Monitoring in dynamic systems with tipping based on the principle of large deviations

Sergey Dubovik
Robotics and Intelligent Control Systems Laboratory
Sevastopol State University
Sevastopol, Russian Federation

Ivan Lipko
Robotics and Intelligent Control Systems Laboratory
Sevastopol State University
Sevastopol, Russian Federation
ivanlipko@yandex.ru

***Abstract*— The problem of predictability of the El Niño phenomenon on the basis of the well-known Gin-Timmerman model is considered. The presence of uncompensated instability in the model against the background of statistical data accumulated over the entire time of observations on the problem leads to the idea of the presence of some hidden damping mechanism, however small: critical events of sharp temperature increase do not occur too often and between them, on average, there are 7-12 years of rather stable behavior. Without fully revealing what this mechanism is, some small noise can be introduced into the system and an attempt is made to use this for research. An attempt is made to create a prediction algorithm by using the principle of large deviations in the vicinity of the equilibrium state in combination with global deterministic analysis. The paper applies such methods as the Runge-Kutta method of the 4th order, the search of the instanton by methods of the large deviation theory. The explicit analytical formulas for calculating the most probable trajectories of realization of the event of exceeding the high temperature difference between the eastern and western surface zones of the equatorial part of the Pacific Ocean are shown. The corresponding results for several levels are given. An example with identification of a linear model, which is used for local forecasting of a hazardous event, is shown. We have shown that even small changes in the initial conditions can lead to a sufficiently large difference in the time required for the temperature ejection phenomenon to occur. This complicates the process of El Niño research and reduces the window of time for forecasting.**

Keywords—El-Nino, large deviation, prediction, Gin-Timmerman model, random perturbation, small parameter.

I. Introduction

Recently, there has been a renewed interest in the qualitative theory of nonlinear systems. In the framework of control theory, we are talking about the control of critical changes in the dynamics of the system under the action of small (or slow) changes in its parameters. Even appropriate terminology has appeared: for example, phenomena well known as bifurcations (in biology back in the 60's they were studied in connection with the Belousov-Zhabotinsky reactions) are now sometimes referred to as "B-tipping", i.e., "tipping", meaning exactly qualitative changes in behavior [1]. In contrast, in "N-tipping" (N - noise) the parameter is modeled by "white noise" [2]. If now we include one of the parameters in the state vector and focus on the study of "tipping" depending on the rate of its change (usually, however, rather small), we obtain "R-tipping" characterized by a linear change of the parameter in time [1, 3]. Generalization of this approach to arbitrary velocities, including sufficiently large ones, and the desire for asymptotic parameter estimates led to the consideration of systems with separation of motions, i.e., to the active use of singular perturbations. Since many practically interesting problems with "tipping" contain nonlinearities of the same type as in the Van der Polye equation, i.e., with alternation of stable and unstable manifolds, relaxation-type oscillations arise in such systems [4]. In this case, the parameter-limited solutions are a composition of curve sections (arcs) on stable regions and sections of rapid motion ("jump") from one of the stable regions to another. More and more often, however, such an asymptotic analysis apparatus as in [4] turns out to be insufficient: unstable arcs appear in the compositions constituting phase curves when the parameter approaches zero. Because of their shape or because of their initial disbelief in the news, these limiting curves have come to be called "ducks". Such modes are established and analyzed with the help of geometric singular perturbation theory (GSPT), the origin of which is the theorems of N. Fenichel [5, 6]. Such dynamics is characterized by oscillations with alternation of small and large oscillations with different velocities ("mixed-mode oscillations with multiple time scales") [7]; these oscillations are very sensitive to small variations of parameters and initial conditions, which is well illustrated by the results of studies of practical problems. One of such applications of GSPT theory, which has a long history, is the problem of modeling and forecasting the phenomenon known in meteorology as El-Nino-Southern Oscillation (ENSO) [7-9]. In all these works, the ENSO study is based on a low-dimensional model of the Gin-Timmerman phenomenon, while the later ones, [8, 9], just note the strong sensitivity of the results to variations in the initial data. Therefore, in [9] (as in [2]), for example, the stochastic Gin-Timmerman model is already used. This shows once again that the use of stochastic models instead of deterministic models in practical problems is quite justified in a number of cases.

In this paper, an attempt is made to consider one of the "rollover" problems from a probabilistic point of view, basing the algorithm for estimating the probability of a critical event (CE) on the principle of large deviations [10]. The problem is formulated in the next section, and Section 3 presents the solution algorithm as applied to the ENSO problem, where a Gaussian perturbation of the deterministic Gin-Timmerman model is chosen as the object of study. Sections 4 and 5 are devoted to two levels of consideration of the model, local and global, within which the results of the numerical study are given. The conclusion contains the conclusions of the paper.

II. The Problem Statement

A. *Estimation of large deviations*

Let's consider the system

$$\dot{\tilde{x}} = a(\tilde{x}) + \varepsilon\sigma\dot{w},\ \ \tilde{x}(t_0) = x_0 \in R^n, \tag{1}$$

where $\dot{w}$ is k-vector of "white noise", a is a smooth vector function, $\varepsilon > 0$ is a small parameter, σ is $n \times k$ matrix. We will assume that the corresponding unperturbed system (derived from (1) when $\varepsilon = 0$):

$$\dot{x} = a(x), \tag{2}$$

has a single solution for each x_0 in a given (and small enough) neighborhood of its equilibrium state $\chi,\ a(\chi)=0$.

Together with equation (1), consider a deterministic path system [11, 12]:

$$\dot{\phi}=f(\phi,v)=a(\phi)+\sigma v, \tag{3}$$

where v is the summable function for which the action functional [10] (AF) is defined on the solutions of (3):

$$S_{t_0t_f}(\phi,v)=\frac{1}{2}\int_{t_0}^{t_f}v^T v dt, \tag{4}$$

taking finite values for absolutely continuous functions on $[t_0,t_f],\ \phi(t_0)=x_0\in R^n$.

Let us introduce the "tipping" condition:

$$l(t_f,\phi(t_f))=C\phi(t_f)-y=0 \tag{5}$$

and a region D with regular boundary ∂D, containing all regular states of the system (3), i.e., the states "before the tipping"; in particular $x_0\in D$ and the set of paths leaving D as part of functions continuous on the segment $[t_0,t_f]$, is :

$$F=\{\phi\in C_{t_0t_f}(R^n):\phi_\tau\in D,\tau\in[t_0,t_f),\phi_{t_f}\in R^n\backslash D\}.$$

Then, according to the principle of large deviations [10], the equality is true:

$$\lim_{\varepsilon\to 0}\varepsilon^2 \ln P\{\tilde{x}_t\in R^n\backslash D\}=-\min_{\phi\in F}S_{t_0t_f}(\phi,v), \tag{6}$$

where the functional $S_{t_0t_f}=S_{t_0t_f}(\phi,t)$ is defined in accordance with (4) on the solutions of the controlled system (3). The optimal control problem (3-5) will be referred to as the Lagrange-Pontryagin (LP) problem, and its solutions (extrema of the AF) as the CE profiles.

Additional conditions in (3) $\phi(t_f)=x_f$ are defined by the functional set and relations (5),(6), i.e. $Cx_f-y=0$.

In [12, 14, 15], examples of solution construction in problems for a linear or nearly linear object stabilized in the vicinity of the equilibrium point are given: in the linear case, the solution exists in the form of an (explicit) a-profile of the CE – a single curve leading from the attractor to the CE, to which the profiles originating from the points $x_0=x(t_0)$ at $t_0\to-\infty$. In contrast to this kind of problems, the dynamics of eco- and meteo-monitoring objects is characterized by the absence of Hurvicness of the zero linearized unperturbed equations (2). In the case of the Gin-Timmerman model, the object has a saddle-focus with a negative real root as an equilibrium point; then the LP problem is formed as a reduction problem for some damped system. Let us pass to concrete models.

B. *ENSO mathematical model: weakly perturbed and weakly stable system*

As a mathematical model of the ENSO phenomenon, i.e., as equation (2), we take system (3) from [7] with state vector $\phi\in R^3$. We take the state designation immediately with respect to the path system to preserve the component designations as in the original (dimensionless) Jin-Timmerman model: $\phi=(x,\ \ v,\ \ z)^T=(\phi_1,\ \ \phi_2,\ \ \phi_3)^T$, with respect to which we specify that the coordinate is the normalized temperature difference between the eastern and western surface zones of the equatorial Pacific Ocean.

The right-hand side in (2) is of the form:

$$a(\phi)=\begin{pmatrix}\rho\delta(x^2-ax)+x[x+y+c-\mathrm{cth}(x+z)]\\ -\rho\delta(ay+x^2)\\ \delta(k-z-x/2)\end{pmatrix}$$

We will also choose the values of the model parameters according to one of the variants in [7]:

$$\begin{gathered}\delta=0.2254,\rho=0.322,a=7.394,\\ k=0.403,c=2.395.\end{gathered} \tag{12}$$

These values correspond to the equilibrium state of the unperturbed system ϕ_e:

$$\phi_e=(x_e,y_e,z_e)^T=(-2.484,\quad -0.8345,\quad 1.645)^T \tag{13}$$

and the Jacobi matrix

$$A_e=A(\phi_e)=\frac{da(\phi)}{d\phi}|_{\phi=\phi_e}=\begin{pmatrix}\alpha_{1e} & x_e & -cx_e\cdot c_{1e}\\ -2\rho\delta x_e & -\rho\delta\cdot a & 0\\ -\delta/2 & 0 & -\delta\end{pmatrix},$$

where $\alpha_{1e}=\alpha_{1e}(\phi_e)$, $c_{1e}=c_{1e}(\phi_e)$ have the form:

$$\begin{gathered}c_{1e}=1-[th(x_e+y_e)]^2,\\ \alpha_{1e}=\rho\delta(2x_e-a)+2x_e+y_e+c[1-th(x_e+y_e)-x_ec_{1e}].\end{gathered}$$

For the variant (12) the matrix has eigenvalues

$$\lambda_{1,2}=0.029\pm j1.008,\lambda_3=-0.328,$$

that is, the fixed point (13) is a saddle-focus [16] with an unstable oscillatory component and a stable aperiodic motion.

The presence of uncompensated instability in the model does not seem quite convincing against the background of the statistical data accumulated over the entire time of observations on the problem: the conclusion is that CEs do not occur too often and between them, on average, there are 7-12 years of rather stable behavior. This leads to the idea of the presence of some hidden damping mechanism, however small. In this connection, in addition to the small perturbation, we will consider in (1)

$$\dot{\tilde{x}}=a_\varepsilon(\tilde{x})+\varepsilon\sigma\dot{w}, \tag{14}$$

where

$$a_\varepsilon(x)=a(x)+\varepsilon\vartheta x,\quad \vartheta\in R^{3\times 3}.$$

Selecting in $\vartheta=\{\vartheta_{i,j}\}$ elements

$$\begin{gathered}\vartheta_{1,1}=0.045[1-th(x_e+y_e)-x_ec_{1e}],\\ \vartheta_{1,3}=-0.045x_ec_{1e},\end{gathered} \tag{15}$$

and others are zeros, we get when $\varepsilon=1$ the system (14) have the same Jacobi matrix as above; and if in A_e change $c=2.395$ to $c=2.440$. In this case A_e acquires eigenvalues

$$\lambda_{1,2}=-0.0125\pm j1.048,\lambda_3=-0.3224.$$

III. A-PROFILE AND SITUATIONAL FORECAST OF LOCAL PROCESSES

Let us consider the system (14),(15) in the linear approximation and for it the solution of the system in the form of an a-profile [11], that is, at $t_0 \to -\infty$. This corresponds to the path system

$$\dot{\varphi} = \mathrm{A_e}\varphi + \sigma\sigma^T e^{A_e^T(t_f - t)} D^{-1} x_f,\ C\phi(t_f) = y,$$

providing (asymptotically on $t_0 \to -\infty$) the minimum value of AF (quasipotential [10])

$$V(0, x_f) = \tfrac{1}{2} x_f^T D^{-1} x_f.$$

Here $D > 0$ is the only solution of the Lyapunov equation $\sigma\sigma^T = -A_e D - D A_e^T$. According to Lagrange's principle, as part of the necessary conditions of extremum we have the minimization problem $V(0, x_f)$ under constraint (5), resulting in $x_f = DC^T(CDC^T)^{-1}y$.

As a result, for the A-profile we obtain

$$\tilde{\phi}(t) = De^{A_e^T(t_f - t)} C^T (CDC^T)^{-1} y. \qquad (16)$$

For the minimum value of FD on the segment $[t_S, t_f]$, $t_0 < t_S < t_f$, there is the equality [11]

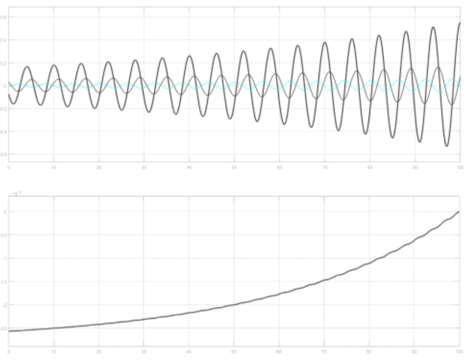
$$\tilde{S}_{t_S t_f} = V(0, x_f) - V(0, \tilde{\phi}(t_S)). \qquad (17)$$

Remark 1. Taking into account the asymptotic relation (6), relations (16) and (17) present the following possibility of situational forecast: the a-profile forms in the phase space of an unperturbed system curve (16) leading from $\chi = 0$ into point $x_f \in \partial D$, and as soon as once at some moment t_S the trajectory of the perturbed system (1) (or (14),(15)) enters a small neighborhood of the a-profile curve, the difference of quasipotentials (17) gives an estimate of probability

$$P_{t_S t_f} = P\{\tilde{x}_t \in R^n \backslash D, t_S \le t \le t_f\}$$

of exiting the region D. The details can be found in [17], and here we will give the results of computing (16),(17) and illustrate the possibility of predicting the ENSO phenomenon.

Fig. 1a shows the a-profile (16) for model (14), (15), and Fig. 1b shows the AF (17) in the functions $t_S = t$: $\tilde{S}(t) = \tilde{S}_{t_S t_f | t_S = t}$, which by virtue of (6) gives an estimate (at $\varepsilon = 1$) $P_{t_S t_f}$, ore precisely $\tilde{S}(t) \cong ln\, P_f\,(t) = ln\, P_{t t_f}$.



a)

b)

Fig. 1. A-profile and action functional as a function of time.

IV. A-PROFILE AND SITUATIONAL FORECAST OF LOCAL PROCESSES

An alternative method is to reconstruct the model from data corresponding to stationary dynamics in a small neighborhood of the equilibrium state. For this purpose, a section without emissions of the El Niño phenomenon was selected and used to identify a linear model of the 3rd order state space. The purpose of the identification was to obtain a model of the behavioral dynamics followed by the creation of an A-profile of the critical situation associated with the exceeding of the temperature level threshold.

As a result of parametric identification by error [19] with initialization of parameters by the method of subspaces [20] (functions *pem* and *ssest* of Matlab software package), a linear stationary model in the form of state space was obtained

$$\begin{cases} \frac{dv}{dt} = Av + Bu, \\ z = Cx, \end{cases} \qquad (18)$$

where $A \in R^{3\times3}$, $B \in R^{3\times3}$, C is an eye matrix.

The obtained model describes the training data with sufficiently acceptable quality. It is important to note that the eigenvalues of the system A matrix are very close to the linearized Gin-Timmerman model and equal to $\lambda_{1,2} = -0.0002 \pm 0.8177i$, $\lambda_3 = -0.4473$. Thus, a model with dynamics similar to the pre-crisis model was obtained from the experimental data. This model was also used to construct the a-profile using the same scheme as above.

Remark 2. Due to the monotonicity of AF in time, both the probability value $P_f(t)$ (or its logarithm) and the estimate of the remaining time $t_f - t$ (Fig. 1b) can characterize the proximity of the current moment to the critical one.

V. GLOBAL CONTROL

In accordance with Remark 1, the neighborhood of the A-profile in the considered monitoring problem is a critical set between small fluctuations in the vicinity of the equilibrium state and global movements in the direction of "overturning". Small fluctuations can be considered as fluctuations at which the changes of the normalized difference are limited by the interval $M_0 = \{-3 < x < y < 0\}$, , and it is in this range that the role of random perturbations is essential; then $M = \{-3 < x < 0\}$ corresponds to the global level and the approach of a strong El Niño event means $x \in M_1 = M \backslash M_0$. This motion at $x(t) \in M_1$ is determined by global instability and is independent of noise, so a nonlinear deterministic system (2) is used as a model. In turn, this means that, unlike the local LP problem, there is no probabilistic criterion for the proximity of a crisis at the global level. However, the criterion of time to crisis remains (see Remark 2) and this is illustrated in Fig. 2; the solutions of system (2) shown there differ only in the initial conditions. A small change in the initial conditions leads to a rather large difference in the time until the phenomenon is realized.

This shows the strong sensitivity of the model in this regime and places increased demands on the constant updating of its parameter estimates.

VI. CONCLUSION

Systems with complex dynamics arising in the description of some processes of meteorology and ecology are considered. Their peculiarity is that, being unstable, they retain some properties close to stable behavior in the vicinity of the equilibrium state during sufficiently long time intervals. This allows us to use for their study various tools for analysis in the local zone and outside it, which is done in this paper with

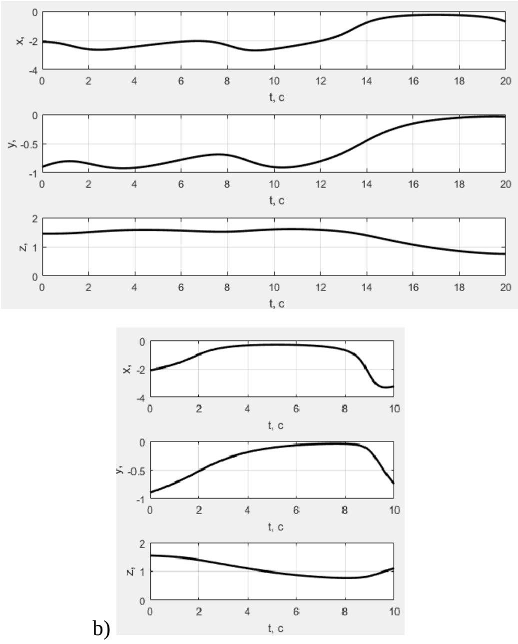
a)
b)


Fig. 2. Solutions for nonlinear Gin-Timmerman system with (a) $x_0 = (-2.1, -0.9, \ 1.45)^T$ and (b) $x_0 = (-2.1, -0.9, 1.55)^T$.

application to the problem of modeling and forecasting of the El Niño-Southern Oscillation process. An attempt is made to use the principle of large deviations in the neighborhood of the equilibrium state in combination with a global deterministic analysis based on the well-known Gin-Timmerman model.

REFERENCES

[1] P. Ashwin, S. Wieczorek, R. Vitolo and PM. Cox, “Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system”. Phil. 23 Trans. R. Soc. A 370, 1166–1184. (doi:10.1098/rsta.2011.0306).

[2] D. Lucente, S. Duffner, C. Herbert, J.Rolland and F. Bouchet, “Machine learning of committor functions for predicting high impact climate events”, Climate Informatics, Oct 2019.

[3] S. Wieczorek, C. Xie and P. Ashwin, “Rate-Induced Tipping: Thresholds, Edge States and Connecting”, Orbits, 2021.

[4] E.F. Mishchenko and N.Kh. Rozov, Differential Equations with Small Parameters and Relaxation Oscillations, ISBN: 978-1-4615-9047-7, Published: 09 March 2013, 228 p.

[5] N. Fenichel, “Geometric singular perturbation theory for ordinary differential equations”, Journal of Differential Equations 31 (1979), no. 1, 53–98. MR 0524817.

[6] C.K. Jones, “Geometric singular perturbation theory”. In: Johnson R. (eds) Dynamical Systems. Lecture Notes in Mathematics, 1609:44–118, 1995.

[7] A. Roberts, J. Guckenheimer, E. Widiasih, A. Timmermann, and C. K. Jones, “Mixed-mode oscillations of el nino–southern oscillation,” Journal of the Atmospheric Sciences, vol. 73, no. 4, pp. 1755–1766, 2016.

[8] J. Guckenheimer, Timmermann A., Dijkstra H. and A. Roberts, (Un) predictability of strong El Niño events. Dynamics and Statistics of the Climate System, 2 (1), 2017. DOI: 10.1093/climsys/dzx004.

[9] D. Lucente, C. Herbert and F. Bouchet. “Committor Functions for Climate Phenomena at the Predictability Margin: The example of El Niño Southern Oscillation in the Jin and Timmerman model”. Journal of the Atmospheric Sciences, American Meteorological Society, 2022

[10] M. I. Freidlin and A. D. Wentzell, Random Perturbations of Dynamical Systems, Springer New York, 2012, ISBN: 1468401777, 9781468401776, 328 p.

[11] S.A. Dubovik , Asymptotic Semantization of Data in Control Systems. Mekhatronika, Avtomatizatsiya, Upravlenie. 2019;20(8):461-471. (In Russ.) DOI: 10.17587/mau.20.461-471.

[12] S. A. Dubovik and A. A. Kabanov, Funkcional'no ustojchivye sistemy upravleniya: asimptoticheskie metody sinteza [Functionally stable control systems: asymptotic methods of synthesis]. Moscow, INFRA-M, [In Russian]. 2019. 249 p. DOI: 10.12737/ monography_5b446a985cf9a5.11626044.

[13] V.M. Alekseev, V.M. Tikhomirov and S.V. Fomin, Optimal'noe upravlenie [Optimal control]. Moscow, Nauka, [In Russian] 1979. 430 p.

[14] S.A. Dubovik, A.A. Kabanov and I. Lipko, Sistema operativnogo kontrolia krena sudna na volnenii. Patent na poleznuiu model' [The system of operational control of vessel roll on waves], № RU 195 040 U1. – 2019.

[15] Dubovik S.A. Use of Quasipotentials for Monitoring of Large Deviations in the Control Processes. Mekhatronika, Avtomatizatsiya, Upravlenie. 2016;17(5):301-307. (In Russ.).

[16] N.V. Butenin, Iu.I. Neimark and N.A. Fufaev, Vvedenie v teoriiu nelineinykh kolebanii [Introduction in the theory of nonlinear oscilations]. Moscow. Nauka, 1976. 384 p.

[17] S.A. Dubovik and A.A. Kabanov, An Asymptotic Method for Predicting Risks in Problems of Stochastic Monitoring and Control. Mekhatronika, Avtomatizatsiya, Upravlenie. 2022;23(8):395-405. (In Russ.). DOI: 10.17587/mau.23.395-405.

[18] L. Ljung , System Identification: Theory For the User, Second Edition, Upper Saddle River, N.J: Prentice Hall, 1999.

[19] P. van Overschee and B. De Moor. Subspace Identification of Linear Systems: Theory, Implementation, Applications. Springer Publishing: 1996.