A,2}\Phi_p)]\}.$$

Depending on the value A we find:

$$A=1,\quad \frac{1}{\sqrt{2}}\{(\gamma_1-i\gamma_2)\Phi_1+\frac{1}{\sqrt{3}}[\gamma_1\Phi_1-\gamma_p\Phi_p-i\gamma_1\Phi_2]\}=\frac{1}{2\sqrt{6}}(\sqrt{3}+1)(\gamma_1-i\gamma_2)(f_1-if_2);$$

$$A=2,\quad \frac{1}{\sqrt{2}}\{(\gamma_1-i\gamma_2)\Phi_2+\frac{1}{\sqrt{3}}[\gamma_2\Phi_1-i\gamma_2\Phi_2+i\gamma_p\Phi_p]\}=$$

$$=\frac{1}{\sqrt{2}}\{(\frac{i}{2}+\frac{i}{2\sqrt{3}})(\gamma_1-i\gamma_2)(f_1-if_2)\}=\frac{i}{2\sqrt{6}}(\sqrt{3}+1)(\gamma_1-i\gamma_2)(f_1-if_2);$$

$$A=3,\quad \frac{1}{\sqrt{2}}\{(\gamma_1-i\gamma_2)\Phi_3+\frac{1}{\sqrt{3}}[\gamma_3\Phi_1-i\gamma_3\Phi_2]\}=$$

$$=\frac{1}{\sqrt{6}}\gamma_3\frac{1}{2}(1-i\gamma_1\gamma_2)(f_1-if_2)=\frac{1}{2\sqrt{6}}\gamma_3(1-i\gamma_1\gamma_2)(f_1-if_2);$$

$$A=4,\quad \frac{1}{\sqrt{2}}\{(\gamma_1-i\gamma_2)\Phi_4+\frac{1}{\sqrt{3}}[\gamma_4\Phi_1-i\gamma_4\Phi_2]\}=\frac{1}{2\sqrt{6}}\gamma_4(1-i\gamma_1\gamma_2)(f_1-if_2).$$

Therefore, we obtain the following relation

$$-b_{m-3/2}\Gamma_- f_{+3/2}=-b_{m-3/2}\frac{1}{2\sqrt{6}}\begin{vmatrix}(\sqrt{3}+1)(\gamma_1-i\gamma_2)(f_1-if_2)\\ i(\sqrt{3}+1)(\gamma_1-i\gamma_2)(f_1-if_2)\\ \gamma_3(1-i\gamma_1\gamma_2)(f_1-if_2)\\ \gamma_4(1-i\gamma_1\gamma_2)(f_1-if_2)\end{vmatrix}. \qquad (45)$$

Now consider the term $\Gamma_+ f_{-1/2}$, from

$$\Gamma_+\Phi|_A=\frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\delta_{A,p}\Phi_p+\frac{1}{\sqrt{3}}\gamma_p[\delta_{A,p}\Phi_1-\delta_{A,1}\Phi_p+i(\delta_{A,p}\Phi_2-\delta_{A,2}\Phi_p)]\};$$

depending on the value A we get:

$$A=1,\quad \frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_1+\frac{1}{\sqrt{3}}[\gamma_1\Phi_1-\gamma_p\Phi_p+i\gamma_1\Phi_2]\}=-\frac{1}{2\sqrt{6}}(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4);$$

$$A=2,\quad \frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_2+\frac{1}{\sqrt{3}}[\gamma_2\Phi_1+i\gamma_2\Phi_2-i\gamma_p\Phi_p]\}=$$

$$=\frac{1}{2\sqrt{6}}(\gamma_2+i\gamma_1)(f_1+if_2)-\frac{i}{2\sqrt{6}}(\gamma_1-i\gamma_2)(f_1+if_2)-$$

$$-\frac{i}{2\sqrt{6}}(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)=-\frac{i}{2\sqrt{6}}(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4);$$

$$A=3,\quad \frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_3+\frac{1}{\sqrt{3}}[\gamma_3\Phi_1+i\gamma_3\Phi_2]\}=$$

$$=\frac{1}{2\sqrt{6}}\{2\sqrt{3}(\gamma_1+i\gamma_2)f_3+\gamma_3(1-i\gamma_1\gamma_2)(f_1+if_2)\};$$

$$A=4,\quad \frac{1}{\sqrt{2}}\{(\gamma_1+i\gamma_2)\Phi_4+\frac{1}{\sqrt{3}}[\gamma_4\Phi_1+i\gamma_4\Phi_2]\}=$$

$$=\frac{1}{2\sqrt{6}}\{2\sqrt{3}(\gamma_1+i\gamma_2)f_4+\gamma_4(1-i\gamma_1\gamma_2)(f_1+if_2)\}.$$

Thus, we find the identity

$$a_{m+1/2}\Gamma_+ f_{-1/2} = a_{m+1/2}\,\frac{1}{2\sqrt{6}}\begin{vmatrix} -(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4) \\ -i(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4) \\ 2\sqrt{3}(\gamma_1+i\gamma_2) f_3+\gamma_3(1-i\gamma_1\gamma_2)(f_1+if_2) \\ 2\sqrt{3}(\gamma_1+i\gamma_2) f_4+\gamma_4(1-i\gamma_1\gamma_2)(f_1+if_2) \end{vmatrix}. \qquad (46)$$

Therefore, we have four equations:

$$S_3=+1/2, \quad (ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)+\frac{i}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{1+\sqrt{3}}{2\sqrt{6}}b_{m-3/2}(\gamma_1-i\gamma_2)(f_1-if_2)-\frac{1}{2\sqrt{6}}a_{m+1/2}(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)=0, \qquad (47)$$

$$i(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)-\frac{1}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{6}}(1+\sqrt{3})b_{m-3/2}(\gamma_1-i\gamma_2)(f_1-if_2)-\frac{i}{2\sqrt{6}}a_{m+1/2}(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)=0 \qquad (48)$$

(eq. (48) coincides with (47)),

$$(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3[(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]-\frac{1}{2\sqrt{6}}b_{m-3/2}\gamma_3(1-i\gamma_1\gamma_2)(f_1-if_2)+$$

$$+\frac{1}{2\sqrt{6}}a_{m+1/2}[2\sqrt{3}(\gamma_1+i\gamma_2)f_3+\gamma_3(1-i\gamma_1\gamma_2)(f_1+if_2)]=0, \qquad (49)$$

$$(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1-i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4[(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-3/2}\gamma_4(1-i\gamma_1\gamma_2)(f_1-if_2)+\frac{1}{2\sqrt{6}}a_{m+1/2}[2\sqrt{3}(\gamma_1+i\gamma_2)f_4+\gamma_4(1-i\gamma_1\gamma_2)(f_1+if_2)]=0. \qquad (50)$$

So we have only 3 different equations.

6. Equation related to $S=-1/2$

Let us consider the fourth equation in (12):

$$(iP+M)f_{-1/2}-b_{m-1/2}\Gamma_- f_{+1/2}+a_{m+1/2}\Gamma_+ f_{-3/2}=0.$$

For the term $(iP+M)\Phi|_A$, depending on the value A we get:

$$A=1,\quad (ip+M)\Phi_1+\frac{i}{\sqrt{3}}[\gamma_1 p_\mu\Phi_\mu - p_1\gamma_\mu\Phi_\mu]=$$

$$=(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{i}{\sqrt{3}}\gamma_1\frac{1}{2}(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4);$$

$$A=2,\quad (ip+M)\Phi_2+\frac{i}{\sqrt{3}}[\gamma_2 p_\mu\Phi_\mu - p_2\gamma_\mu\Phi_\mu]=$$

$$=-i(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{1}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3f_3+p_4f_4);$$

$$A=3,\quad (ip+M)\Phi_3+\frac{i}{\sqrt{3}}[\gamma_3 p_\mu\Phi_\mu - p_3\gamma_\mu\Phi_\mu]=$$

$$=(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)];$$

$$A=4,\quad (ip+M)\Phi_4+\frac{i}{\sqrt{3}}[\gamma_4 p_\mu\Phi_\mu - p_4\gamma_\mu\Phi_\mu]=$$

$$=(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)].$$

So we arrive at the formula

$$(iP+M)f_{-1/2}=\left|\begin{array}{l} (ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{i}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3f_3+p_4f_4) \\ \hline -i(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{1}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3f_3+p_4f_4) \\ \hline (ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)- \\ -\frac{i}{2\sqrt{3}}p_3\{(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)\} \\ \hline (ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)- \\ -\frac{i}{2\sqrt{3}}p_4\{(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)\} \end{array}\right|. \qquad (51)$$

Consider the term

$$\Gamma_- f_{+1/2} = \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\delta_{A,p}\Phi_p + \frac{1}{\sqrt{3}}\gamma_p[\delta_{A,p}\Phi_1 - \delta_{A,1}\Phi_p - i(\delta_{A,p}\Phi_2 - \delta_{A,2}\Phi_p)]\};$$

depending on the value A we obtain:

$$A=1, \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_1 + \frac{1}{\sqrt{3}}[\gamma_1\Phi_1 - \gamma_p\Phi_p - i\gamma_1\Phi_2]\} = -\frac{1}{2\sqrt{6}}(1 - i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4);$$

$$A=2, \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_2 + \frac{1}{\sqrt{3}}[\gamma_2\Phi_1 - i\gamma_2\Phi_2 + i\gamma_p\Phi_p]\} = \frac{i}{2\sqrt{6}}(1 - i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4);$$

$$A=3, \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_3 + \frac{1}{\sqrt{3}}[\gamma_3\Phi_1 - i\gamma_3\Phi_2]\} = \frac{1}{2\sqrt{6}}\{2\sqrt{3}(\gamma_1 - i\gamma_2) f_3 + \gamma_3(1 + i\gamma_1\gamma_2)(f_1 - if_2)\};$$

$$A=4, \frac{1}{\sqrt{2}}\{(\gamma_1 - i\gamma_2)\Phi_4 + \frac{1}{\sqrt{3}}[\gamma_4\Phi_1 - i\gamma_4\Phi_2]\} = \frac{1}{2\sqrt{6}}\{2\sqrt{3}(\gamma_1 - i\gamma_2) f_4 + \gamma_4(1 + i\gamma_1\gamma_2)(f_1 - if_2)\}.$$

Thus, we have the formula

$$\Gamma_- f_{+1/2} = \frac{1}{2\sqrt{6}}\left|\begin{matrix} -(1 - i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) \\ i(1 - i\gamma_1\gamma_2)(\gamma_3 f_3 + \gamma_4 f_4) \\ 2\sqrt{3}(\gamma_1 - i\gamma_2) f_3 + \gamma_3(1 + i\gamma_1\gamma_2)(f_1 - if_2) \\ 2\sqrt{3}(\gamma_1 - i\gamma_2) f_4 + \gamma_4(1 + i\gamma_1\gamma_2)(f_1 - if_2) \end{matrix}\right|.$$

Consider the term $\Gamma_+ f_{-3/2}$:

$$\Gamma_+\Phi|_A = \frac{1}{\sqrt{2}}\{(\gamma_1 + i\gamma_2)\delta_{A,\wp}\Phi_\wp + \frac{1}{\sqrt{3}}\gamma_\wp[\delta_{A,\wp}\Phi_1 - \delta_{A,1}\Phi_\wp + i(\delta_{A,\wp}\Phi_2 - \delta_{A,2}\Phi_\wp)]\}.$$

Depending on the value A we get:

$$A=1, \frac{1}{\sqrt{2}}\{(\gamma_1 + i\gamma_2)\Phi_1 + \frac{1}{\sqrt{3}}[\gamma_1\Phi_1 - \gamma_p\Phi_p + i\gamma_1\Phi_2]\} = \frac{1}{2\sqrt{6}}(\sqrt{3}+1)(\gamma_1 + i\gamma_2)(f_1 + if_2);$$

$$A=2, \frac{1}{\sqrt{2}}\{(\gamma_1 + i\gamma_2)\Phi_2 + \frac{1}{\sqrt{3}}[\gamma_2\Phi_1 + i\gamma_2\Phi_2 - i\gamma_p\Phi_p]\} = -\frac{i}{2\sqrt{6}}(\sqrt{3}+1)(\gamma_1 + i\gamma_2)(f_1 + if_2);$$

$$A=3, \frac{1}{\sqrt{2}}\{(\gamma_1 + i\gamma_2)\Phi_3 + \frac{1}{\sqrt{3}}[\gamma_3\Phi_1 + i\gamma_3\Phi_2]\} = \frac{1}{2\sqrt{6}}\gamma_3(1 + i\gamma_1\gamma_2)(f_1 + if_2);$$

$$A=4, \frac{1}{\sqrt{2}}\{(\gamma_1 + i\gamma_2)\Phi_4 + \frac{1}{\sqrt{3}}[\gamma_4\Phi_1 + i\gamma_4\Phi_2]\} = \frac{1}{2\sqrt{6}}\gamma_4(1 + i\gamma_1\gamma_2)(f_1 + if_2).$$

Thus, we obtain the formula

$$\Gamma_+ f_{-3/2} = \frac{1}{2\sqrt{6}}\left|\begin{matrix} (\sqrt{3}+1)(\gamma_1 + i\gamma_2)(f_1 + if_2) \\ -i(\sqrt{3}+1)(\gamma_1 + i\gamma_2)(f_1 + if_2) \\ \gamma_3(1 + i\gamma_1\gamma_2)(f_1 + if_2) \\ \gamma_4(1 + i\gamma_1\gamma_2)(f_1 + if_2) \end{matrix}\right|.$$

Therefore, we get 4 equations:

$$S_3 = -1/2, \quad (ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{i}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3 f_3+p_4 f_4)+$$

$$+\frac{1}{2\sqrt{6}}b_{m-1/2}(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)+\frac{\sqrt{3}+1}{2\sqrt{6}}a_{m+3/2}(\gamma_1+i\gamma_2)(f_1+if_2)=0; \quad (52)$$

$$-i(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{1}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{6}}b_{m-1/2}(1-i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)-i\frac{\sqrt{3}+1}{2\sqrt{6}}a_{m+3/2}(\gamma_1+i\gamma_2)(f_1+if_2)=0 \quad (53)$$

(they differ only in the multiplier i),

$$(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1+i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-1/2}[2\sqrt{3}(\gamma_1-i\gamma_2)f_3+\gamma_3(1+i\gamma_1\gamma_2)(f_1-if_4)]+\frac{1}{2\sqrt{6}}a_{m+3/2}\gamma_3(1+i\gamma_1\gamma_2)(f_1+if_2)=0; \quad (54)$$

$$(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1+i\gamma_1\gamma_2)(p_3 f_3+p_4 f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3 f_3+\gamma_4 f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-1/2}[2\sqrt{3}(\gamma_1-i\gamma_2)f_4+\gamma_4(1+i\gamma_1\gamma_2)(f_1-if_4)]+\frac{1}{2\sqrt{6}}a_{m+3/2}\gamma_4(1+i\gamma_1\gamma_2)(f_1+if_2)=0. \quad (55)$$

So, we have only 3 different equations.

7. The system of equations in the variable r

Let us collect together equations (31), (41), (47)–(50), (52)–(55). We divide 8 equations into two groups. The first group is as follows:

$$(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)(f_1-if_2)+a_{m-1/2}\frac{3M^2}{3M^2-(1+\sqrt{3})^2 B}\gamma_+(f_1-if_2)=0,$$

$$(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)(f_1+if_2)-\frac{3M^2}{3M^2+(1+\sqrt{3})^2 B}b_{m+1/2}\gamma_-(f_1+if_2)=0,$$

or shortly

$$(ip+M)F_1+\frac{3M^2}{3M^2-(1+\sqrt{3})^2 B}a_{m-1/2}\gamma_+ F_2=0, \quad (56)$$

$$(ip+M)F_4-\frac{3M^2}{3M^2+(1+\sqrt{3})^2 B}b_{m+1/2}\gamma_- F_3=0; \quad (57)$$

the third equation

$$(ip+M)\frac{1}{4}(1+i\gamma_1\gamma_2)(f_1-if_2)+\frac{i}{2\sqrt{3}}(\gamma_1-i\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{1+\sqrt{3}}{2\sqrt{6}}b_{m-3/2}(\gamma_1-i\gamma_2)(f_1-if_2)-\frac{1}{2\sqrt{6}}a_{m+1/2}(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)=0$$

is transformed to other form (with the use of the above constraint)

$$\{(ip+M)\frac{3M^2-2B}{3M^2-(1+\sqrt{3})^2B}-\frac{2BM}{3M^2-(1+\sqrt{3})^2B}\}F_2-b_{m-3/2}\gamma_-F_1-\frac{2Ba_{m+1/2}\gamma_-}{3M^2+(1+\sqrt{3})^2B}F_3=0; \quad (58)$$

the fourth equation

$$(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-1/2}[2\sqrt{3}(\gamma_1-i\gamma_2)f_4+\gamma_4(1+i\gamma_1\gamma_2)(f_1-if_4)]+\frac{1}{2\sqrt{6}}a_{m+3/2}\gamma_4(1+i\gamma_1\gamma_2)(f_1+if_2)=0$$

is transformed to the form (again with the use of the known constraint)

$$\{(ip+M)\frac{3M^2+2B}{3M^2+(1+\sqrt{3})^2B}+\frac{2BM}{3M^2+(1+\sqrt{3})^2B}\}F_3+a_{m+3/2}\gamma_+F_4-\frac{2Bb_{m-1/2}\gamma_+}{3M^2-(1+\sqrt{3})^2B}F_2=0. \quad (59)$$

In equations in eqs.(56)–(59) we use the notations

$$F_1=\frac{1}{2}(1-i\gamma_1\gamma_2)(f_1-if_2), \quad F_2=\frac{1}{2}(1+i\gamma_1\gamma_2)(f_1-if_2),$$
$$F_3=\frac{1}{2}(1-i\gamma_1\gamma_2)(f_1+if_2), \quad F_4=\frac{1}{2}(1+i\gamma_1\gamma_2)(f_1+if_2), \quad (60)$$

Below we write down the remaining 4 equations:

$$(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1-i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3[(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-3/2}\gamma_3(1-i\gamma_1\gamma_2)(f_1-if_2)+$$

$$+\frac{1}{2\sqrt{6}}a_{m+1/2}[2\sqrt{3}(\gamma_1+i\gamma_2)f_3+\gamma_3(1-i\gamma_1\gamma_2)(f_1+if_2)]=0, \quad (61)$$

$$(ip+M)\frac{1}{2}(1-i\gamma_1\gamma_2)f_4+\frac{i}{2\sqrt{3}}\gamma_4(1-i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_4[(\gamma_1+i\gamma_2)(f_1-if_2)+(1-i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-3/2}\gamma_4(1-i\gamma_1\gamma_2)(f_1-if_2)+$$

$$+\frac{1}{2\sqrt{6}}a_{m+1/2}[2\sqrt{3}(\gamma_1+i\gamma_2)f_4+\gamma_4(1-i\gamma_1\gamma_2)(f_1+if_2)]=0; \quad (62)$$

$$(ip+M)\frac{1}{4}(1-i\gamma_1\gamma_2)(f_1+if_2)+\frac{i}{2\sqrt{3}}(\gamma_1+i\gamma_2)(p_3f_3+p_4f_4)+$$

$$+\frac{1}{2\sqrt{6}}b_{m-1/2}(1-i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)+\frac{\sqrt{3}+1}{2\sqrt{6}}a_{m+3/2}(\gamma_1+i\gamma_2)(f_1+if_2)=0, \quad (63)$$

$$(ip+M)\frac{1}{2}(1+i\gamma_1\gamma_2)f_3+\frac{i}{2\sqrt{3}}\gamma_3(1+i\gamma_1\gamma_2)(p_3f_3+p_4f_4)-$$

$$-\frac{i}{2\sqrt{3}}p_3[(\gamma_1-i\gamma_2)(f_1+if_2)+(1+i\gamma_1\gamma_2)(\gamma_3f_3+\gamma_4f_4)]-$$

$$-\frac{1}{2\sqrt{6}}b_{m-1/2}[2\sqrt{3}(\gamma_1-i\gamma_2)f_3+\gamma_3(1+i\gamma_1\gamma_2)(f_1-if_4)]+$$

$$+\frac{1}{2\sqrt{6}}a_{m+3/2}\gamma_3(1+i\gamma_1\gamma_2)(f_1+if_2)=0. \quad (64)$$

Let us act on eq. (56) by the matrix $(M-ip)$, this gives

$$F_1=-\frac{1}{p^2+M^2}\frac{3M^2}{3M^2-(1+\sqrt{3})B}a_{m-1/2}(M-ip)\gamma_+F_2. \quad (65)$$

Taking into account this expression for F_1, we transform eq. (58) to the form

$$\frac{1}{3M^2-(1+\sqrt{3})B}\{(ip+M)[3M^2-2B+\frac{6M^2b_{m-3/2}a_{m-1/2}}{p^2+M^2}]-2MB\}F_2-\frac{2B\,a_{m+1/2}\gamma_-}{3M^2+(1+\sqrt{3})^2B}F_3=0. \quad (66)$$

Similarly, from eq. (57) it follows

$$F_4=\frac{1}{p^2+M^2}\frac{3M^2}{3M^2+(1+\sqrt{3})^2B}b_{m+1/2}(M-ip)\gamma_-F_3; \quad (67)$$

so from eq. (59) we derive

$$\frac{1}{3M^2+(1+\sqrt{3})^2B}\{(ip+M)[3M^2+2B+\frac{6M^2a_{m+3/2}b_{m+1/2}}{p^2+M^2}]+2BM\}F_3-\frac{2B\,b_{m-1/2}\gamma_+}{3M^2-(1+\sqrt{3})^2B}F_2=0. \quad (68)$$

Thus, we have the system of two linked equations (66), (68) for the variables F_2, F_3. Let us apply the exclusion method. To this end, we act on eq. (68) by the operator $a_{m+1/2}\gamma_-$, which results in

$$\frac{1}{2B}\{(-ip+M)[(3M^2+2B)+\frac{6M^2}{p^2+M^2}(a_{m+1/2}b_{m-1/2}-B)]+2BM\}\times$$

$$\times\frac{2B}{3M^2+(1+\sqrt{3})^2B}a_{m+1/2}\gamma_-F_3-\frac{4B}{3M^2-(1+\sqrt{3})B}a_{m+1/2}b_{m-1/2}F_2=0. \quad (69)$$

Equation (69), with the use of (66), is transformed to the 4-order equation for F_2:

$$\{(-ip+M)[(3M^2+2B)+\frac{6M^2}{p^2+M^2}(a_{m+1/2}b_{m-1/2}-B)]+2BM\}\times$$

$$\times\{(ip+M)[(3M^2-2B)+\frac{6M^2}{p^2+M^2}(a_{m+1/2}b_{m-1/2}+B)]-2BM\}F_2-8(B)^2a_{m+1/2}b_{m-1/2}F_2=0; \quad (70)$$

we have taken into account the identity $b_{m-3/2}a_{m-1/2}-a_{m+1/2}b_{m-1/2}=B.$

Similarly, let us act on eq. (66) by the operator $b_{m-1/2}\gamma_+$, this gives

$$\{(-ip+M)[(3M^2-2B)+\frac{6M^2}{p^2+M^2}(b_{m-1/2}a_{m+1/2}+B)]-2BM\}\times$$

$$\times\frac{2B}{3M^2-(1+\sqrt{3})^2B}b_{m-1/2}\gamma_+F_2-\frac{8(B)^2}{3M^2+(1+\sqrt{3})^2B}b_{m-1/2}a_{m+1/2}F_3=0.$$

Using eq. (66), we can eliminate the variable F_2:

$$\{(-ip+M)[(3M^2-2B)+\frac{6M^2}{p^2+M^2}(b_{m-1/2}a_{m+1/2}+B)]-2BM\}\times$$

$$\times\{(ip+M)[(3M^2+2B)+\frac{6M^2}{p^2+M^2}(b_{m-1/2}a_{m+1/2}-B)]+2BM\}F_3-8(B)^2b_{m-1/2}a_{m+1/2}F_3=0. \quad (71)$$

Thus, we have two 4-order differential equations, (70) and (71), for functions F_2 and F_3; after regrouping the terms they take on the form:

$$\{(-ip+M)[3M^2(1+\frac{2a_{m+1/2}b_{m-1/2}}{p^2+M^2})+2B(1-\frac{3M^2}{p^2+M^2})]+2BM\}\times$$

$$\times\{(ip+M)[3M^2(1+\frac{2a_{m+1/2}b_{m-1/2}}{p^2+M^2})-2B(1-\frac{3M^2}{p^2+M^2})]-2BM\}F_2=8B^2a_{m+1/2}b_{m-1/2}F_2, \quad (72)$$

$$\{(-ip+M)[3M^2(1+\frac{2b_{m-1/2}a_{m+1/2}}{p^2+M^2})-2B(1-\frac{3M^2}{p^2+M^2})]-2BM\}\times$$

$$\times\{(ip+M)[3M^2(1+\frac{2b_{m-1/2}a_{m+1/2}}{p^2+M^2})+2B(1-\frac{3M^2}{p^2+M^2})]+2BM\}F_3=8B^2b_{m-1/2}a_{m+1/2}F_3. \quad (73)$$

From equations (72) and (73), we derive

$$\{9M^4\beta^2+12BM^3ip\beta-4B^2(p^2+M^2)\beta+12B^2p^2M^2\}F_2=0, \quad (74)$$

$$\{9M^4\lambda^2-12eBM^3ip\lambda-4B^2(p^2+M^2)\lambda+12B^2p^2M^2\}F_3=0, \quad (75)$$

where the notations β and λ are used:

$$\beta=p^2+M^2+2a_{m+1/2}b_{m-1/2}, \quad \lambda=p^2+M^2+2b_{m-1/2}a_{m+1/2}. \quad (76)$$

Below we need the explicit form of the matrices

$$\gamma_1 = \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \gamma_2 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \gamma_3 = \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}, \gamma_4 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix},$$

$$i\gamma_1\gamma_2 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \frac{1}{2}(1+i\gamma_1\gamma_2) = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \frac{1}{2}(1-i\gamma_1\gamma_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$ip = \begin{vmatrix} 0 & 0 & p_3+ip_4 & 0 \\ 0 & 0 & 0 & -p_3+ip_4 \\ -p_3+ip_4 & 0 & 0 & 0 \\ 0 & p_3+ip_4 & 0 & 0 \end{vmatrix}.$$

Bearing in mind these expressions, we find the explicit structure of F_2 and F_3:

$$F_2(r) = \begin{vmatrix} 0 \\ f_{1,2}-if_{2,2} \\ 0 \\ f_{1,4}-if_{2,4} \end{vmatrix} = \begin{vmatrix} 0 \\ \chi_2 \\ 0 \\ \chi_4 \end{vmatrix}, \quad F_3(r) = \begin{vmatrix} f_{1,1}+if_{2,1} \\ 0 \\ f_{1,3}+if_{2,3} \\ 0 \end{vmatrix} = \begin{vmatrix} \varphi_1 \\ 0 \\ \varphi_3 \\ 0 \end{vmatrix}. \tag{77}$$

Correspondingly, eqs. (75) reduce to two linked equations

$$\begin{gathered} \{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}\varphi_1 - 12BM^3(+p_3+ip_4)\lambda\varphi_3 = 0, \\ \{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}\varphi_3 - 12BM^3(-p_3+ip_4)\lambda\varphi_1 = 0. \end{gathered} \tag{78}$$

Let us multiply the first equation in (78) by $(-p_3+ip_4)$, then we get

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}(-p_3+ip_4)\varphi_1 + 12BM^3p^2\lambda\varphi_3 = 0,$$

whence it follows

$$\lambda\varphi_3 = -\frac{1}{12BM^3p^2}\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}(-p_3+ip_4)\varphi_1. \tag{79}$$

Acting on the second equation in (78) by operator λ, we get

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}\lambda\varphi_3 - 12BM^3(-p_3+ip_4)\lambda^2\varphi_1 = 0.$$

Further, bearing in mind (79), we obtain the 4-order equation for function φ_1:

$$\begin{gathered} \{(9M^4)^2\lambda^4 - 72M^4B^2(p^2+M^2)\lambda^3 + 16B^4(p^2+M^2)^2\lambda^2 + 360M^6B^2p^2\lambda^2 - \\ -96B^4p^2M^2(p^2+M^2)\lambda + 144B^4(p^2)^2M^4\}\varphi_1 = 0. \end{gathered}$$

The 4-order operator is factorized in the product of two commuting multipliers

$$\begin{gathered} \{9M^4\lambda^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\times \\ \times\{9M^4\lambda^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_1 = 0. \end{gathered} \tag{80}$$

We should find solutions of both 2-order equations

$$\{9M^4\lambda^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_1 = 0, \tag{81}$$

$$\{9M^4\lambda^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_1 = 0. \tag{82}$$

Similarly we can derive equations for the variable φ_3. To this end, let us multiply the second equation in (78) by $(-p_3 - ip_4)$, this gives

$$\lambda\varphi_1 = -\frac{1}{12BM^3p^2}\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}(p_3+ip_4)\varphi_3.$$

This expression for $\lambda\varphi_1$ should be taken into account in (78), in this way we arrive at 4-order equation for the variable φ_3, which also is factorized as follows

$$\{9M^4\lambda^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\times$$
$$\times\{9M^4\lambda^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_3 = 0. \tag{83}$$

There exist two equations for φ_3:

$$\{9M^4\lambda^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_3 = 0, \tag{84}$$

$$\{9M^4\lambda^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}\,]\lambda + 12B^2p^2M^2\}\varphi_3 = 0. \tag{85}$$

Let us derive the 2-order differential equations for the variables φ_1 and φ_3. To this end, first consider equation (see (81)), presented in the form

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda\}\varphi_1 = \{12M^3\sqrt{-B^2p^2}\,\lambda - 12B^2p^2M^2\}\varphi_1,$$

and take it into account in the first equation in (78):

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}\varphi_1 - 12BM^3(+p_3+ip_4)\lambda\varphi_3 = 0,$$

in this way we obtain

$$\lambda\{\sqrt{-B^2p^2}\,\varphi_1 - B(p_3+ip_4)\varphi_3\} = 0. \tag{86}$$

Now let us turn to eq. (84), presented in the form

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda\}\varphi_3 = \{12M^3\sqrt{-B^2p^2}\,\lambda - 12B^2p^2M^2\}\varphi_3, \tag{87}$$

and take it into account in the second equation in (78):

$$\{9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda + 12B^2p^2M^2\}\varphi_3 - 12BM^3(-p_3+ip_4)\lambda\varphi_1 = 0,$$

so we obtain

$$\lambda\{-B(-p_3+ip_4)\varphi_1 + \sqrt{-B^2p^2}\,\varphi_3\} = 0. \tag{88}$$

Let us joint equations (86) and (88) into one system:

$$\lambda\{\sqrt{-B^2p^2}\,\varphi_1 - B(p_3+ip_4)\varphi_3\} = 0,\ \lambda\{-B(-p_3+ip_4)\varphi_1 + \sqrt{-B^2p^2}\,\varphi_3\} = 0. \tag{89}$$

We may conclude that both functions φ_1 and φ_3 satisfy one and the same structure

$$\lambda\varphi_1 = 0, \quad \lambda\varphi_3 = 0, \ \lambda F_1 = 0. \qquad (90)$$

The same is valid for function F_3. Indeed, let us turn to eq. (82) for function φ_1, presented in the form

$$[9M^4\lambda^2 - 4B^2(p^2+M^2)\lambda]\varphi_1 = -12M^3\sqrt{-B^2p^2}\lambda\varphi_1 - 12B^2p^2M^2\varphi_1;$$

taking into account this equality, from eq. (78) we derive

$$\lambda[\sqrt{-B^2p^2}\varphi_1 + B(p_3+ip_4)\varphi_3] = 0. \qquad (91)$$

Similarly, using eq. (85) in the form

$$[9M^4\lambda^2 - 4B^2(p^2+M^2)]\varphi_3 = -12M^3\sqrt{-B^2p^2}\lambda\varphi_3 - 12B^2p^2M^2\varphi_3$$

we derive from eq. (78) the following result

$$\lambda[B(-p_3+ip_4)\varphi_1 + \sqrt{-B^2p^2}\varphi_3] = 0. \qquad (92)$$

Considering eqs. (91) and (92) as a system, we conclude that functions φ_1, φ_3, F_3 satisfy one of the same equation $\lambda\varphi_1 = 0, \quad \lambda\varphi_3 = 0, \quad F_3 = 0.$

8. Solutions for functions $F_3 - \{\varphi_1, \varphi_3\}$

Let us study equation $\lambda F_3 = 0$. Explicitly, it reads

$$\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1/2)^2}{r^2} + B(m-1/2) - (\frac{Br}{2})^2 - (p^2+M^2)\}F_3 = 0. \qquad (93)$$

In the variable $x = |B_0|r^2$, where $B_0 = \dfrac{B}{2}$; this equation reads

$$\{x\frac{d^2}{dx^2} + \frac{d}{dx} - [\frac{(m+1/2)^2}{4x} - \frac{B_0}{4|B_0|}(2m-1) + \frac{x}{4} + \frac{p^2+M^2}{4|B_0|}]\}F_3 = 0. \qquad (94)$$

We search solutions in the form $F_3(x) = x^A e^{-Cx} g_3(x)$; for function $g_3(x)$ we get the equation

$$\{x\frac{d^2}{dx^2} + (2A+1-2Cx)\frac{d}{dx} + \frac{1}{x}[A^2 - \frac{(m+1/2)^2}{4}] + (C^2 - 1/4) -$$
$$-[2AC + C - \frac{B_0}{4|B_0|}(2m-1) + \frac{p^2+M^2}{4|B_0|}]\}g_3 = 0.$$

Fixing parameters as follows $A = |m+1/2|/2, C = 1/2,$ we get

$$\{x\frac{d^2}{dx^2} + (|m+1/2|+1-x)\frac{d}{dx} - [\frac{|m+1/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m-1) + \frac{p^2+M^2}{4|B_0|}]\}g_3 = 0, \qquad (95)$$

which is the confluent hypergeometric equation with parameters

$$\gamma = |m+1/2|+1, \quad \alpha = \frac{|m+1/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m-1) + \frac{p^2+M^2}{4|B_0|}.$$

The quantization rule is $\alpha = -n$, which leads to

$$\varepsilon^2 - p_3^2 - M^2 = 4|B_0|n + 2|B_0|(|m+1/2|+1) - B_0(2m-1); \quad (96)$$

this spectrum corresponds to the functions $\{\varphi_1, \varphi_3\} - F_3$.

9. Equations for $F_2 - \{\chi_2, \chi_4\}$

Let us consider eq. (74):

$$\{9M^4\beta^2 + 12BM^3 ip\beta - 4B^2(p^2+M^2)\beta + 12B^2p^2M^2\}F_2 = 0. \quad (97)$$

From (97) it follows the system of two linked equations

$$\{9M^4\beta^2 - 4B^2(p^2+M^2)\beta + 12B^2p^2M^2\}\chi_2 + 12BM^3(-p_3+ip_4)\beta\chi_4 = 0, \quad (98)$$

$$12BM^3(p_3+ip_4)\beta\chi_2 + \{9M^4\beta^2 - 4B^2(p^2+M^2)\beta + 12B^2p^2M^2\}\chi_4 = 0. \quad (99)$$

From eq. (98) we can express $\beta\chi_4$:

$$\beta\chi_4 = \frac{1}{12BM^3p^2}\{9M^4\beta^2 - 4B^2(p^2+M^2)\beta + 12B^2p^2M^2\}(p_3+ip_4)\chi_2. \quad (100)$$

With this in mind, from eq. (99) we derive

$$\{(9M^4)^2\beta^4 - 72M^4B^2(p^2+M^2)\beta^3 + 16B^4(p^2+M^2)^2\beta^2 +$$
$$+360M^6B^2p^2\beta^2 - 96B^4p^2M^2(p^2+M^2)\beta + 144B^4(p^2)^2M^4\}\chi_2 = 0.$$

The fourth order operator is factorized in the product of two commuting multipliers

$$\{9M^4\beta^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\times$$
$$\times\{9M^4\beta^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\chi_2 = 0, \quad (101)$$

we should find solutions of both equations

$$\{9M^4\beta^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\chi_2 = 0, \quad (102)$$

$$\{9M^4\beta^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\chi_2 = 0. \quad (103)$$

Let us derive similar equations for function χ_4. To this end, we should start with eq. (99) written in the form

$$\beta\chi_2 = \frac{1}{12BM^3p^2}\{9M^4\beta^2 - 4B^2(p^2+M^2)\beta + 12B^2p^2M^2\}(-p_3+ip_4)\chi_4. \quad (104)$$

Taking into account (104), from eq. (98) we derive

$$\{(9M^4)^2\beta^4 - 72M^4B^2(p^2+M^2)\beta^3 + 16B^4(p^2+M^2)^2\beta^2 +$$
$$+360M^6B^2p^2\beta^2 - 96B^4p^2M^2(p^2+M^2)\beta + 144B^4(p^2)^2M^4\}\chi_4 = 0.$$

The fourth order operator is factorized into the product of two commuting multipliers

$$\{9M^4\beta^2 + [-4B^2(p^2+M^2) + 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\times$$
$$\times\{9M^4\beta^2 + [-4B^2(p^2+M^2) - 12M^3\sqrt{-B^2p^2}]\beta + 12B^2p^2M^2\}\chi_4 = 0. \quad (105)$$

We should find solutions of both equations

$$\{9M^4\beta^2+[-4B^2(p^2+M^2)-12M^3\sqrt{-B^2p^2}\,]\beta+12B^2p^2M^2\}\chi_4=0, \tag{106}$$

$$\{9M^4\beta^2+[-4B^2(p^2+M^2)+12M^3\sqrt{-B^2p^2}\,]\beta+12B^2p^2M^2\}\chi_4=0. \tag{107}$$

By analogy with previous section, we can prove that the functions χ_2,χ_4,F_2 obey one and the same equation

$$\beta\chi_4=0,\quad \beta\chi_2=0,\quad \beta F_2=0. \tag{108}$$

10. Solutions for functions $F_2-\{\chi_2,\chi_4\}$

Let us consider eq. (108)

$$\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m-1/2)^2}{r^2}+B(m+1/2)-(\frac{Br}{2})^2-(p^2+M^2)\}F_2=0. \tag{109}$$

In the variable $x=|B_0|r^2$, it reads

$$\{x\frac{d^2}{dx^2}+\frac{d}{dx}-[\frac{(m-1/2)^2}{4x}-\frac{B_0}{4|B_0|}(2m+1)+\frac{x}{4}+\frac{p^2+M^2}{4|B_0|}]\}F_2=0. \tag{110}$$

We search solutions in the form $F_2(x)=x^Ae^{-Cx}g_2(x)$, where $A=\frac{|m-1/2|}{2}$, $C=1/2$, further we get

$$\{x\frac{d^2}{dx^2}+(|m-1/2|+1-x)\frac{d}{dx}-[\frac{|m-1/2|}{2}+\frac{1}{2}-\frac{B_0}{4|B_0|}(2m+1)+\frac{p^2+M^2}{4|B_0|}]\}g_2=0, \tag{111}$$

which is the confluent hypergeometric equation with parameters

$$\gamma=|m-1/2|+1,\quad \alpha=\frac{|m-1/2|}{2}+\frac{1}{2}-\frac{B_0}{4|B_0|}(2m+1)+\frac{p^2+M^2}{4|B_0|}.$$

The quantization rule $\alpha=-n$ leads to

$$\varepsilon^2-p_3^2-M^2=4|B_0|n+2|B_0(|m-1/2|+1)-B_0(2m+1); \tag{112}$$

this spectrum corresponds to the functions $\{\chi_2,\chi_4\}-F_2$.

11. Equations for concomitant components F_1 and F_4

Let us consider eq. (67):

$$F_4=\frac{1}{p^2+M^2}\frac{3M^2}{3M^2+(1+\sqrt{3})^2B}b_{m+1/2}(M-ip)\gamma_-F_3.$$

Acting on it by operator $(M-ip)a_{m+3/2}\gamma_+$, we obtain

$$(M-ip)a_{m+3/2}\gamma_+F_4=\frac{6M^2}{3M^2+(1+\sqrt{3})^2B}[b_{m-1/2}a_{m+1/2}-B]F_3=$$

$$=\frac{6M^2}{3M^2+(1+\sqrt{3})^2B}\{\frac{1}{2}[\lambda-(p^2+M^2)]-B\}F_3=-\frac{3M^2}{3M^2+(1+\sqrt{3})^2B}[(p^2+M^2)+2B]F_3,$$

where $\lambda F_3 = 0$ is taken into account. Consequently, we get equation for the variable F_3 in the form

$$F_3 = -\frac{3M^2 + (1+\sqrt{3})^2 B}{3M^2[(p^2 + M^2) + 2B]}(M - ip)a_{m+3/2}\gamma_+ F_4. \qquad (113)$$

With the use of (113), equation (57) is transformed to the following form

$$\{(p^2 + M^2) + 2B + 2b_{m+1/2}a_{m+3/2}\}F_4 = 0;$$

or explicitly

$$\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+3/2)^2}{r^2} + B(m-3/2) - (\frac{Br}{2})^2 - (p^2 + M^2)\}F_4 = 0;$$

in the variable $x = |B_0| r^2$, it reads

$$\{x\frac{d^2}{dx^2} + \frac{d}{dx} - [\frac{(m+3/2)^2}{4x} - \frac{B_0}{4|B_0|}(2m-3) + \frac{x}{4} + \frac{p^2 + M^2}{4|B_0|}]\}F_4 = 0. \qquad (114)$$

Let us search solutions in the form $F_4(x) = x^A e^{-Cx} f_4(x)$, where $A = \frac{|m+3/2|}{2}$, $C = 1/2$; for the variable g_4 we find an equation of the confluent hypergeometric type

$$\{x\frac{d^2}{dx^2} + (|m+3/2| + 1 - x)\frac{d}{dx} - [\frac{|m+3/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m-3) + \frac{p^2 + M^2}{4|B_0|}]\}g_4 = 0 \qquad (115)$$

with parameters

$$\gamma = |m+3/2| + 1, \alpha = \frac{|m+3/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m-3) + \frac{p^2 + M^2}{4|B_0|}.$$

The quantization rule is $\alpha = -n$ gives

$$\varepsilon^2 - p_3^2 - M^2 = 4|B_0|n + 2|B_0|(|m+3/2| + 1) - B_0(2m-3); \qquad (116)$$

this spectrum corresponds to the function F_4.

Let us find equation for the variable F_1:

$$F_1 = \frac{1}{p^2 + M^2}\frac{3M^2}{3M^2 - (1+\sqrt{3})^2 B}a_{m-1/2}(M - ip)\gamma_+ F_2.$$

Acting on this by operator $(M - ip)b_{m-3/2}\gamma_-$, we obtain

$$(M - ip)b_{m-3/2}\gamma_- F_1 =$$

$$= -\frac{6M^2}{3M^2 - (1+\sqrt{3})^2 B}b_{m-3/2}a_{m-1/2}F_2 = -\frac{6M^2}{3M^2 - (1+\sqrt{3})^2 B}[a_{m+1/2}b_{m-1/2} + B]F_2 =$$

$$= -\frac{6M^2}{3M^2 - (1+\sqrt{3})^2 B}\{\frac{1}{2}[\beta - (p^2 + M^2)] + B\}F_2 = \frac{3M^2}{3M^2 - (1+\sqrt{3})^2 B}[(p^2 + M^2) - 2B]F_2,$$

where eq. $\lambda F_2 = 0$ is taken into account. Thus, we have the expression for the variable F_2:

$$F_2 = \frac{3M^2 - (1+\sqrt{3})^2 B}{3M^2[(p^2 + M^2) - 2B]}(M - ip)b_{m-3/2}\gamma_- F_1. \qquad (117)$$

With this in mind, from eq. (56) we obtain the equation

$$\{2a_{m-1/2}b_{m-3/2} + p^2 + M^2 - 2B\}F_1 = 0. \qquad (118)$$

Explicitly, it reads

$$\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-3/2)^2}{r^2} + B(m+3/2) - (\frac{Br}{2})^2 - (p^2 + M^2)\}F_1 = 0. \qquad (119)$$

In the variable $x = |B_0| r^2$, we have

$$\{x\frac{d^2}{dx^2} + \frac{d}{dx} - [\frac{(m-3/2)^2}{4x} - \frac{B_0}{4|B_0|}(2m+3) + \frac{x}{4} + \frac{p^2 + M^2}{4|B_0|}]\}F_1 = 0.$$

Its solutions are searching in the form $F_1(x) = x^A e^{-Cx} g_1(x)$, where $A = \frac{|m-3/2|}{2}$, $C = 1/2$:

$$\{x\frac{d^2}{dx^2} + (|m-3/2|+1-x)\frac{d}{dx} - [\frac{|m-3/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m+3) + \frac{p^2 + M^2}{4|B_0|}]\}g_1 = 0;$$

which is an equation of the confluent hypergeometric type with the parameters

$$\gamma = |m-3/2|+1, \alpha = \frac{|m-3/2|}{2} + \frac{1}{2} - \frac{B_0}{4|B_0|}(2m+3) + \frac{p^2 + M^2}{4|B_0|}.$$

The quantization rule is $\alpha = -n$ leads to

$$\varepsilon^2 - p_3^2 - M^2 = 4|B_0|n + 2|B_0|(|m-3/2|+1) - B_0(2m+3); \qquad (120)$$

this spectrum corresponds to the variable F_1.

12. Conclusion

Thus, we have found 4 possible energy spectra for the spin 3/2 particle in external uniform magnetic field (assume that $B_0 > 0$)

$$\begin{aligned}
&F_3, \quad \varepsilon^2 - p_3^2 - M^2 = 2B_0\{2n+1+|m-\frac{3}{2}|-(m+\frac{3}{2})\};\\
&F_2, \quad \varepsilon^2 - p_3^2 - M^2 = 2B_0\{2n+1+|m+\frac{3}{2}|-(m-\frac{3}{2})\};\\
&F_4, \quad \varepsilon^2 - p_3^2 - M^2 = 2B_0\{2n+1+|m-\frac{1}{2}|-(m+\frac{1}{2})\};\\
&F_1, \quad \varepsilon^2 - p_3^2 - M^2 = 2B_0\{2n+1+|m+\frac{1}{2}|-(m-\frac{1}{2})\}.
\end{aligned} \qquad (121)$$

It may be proved that equations from the second group (61)–(64) lead to the one same energy spectra.

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