BLOCK METHODS OF MULTIDIMENSIONAL-MATRIX DATA ANALYSIS

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The paper is devoted to the substantiation and the software implementation of the block method for the solving of the multidimensional-matrix problems. For this purpose, the notion of the cell is introduced as the ordered set of the multidimensional matrices. The two-dimensional block matrices are considered along with the cells as the multidimensional-matrix block constructions. It is shown that the multidimensional-matrix problem formulated in terms of the cells is transformed in the natural way to the block two-dimensional problem. As an application, the problem of the estimation of the coefficients of the multidimensional-matrix polynomial regression function is considered. This problem is reduced to the numerical solution of the system of the multidimensional-matrix linear algebraic equations. The two methods to the solution of the system are developed: the block Gauss method and the block method of the matrix inverse. Both of them are realized programmatically as standard functions.

Introduction

A block matrix is the ordinary (two-dimensional) matrix consisting of other ordinary matrices which are called the elements of the block matrix [1-5]. It is possible to perform the operations on block matrices similar to operations on the ordinary matrices. This circumstance led to the emergence of the block methods for solving the matrix problems. It is supposed that the matrices of the initial problem divide into the blocks, after that, the block solution methods are used. At the same time, such problems as choosing the number and nature of the blocks and the developing of the divide algorithm occur. In addition, the benefits of the block methods compared with the usual methods are not determined explicitly. In connection with these disadvantages, the block methods are not widely used in practice. As a consequence, there are no standard programs for the block matrix methods in the programming systems.

The situation changes radically, if the problem is generated initially in block form, i.e. if it is known that the matrices consist of the given blocks. The block approach is becoming the unique right approach in this case. It turned out, that the block form problems are generated in framework of the multidimensional-matrix approach. The purpose of this report is the statement of the natural connection between the multidimensional-matrix mathematical approach and block methods on the instance of the multidimensional-matrix regression analysis.

I. Cells and block matrices

Let $A_1, A_2, \ldots, A_m$ be $q_1, q_2, \ldots, q_m$ – dimensionality matrices respectively [6,7]. Let us call this ordered by one index $i = 1, 2, ..., m$ set as one-dimensional cell, in accordance with the data type "cell array" of the Matlab programming system [8]. It is possible to order the set of the multidimensional matrices by the set of indexes. The set of the multidimensional matrices ordered by p indices $i_1, i_2, ..., i_p$ is denoted $A = \{A_{i_1,i_2,...,i_p}\}$, $i_\alpha = 1, 2, ..., m_\alpha$, $\alpha = 1, 2, ..., p$, and is called p – dimensional cell.

The multiplication of the cells is determined by analogy with the multiplication of the multidimensional matrices [6, 7]. If $A = \{A_{i_1,i_2,...,i_p}\} = \{A_{l,s,c}\}$ is the p-dimensional cell, where $l = (l_1, l_2, ..., l_\kappa)$, $s = (s_1, s_2, ..., s_\lambda)$, $c = (c_1, c_2, ..., c_\mu)$ are multi-indices, $\kappa + \lambda + \mu = p$, and $B = \{A_{j_1,j_2,...,j_q}\} = \{B_{c,s,m}\}$ is the q-dimensional cell, where $m = (m_1, m_2, ..., s_\nu)$, $\lambda + \mu + \nu = q$, then the $(\kappa + \lambda + \nu)$-dimensional cell $D = \{D_{s,c,m}\}$ is called the (λ, μ)-rolled product of the cells A and B, if its elements are defined by the expression $D_{l,s,m} = \sum_c {}^{\lambda_1,\mu_1}(A_{l,s,c}B_{c,s,m})$. We will denote the (λ, μ)-rolled product of the cells A and B as ${}^{\lambda,\mu}\{AB\}$:

$$D =^{\lambda,\mu} \{AB\} = \{\sum_c {}^{\lambda',\mu'}(A_{l,s,c}B_{c,s,m})\} = \{D_{l,s,m}\}$$

.

We will consider the two-dimensional block matrices along with the cells. We will call them simply block matrices. It is advisable to transform the solution of the multidimensional-matrix problems to the solution of the two-dimensional-matrix problems due to the fact that the last problems are well developed. Such a transformation is performed on the base of the theorem on the associated matrices [6, 7] by replacing the multidimensional matrices with the associated with them two-dimensional ones. The received two-dimensional-matrix solution is then transformed back into the multidimensional-matrix solution. The forming of the associated matrices and the back transformation are performed on the program level. So the multidimensional-matrix problem formulated in terms of the cells creates the two-dimensional block problem.

II. equation for the coefficients of the

MULTIDIMENSIONAL-MATRIX POLYNOMIAL REGRESSION

It is assumed that the hypothetical regression function $y = \phi(x)$ is polynomial of the degree m [9, 11]:

$$y = \phi(x) = \sum_{k=0}^{m} {}^{0,kq}(C_{p,kq}x^k) = \sum_{k=0}^{m} {}^{0,kq}(x^k C_{kq,p}),$$

where x is the q-dimensional matrix input variable, y is the p-dimensional-matrix output variable, $x^k =^{0,0} x^k$ is the $(0,0)$-rolled k-th degree of the matrix x, $C_{(p,kq)}$ are the $(p+kq)$-dimensional matrices of the coefficients,

$$C_{p,kq} = (c_{i,\bar{j}_k}), i = (i_1, ..., i_p), \bar{j}_k = (j_1, ..., j_k),$$

satisfied the conditionals

$$C_{(p,kq)} = C_{(kq,p)}^{H_{p+kq,kq}}, C_{(kq,p)} = C_{(p,kq)}^{B_{p+kq,kq}},$$

$H_{p+kq,kq}$ and $B_{p+kq,kq}$ are transpose substitutions of the type "back" and "forward" respectively [6], and ${}^{0,kq}(C_{(p,kq)}x^k) =^{0,kq} (x^k C_{(kq,p)})$ are $(0,kq)$-rolled productions of the multidimensional matrices. It is necessary to find the estimations $\hat{C}_{(p,0)}$, $\hat{C}_{(p,q)}$,...,$\hat{C}_{(p,mq)}$ of the unknown coefficients $C_{(p,0)}$, $C_{(p,q)}$,...,$C_{(p,mq)}$ on the base of the observations $(x_1, y_{o,1}), (x_2, y_{o,2}), ..., (x_n, y_{o,n})$.

The following system of the linear algebraic equations for the coefficients $C_{(0,p)}$, $C_{(q,p)}$,...,$C_{(mq,p)}$ is obtained in the work [10]:

$$\sum_{k=0}^{m} {}^{0,kq}(s_{x^{l+k}} C_{(kq,p)}) = s_{x^l y_o}, l = 0, 1, ..., m \quad (1)$$

where

$$s_{x^{l+k}} = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu^l x_\mu^k = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu^{l+k},$$

$$s_{x^l y_o} = \frac{1}{n}\sum_{\mu=1}^{n} x_\mu^l y_{o,\mu}, y_{o,\mu} = (y_{o,i,\mu}), i = (i_1, ..., i_p).$$

Let us note that the analytical (formula) solution to the system (1) is received for the constant, affine and square regression functions only [10].

The coefficients $s_{x^{l+k}}$ of the system (1) form the cell

$$A = \{a_{l,k}\}, a_{l,k} = s_{x^{l+k}}, l, k = 0, 1, ..., m, \quad (2)$$

which elements $a_{l,k} = s_{x^{l+k}}$ are $(lq + kq)$-dimensional matrices, the terms $s_{x^l y_o}$ form the one-dimensionality cell

$$B = \{b_l\}, b_l = s_{x^l y_o}, l = 0, 1, ..., m, \quad (3)$$

which elements $b_l = s_{x^l y_o}$ are $(lq + p)$-dimensional matrices, and the unknown variables $C_{(kq,p)}$ form the one-dimensionality cell

$$Z = \{z_k\}, z_k = C_{(kq,p)}, k = 0, 1, ..., m, \quad (4)$$

which elements $z_k = C_{(kq,p)}$ are $(kq+p)$-dimensional matrices.

The multidimensional-matrix system (1) can be written in form of one equation in terms of cells A, B, Z (2), (3), (4):

$${}^{0,1}\{AZ\} = B, \quad (5)$$

i.e. as $(0,1)$ – rolled product of the cells, particularly, ${}^{0,1}\{AZ\} = \sum_{k=0^m}^{0,kq}(a_{l,k}z_k)$. We will solve the equation (5) and propose two algorithms for the solution.

III. Block algorithms

The first algorithm to the solution of the equation (5) consists of use the block (generalized, in terminology of the work [1]) Gauss elimination algorithm. The equations (5) is equivalent, in accordance with the theorem on the associated matrices, to the following system of the equations in terms of associated two-dimensional matrices:

$$\sum_{k=0^m}^{0,1} ((\tilde{s}_{x^{l+k}})_{(lq,0,kq)}(\tilde{C}_{(kq,p)})_{(kq,0,p)}) = (\tilde{s}_{x^l y_o})_{(lq,0,p)}, \quad (6)$$

where $l = 0, 1, ..., m$, $(\tilde{s}_{x^{l+k}})_{(lq,0,kq)}$ is the two-dimensional matrix $(lq, 0, kq)$-associated with the $(lq + kq)$-dimensional matrix $s_{x^{l+k}}$, $(\tilde{C}_{(kq,p)})_{(kq,0,p)}$ is the two-dimensional matrix $(kq, 0, p)$-associated with the $(kq + p)$-dimensional matrix $C_{(kq,p)}$, and $(\tilde{s}_{x^l y_o})_{(lq,0,p)})$ is the two-dimensional matrix $(lq, 0, p)$-associated with the $(lq+p)$-dimensional matrix $s_{x^l y_o}$. The system of the equations (6) is represented by one equation

$${}^{0,1}[\tilde{A}\tilde{Z}] = \tilde{B} \quad (7)$$

with two-dimensional block matrix of the coefficients

$$\tilde{A} = [\tilde{a}_{l,k}], \tilde{a}_{l,k} = (\tilde{s}_{x^{l+k}})_{(lq,0,kq)}, l, k = 0, 1, ..., m, \quad (8)$$

one-dimensional block matrix of the constant terms

$$\tilde{B} = [\tilde{b}_l], \tilde{b}_l = (\tilde{s}_{x^l y_o})_{(lq,0,p)}), l = 0, 1, ..., m, \quad (9)$$

and one-dimensional block matrix of the unknown variables $\tilde{Z} = [\tilde{z}_k], \tilde{z}_k = (\tilde{C}_{(kq,p)})_{(kq,0,p)}), k = 0, 1, ..., m,$. The product ${}^{0,1}[\tilde{A}\tilde{Z}]$ in (7) means the block multiplication of the matrices. The received equation (7) is solved by Gauss elimination algorithm [1].

The second algorithm to the solution of the equation (7) uses the block inversion of the matrix $\tilde{A}$. The block solution of the equation (7) is $\tilde{Z} =^{0,1} [\tilde{A}^{-1}\tilde{B}]$, where $\tilde{A}^{-1}$ is the matrix block-inverse to the matrix $\tilde{A}$.

The forming of the associate block matrices and back transfer to the cells are performed in accordance with the definition of the associate matrix [6, 7].

The block inversion algorithm is programmed as the block variant of the ordinary matrix inversion by the Gauss elimination [11]. The block multiplication follows the known rule [1]. All of the programs have the algorithmic generality, i.e. they allow solving the arbitrary order systems of the equations.

IV. Conclusion

It is showed that the block methods receive the new impulse for their application in the framework of the multidimensional-matrix mathematical approach. The problem of the numerical calculation of the coefficients of the multidimensional-matrix regression function is solved as the important application. The block methods can be used in other applications, for instance, in the problem of calculation of the coefficients of the orthonormal polynomials of the vector variable [7, 12].

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