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Value Functions and Their Directional Derivatives in Parametric Nonlinear Programming

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Abstract We study questions of existence and calculation of directional derivatives of value functions of nonlinear mathematical programming problems depending on parameters. To this end, we use the directional derivatives of the multivalued mappings, defined by the constraints of the problems; this approach was pioneered by Demyanov. We obtain sufficient conditions for existence and explicit formulas for calculating the directional derivatives of the first and second orders, under weaker hypotheses than those traditionally assumed.

Keywords Nonlinear programming · Value function · Constraint qualifications · Directional derivatives

Mathematical Subject Classification 90C31 · 49J52

1 Introduction

It is generally recognized that value functions are among the most important functions in variational analysis, constrained optimization and their numerous applications. Directional differentiability of value functions plays an important role in stability and sensitivity analysis of optimization problems with respect to the perturbation of their parameters, and it was studied in [1–18]. Questions of existence and calculation of directional derivatives of value functions were investigated in numerous works, where

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key results were obtained by using a variety of different methods [1–16]. Reviews of some results can be found in [3,4,17,18].

Though after the fundamental book [12] by Bonnans and Shapiro, the theory of directional differentiability of value functions in mathematical programming seems to be almost complete; in this paper we would like to demonstrate that some known results can be extended.

One of the first studies of differential properties of value functions was carried out by Demyanov [1]. His approach is focused on the study of directional derivatives of value functions on the basis of derivative-like constructions of multivalued mappings; that was probably one of the first attempts to introduce directional derivatives of multivalued mappings in multivalued analysis. Our paper is based on the Demyanov's approach and utilizes a similar technique.

It is worth mentioning that the differential properties of value functions are closely related to constraint qualifications, or regularity conditions, for optimization problems. For instance, one of the most fruitful approaches to investigation of the directional differentiability of value functions is based on the combination of a directional version (see [9,10]) of the classical Mangasarian–Fromovitz regularity condition (later this version was named *directional Mangasarian–Fromovitz condition*) along with sufficient conditions for the optimal solution to be Lipschitzian at the point under consideration, which were introduced in [8,9,12]. Introduction of new weak constraint qualifications [15,19–21] allows refining some results in this area.

In this paper, we will consider constraint qualification called *relaxed Mangasarian–Fromovitz regularity condition* (or *RMFCQ*), which was introduced in the works [19–21]. This condition is weaker than the Mangasarian–Fromovitz regularity condition, and it is also weaker than the constant rank regularity condition [6], as well as several other regularity conditions [15,20,21]. We introduce a directional version of *RMFCQ*, which we refer to as *relaxed directional Mangasarian–Fromovitz condition* (or, more precisely, *relaxed directional Mangasarian–Fromovitz condition in the direction* $\bar{x}$, whenever we wish to explicitly specify the direction $\bar{x} \in \mathbb{R}^n$). It is not a constraint qualification; however, it guarantees directional differentiability of the multivalued mapping defined by the constraints of the mathematical programming problem and it allows to obtain sufficient conditions for directional differentiability of the value function. In particular, one can replace the directional Mangasarian–Fromovitz condition by the relaxed directional Mangasarian–Fromovitz condition and obtain results, under less restrictive hypotheses, similar to those of Shapiro [8] and Auslender and Cominetti [9] and some of the results of Bonnans and Shapiro [12].

We also introduce a *relaxed directional second-order Mangasarian–Fromovitz condition* and use it to establish sufficient condition for the second-order differentiability of multivalued mappings.

The plan of the paper is as follows. In Sect. 2, we introduce additional notation used throughout the paper and formulate the relaxed directional Mangasarian–Fromovitz condition. Then, we study its relationship to the directional Mangasarian–Fromovitz condition, establish several auxiliary lemmas and prove Theorem 2.1, which provides sufficient conditions for the directional differentiability of multivalued mappings. In Sect. 3, we introduce the relaxed directional second-order Mangasarian–Fromovitz condition and establish a sufficient condition (Theorem 3.1) for the second-order directional differentiability of multivalued mappings. In Sect. 4, we provide sufficient conditions for the directional differentiability of the value function and prove formulas for calculation of its first-order and second-order directional derivatives.

2 Relaxed Directional Mangasarian–Fromovitz Condition

We consider a mathematical programming problem $P(x)$ depending on a parameter $x \in \mathbb{R}^n$:

$$f(x, y) \to \inf_y$$
$$y \in F(x) = \{y \in \mathbb{R}^m \; : \; h_i(x, y) \leq 0 \;\; i \in I, \; h_i(x, y) = 0 \;\; i \in I_0\},$$

where $I = \{1, \ldots, s\}$, $I_0 = \{s+1, \ldots, p\}$, and all functions $f(x, y)$, $h_i(x, y)$, $i = 1, \ldots, p$ are assumed to be twice continuously differentiable.

For the multivalued mapping F defined above by the constraints of $P(x)$, we use the notation

$$domF := \left\{x \in \mathbb{R}^n \; : \; F(x) \neq \varnothing\right\}, \; grF := \{(x, y) \; : \; y \in F(x), \; x \in \mathbb{R}^n\}.$$

Consider the value function

$$\varphi(x) := \inf \{f(x, y) : y \in F(x)\},$$

and the solution set of the problem $P(x)$

$$\omega(x) := \{y \in F(x) \; : \; f(x, y) = \varphi(x)\}, \; x \in \mathbb{R}^n.$$

Let's now fix points $x^0 \in domF$ and $y^0 \in F(x^0)$, for the rest of the paper. In the sequel, for arbitrary chosen points $x \in domF$, $y \in F(x)$, and directions $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, we denote the pairs (x, y) and $(\bar{x}, \bar{y})$ by symbols z and $\bar{z}$, respectively. In particular, specializing x and y to points x^0 and y^0 fixed above, we denote (x^0, y^0) by z^0. Throughout the paper, we assume that the set $\omega(x^0+t\bar{x})$ is nonempty and uniformly bounded for all sufficiently small numbers $t \geq 0$, that is, there exist a number $t_0 > 0$ and a bounded set $Y_0 \subset \mathbb{R}^m$ such that $\omega(x^0 + t\bar{x}) \subset Y_0$ for all $t \in [0, t_0]$. Also, we will freely use the standard "little-o" and "big-O" notations for vector functions into image spaces $\mathbb{R}^k$, $k \in \mathbb{N}$; so, for example, $o(t)$ will denote any such function satisfying $o(t) \to 0$ as $t \downarrow 0$.

Denote the lower and upper Dini derivatives of the function φ at the point x^0 in the direction $\bar{x}$ by

$$D_+\varphi(x^0; \bar{x}) := \liminf_{t \downarrow 0} t^{-1}(\varphi(x^0 + t\bar{x}) - \varphi(x^0)),$$
$$D^+\varphi(x^0; \bar{x}) := \limsup_{t \downarrow 0} t^{-1}(\varphi(x^0 + t\bar{x}) - \varphi(x^0)),$$

respectively, and denote the directional derivative of the function φ in the direction $\bar{x}$ at the point x^0 (when it exists) by

$$\varphi'(x^0;\bar{x}) := \lim_{t\downarrow 0} t^{-1}[\varphi(x^0+t\bar{x})-\varphi(x^0)].$$

Following Demyanov [1], we consider also the lower and upper second-order derivatives of the function φ at the point x^0 in the direction $\bar{x}$:

$$D^2_+\varphi(x^0;\bar{x}) := \liminf_{t\downarrow 0} \frac{2}{t^2}[\varphi(x^0+t\bar{x})-\varphi(x^0)-t\varphi'(x^0;\bar{x})].$$
$$D^{+2}\varphi(x^0;\bar{x}) := \limsup_{t\downarrow 0} \frac{2}{t^2}[\varphi(x^0+t\bar{x})-\varphi(x^0)-t\varphi'(x^0;\bar{x})].$$

We refer to their common value, when it exists, as the *second-order derivative of the function* φ *at* x^0 *in the direction* $\bar{x}$, and denote it by $\varphi''(x^0;\bar{x})$. In other words,

$$\varphi''(x^0;\bar{x}) := \lim_{t\downarrow 0} 2t^{-2}(\varphi(x^0+t\bar{x})-\varphi(x^0)-t\varphi'(x^0;\bar{x})).$$

The goal of the present paper is to obtain sufficient conditions for existence of the derivatives $\varphi'(x^0;\bar{x})$ and $\varphi''(x^0;\bar{x})$ under weaker requirements than those traditionally used, and to calculate them if they exist; otherwise to find estimates on the lower and upper derivatives.

Denote $I(z) := \{i \in I : h_i(z) = 0\}$ and consider the Lagrange function $L(z,\lambda) = f(z) + \langle\lambda, h(z)\rangle$, where $\lambda = (\lambda_1,\ldots,\lambda_p)$, $h = (h_1,\ldots,h_p)$ and the set of Lagrange multipliers

$$\begin{aligned}\Lambda(z) :=\{\lambda \in \mathbb{R}^p \ : \ &\nabla_y f(z) + \sum_{i=1}^{p} \lambda_i \nabla_y h_i(z) = 0,\\ &\lambda_i \geq 0 \ i \in I(z), \lambda_i = 0 \ i \in I \backslash I(z)\}.\end{aligned}$$

We will also need the set

$$\Lambda_0(z) := \{\lambda \in \mathbb{R}^p \ : \ \sum_{i=1}^{p} \lambda_i \nabla_y h_i(z) = 0, \lambda_i \geq 0 \ i \in I(z), \lambda_i = 0 \ i \in I\backslash I(z)\}$$

of singular Lagrange multipliers.

Our sufficient conditions for existence of directional derivatives $\varphi'(x^0;\bar{x})$ and $\varphi''(x^0;\bar{x})$ will be formulated in terms of requirements similar to constraint qualifications (regularity conditions) on the functions $h_i(x,y)$. Recall that constraint qualifications at the point y^0 serve to ensure the validity of the Karush–Kuhn–Tucker conditions, that is, to guarantee that $\Lambda(x^0,y^0) \neq \varnothing$ if y^0 is a local solution of the problem $P(x^0)$. For example, the well-known *Mangasarian–Fromovitz constraint qualification (MFCQ)* [22] requires the linear independence of vectors $\nabla_y h_i(x^0,y^0)$ $i \in I_0$ and the existence of a vector $\bar{y}^0$ such that

$$\langle \nabla_y h_i(x^0, y^0), \bar{y}^0 \rangle = 0, \; i \in I_0, \; \langle \nabla_y h_i(x^0, y^0), \bar{y}^0 \rangle < 0, \; i \in I(x^0, y^0).$$

It is known that *MFCQ* at the point $y^0 \in F(x^0)$ is equivalent to the requirement $\Lambda_0(z^0) = \{0\}$.

To formulate another constraint qualification, introduce the linearized tangent cone to the set $F(x^0)$ at the point $y^0 \in F(x^0)$:

$$\Gamma_{F(x^0)}(y^0) := \{\bar{y} \in \mathbb{R}^m \; : \; \langle \nabla_y h_i(z^0), \bar{y} \rangle \le 0 \; i \in I(z^0), \; \langle \nabla_y h_i(z^0), \bar{y} \rangle = 0 \; i \in I_0\}$$

and denote

$$I^a(z^0) := \{i \in I(z^0) \; : \; \langle \nabla_y h_i(z^0), \bar{y} \rangle = 0 \; \forall \bar{y} \in \Gamma_{F(x^0)}(y^0)\},$$
$$I^-(z^0) := I(z^0) \backslash I^a(z^0).$$

Then, following Minchenko and Stakhovski [19] and Kruger et al. [21], we say that *the relaxed Mangasarian–Fromovitz constraint qualification (RMFCQ)* holds at the point $y^0 \in F(x^0)$ iff $rank\{\nabla_y h_i(x^0, y^0), \; i \in I_0 \cup I^a(x^0, y^0)\} = const$ in some neighbourhood of the point y^0.

It is known [19] that *RMFCQ* is a constraint qualification and it implies *MFCQ*. Some other regularity conditions and their comparison can be found in [23,24].

Lemma 2.1 *The following statements hold:*

(1) *there exists a vector* $\bar{y}^0$ *such that*

$$\langle \nabla_y h_i(z^0), \bar{y}^0 \rangle = 0 \; i \in I_0 \cup I^a(z^0), \langle \nabla_y h_i(z^0), \bar{y}^0 \rangle < 0 \; i \in I^-(z^0);$$

(2) *an index* $i \in I(z^0)$ *belongs to the set* $I^a(z^0)$ *iff there exists a vector* $\lambda \in \Lambda_0(z^0)$ *such that* $\lambda_i > 0$.

Proof The first assertion has been proved in [21], the validity of the second assertion follows immediately from Theorem 17.7 of Gorokhovik [25]. □

Following Luderer et al. [14], introduce the lower and upper Dini derivatives of the multivalued mapping F at the point z^0 in the direction $\bar{x}$:

$$DF(z^0; \bar{x}) := \{\bar{y} \in \mathbb{R}^m \; : \; y^0 + t\bar{y} + o(t) \in F(x^0 + t\bar{x}), \; \forall t > 0\}.$$
$$\widehat{D}F(z^0; \bar{x}) := \{\bar{y} \in \mathbb{R}^m \; : \; \exists t_k \downarrow 0 \; and \; \exists \widehat{y}^k \to \bar{y} \; such \; that$$
$$y_0 + t_k \widehat{y}^k \in F(x^0 + t_k \bar{x}) \; \; for \; \; all \; \; k = 1, 2, \ldots\}.$$

and the set

$$\Gamma(z^0; \bar{x}) := \{\bar{y} \in \mathbb{R}^m \; : \; \langle \nabla h_i(z^0), \bar{z} \rangle \le 0 \; i \in I(z^0),$$
$$\langle \nabla h_i(z^0), \bar{z} \rangle = 0 \; i \in I_0, \; \; \bar{z} = (\bar{x}, \bar{y})\}.$$

Remark 2.1 It is not difficult to check (see, e.g., [14]) that $DF(z^0;\bar{x}) \subset \widehat{DF}(z^0;\bar{x}) \subset \Gamma(z^0;\bar{x})$.

It is known that the condition $DF(z^0;\bar{x}) = \Gamma(z^0;\bar{x}) \neq \varnothing$ plays an important role in studying directional differentiability of the value function. This condition is valid if *MFCQ* holds at the point $y^0 \in F(x^0)$. A weaker requirement which still ensures $DF(z^0;\bar{x}) = \Gamma(z^0;\bar{x}) \neq \varnothing$, called *Mangasarian–Fromovitz condition in the direction* $\bar{x}$ (or, simply, *directional Mangasarian–Fromovitz condition*), was introduced in [9, 10].

Definition 2.1 The *Mangasarian–Fromovitz condition in the direction* $\bar{x}$ (briefly $MF_{\bar{x}}$) holds at the point z^0 iff the family $\{\nabla_y h_i(z^0),\ \ i \in I_0\}$ is linearly independent and there exists a vector $\bar{y}^0$ such that $\langle \nabla h_i(z^0),\ (\bar{x},\bar{y}^0)\rangle = 0$

$$i \in I_0,\ \ \langle \nabla h_i(z^0), (\bar{x},\bar{y}^0)\rangle < 0\ \ i \in I(z^0).$$

It is known [9] that the following assertions are equivalent:

(1) $MF_{\bar{x}}$ holds at the point z^0;
(2) $\sum\limits_{i\in I_0\cup I(z^0)} \lambda_i \langle \nabla_x h_i(z^0), \bar{x}\rangle < 0$ for all $\lambda \in \Lambda_0(z^0)\backslash\{0\}$.

It is not difficult to see that *MFCQ* implies $MF_{\bar{x}}$ for all directions. However, the following example shows that $MF_{\bar{x}}$ may not hold, even in very simple optimization problems.

Example 2.1 Let $F(x) = \{y \in \mathbb{R}^2 : -y_1^2 + y_2 - x \leq 0,\ -y_2 + x \leq 0\}$, $x \in \mathbb{R}$, $x^0 = 0$, $y^0 = (0,0)^T$. Here, the point $z^0 = (x^0, y^0)$ satisfies neither *MFCQ* nor $MF_{\bar{x}}$ for any direction. Indeed, for $h_1(z) = -y_1^2 + y_2 - x$ and $h_2(z) = -y_2 + x$, there does not exist a vector $\bar{y} \in \mathbb{R}^m$ such that $\langle \nabla h_i(z^0), (\bar{x},\bar{y})\rangle < 0$ for $i = 1, 2$, because these inequalities reduce to $\bar{y}_2 < \bar{x}$ and $\bar{y}_2 > \bar{x}$, which cannot be fulfilled. This means that $MF_{\bar{x}}$ does not hold at z^0 for any direction and hence, by the above remark, *MFCQ* does not hold either.

Let $\Gamma(z^0;\bar{x}) \neq \varnothing$ and denote

$$I^a(z^0,\bar{x}) := \{i \in I(z^0)\ :\ \langle \nabla h_i(z^0), (\bar{x},\bar{y})\rangle = 0,\ \ \forall \bar{y} \in \Gamma(z^0;\bar{x})\},$$
$$I^-(z^0,\bar{x}) := I(z^0)\backslash I^a(z^0,\bar{x}).$$

The proof of next lemma follows from Kruger et al. [21] along the same lines as that of the first part of Lemma 2.1.

Lemma 2.2 *Let* $\Gamma(z^0;\bar{x}) \neq \varnothing$. *Then, there exists a vector* $\bar{y}^0$ *such that*

$$\langle \nabla h_i(z^0), (\bar{x},\bar{y}^0)\rangle = 0,\ \ i \in I_0 \cup I^a(z^0,\bar{x}),\ \ \langle \nabla h_i(z^0), (\bar{x},\bar{y}^0)\rangle < 0,\ i \in I^-(z^0,\bar{x}).$$

The validity of next lemma follows from the theory of linear inequalities (see, for example, [25]).

Lemma 2.3 *$\Gamma(z^0;\bar{x}) \neq \varnothing$ iff $\sum\limits_{i\in I_0\cup I(z^0)} \lambda_i\langle\nabla_x h_i(z^0),\bar{x}\rangle \le 0$ for all $\lambda \in \Lambda_0(z^0)$.*

Inequalities $\langle\nabla h_i(z^0),(\bar{x},\bar{y})\rangle \le 0$ with indices $i \in I^a(z^0,\bar{x})$ are called *essentially active* for the set $\Gamma(z^0;\bar{x})$.

Next lemma follows from the results about essentially active linear inequalities (Theorem 17.7 and Corollary 17.3 of Gorokhovik [25]).

Lemma 2.4 *Let $\Gamma(z_0;\bar{x}) \neq \varnothing$. Then,*

(1) *the system $\langle\nabla h_i(z^0),\bar{z}\rangle = 0$ $i \in I_0$, $\langle\nabla h_i(z^0),\bar{z}\rangle < 0$ $i \in I(z^0)$, has no solutions iff $I^a(z^0,\bar{x}) \neq \varnothing$;*
(2) *an inequality with index $i \in I(z^0)$ is essentially active iff there exists a vector $\lambda \in \Lambda_0(z^0)$ such that*

$$\sum_{j\in I_0\cup I(z^0)} \lambda_j\langle\nabla_x h_j(z^0),\bar{x}\rangle = 0 \text{ and } \lambda_i > 0;$$

(3) *if $I^a(z^0,\bar{x}) \neq \varnothing$, then there exists $\lambda \in \Lambda_0(z^0)$, $\sum\limits_{i\in I_0\cup I^a(z^0,\bar{x})} \lambda_i\langle\nabla_x h_i(z^0),\bar{x}\rangle = 0$,*

such that $i \in I^a(z^0,\bar{x})$ iff $\lambda_i > 0$.

Our goal is to find a condition which imposes weaker restrictions than $MF_{\bar{x}}$ but guarantees that $DF(z^0;\bar{x}) = \Gamma(z^0;\bar{x}) \neq \varnothing$. Note that a natural idea of requiring linear independence of the vectors $\{\nabla_y h_i(z^0)\ i \in I_0 \cup I^a(z^0)\}$ is not acceptable because, by virtue of Lemma 2.4, these vectors are always linearly dependent if $I^a(z^0,\bar{x}) \neq \varnothing$.

Definition 2.2 We say that the *relaxed Mangasarian–Fromovitz condition in the direction* $\bar{x}$ (briefly $RMF_{\bar{x}}$) holds at the point z^0 iff $\Gamma(z^0;\bar{x}) \neq \varnothing$, and the system of $(m+1)$-*vectors*

$$\begin{pmatrix} \nabla_y h_i(z) \\ \langle\nabla_x h_i(z),\bar{x}\rangle \end{pmatrix} \quad i \in I_0 \cup I^a(z^0,\bar{x}), \tag{1}$$

has constant rank near z^0.

Note that, in general, $RMF_{\bar{x}}$ (just as $MF_{\bar{x}}$) is not a constraint qualification since it does not guarantee the validity of the Karush–Kuhn–Tucker condition. It is not difficult to see that $MF_{\bar{x}}$ implies $RMF_{\bar{x}}$. Indeed, from the condition $MF_{\bar{x}}$ at the point z^0 it follows that $\sum\limits_{i\in I_0\cup I(z^0)} \lambda_i\langle\nabla_x h_i(z^0),\bar{x}\rangle < 0$ for all $\lambda \in \Lambda_0(z^0)\backslash\{0\}$. At the same time, if $I^a(z^0,\bar{x}) \neq \varnothing$ then, due to Lemma 2.4, there exists a nonzero vector $\lambda \in \Lambda_0(z^0)$ such that

$$\sum_{i\in I_0\cup I(z^0)} \lambda_i\langle\nabla_x h_i(z^0),\bar{x}\rangle = 0.$$

This means that $MF_{\bar{x}}$ can hold at the point z^0 only if $I^a(z^0, \bar{x}) = \varnothing$. However, $I^a(z^0, \bar{x}) = \varnothing$ implies that the rank of (1) is constant near z^0, that is $RMF_{\bar{x}}$ holds at z^0.

Next examples demonstrate that the converse is not true and, in general, $RMF_{\bar{x}}$ does not imply $MF_{\bar{x}}$.

Example 2.2 Let $F(x) = \{y \in \mathbb{R}^2 \;:\; y_2 - x \le 0,\; -y_2 + x \le 0\}$, $x \in \mathbb{R}$,

$$x^0 = 0,\;\; y^0 = (0, 0)^T.$$

It is not difficult to check that $MF_{\bar{x}}$ does not hold at $z^0 = (x^0, y^0)$ in any direction. However, $\Gamma(z^0; \bar{x}) = \{\bar{y} \in \mathbb{R}^2 \;:\; \bar{y}_2 = \bar{x}\}$ and both constraints $h_1(x, y) = y_2 - x$ and $h_2(x, y) = -y_2 + x$ are essentially active, and the rank of $\{\nabla_y h_1(x, y), \nabla_y h_2(x, y)\}$ is constant near z^0. This means $RMF_{\bar{x}}$ holds at z^0 in any direction $\bar{x}$.

Example 2.3 Let

$$F(x) = \left\{y \in \mathbb{R}^3 \;:\; y_1^2 - y_3 + x \le 0,\; -y_1^2 + y_3 - x \le 0,\; y_1 - y_3 = 0\right\},\; x \in \mathbb{R}$$

and $x^0 = 0,\;\; y^0 = (0, 0, 0)^T$ and $\bar{x} = 1$. Then,

$$\Gamma(z^0; \bar{x}) = \left\{y \in \mathbb{R}^3 \;:\; \bar{y}_3 \ge \bar{x},\; \bar{y}_3 \le \bar{x},\; \bar{y}_1 - \bar{y}_3 = 0\right\} = \left\{y \in \mathbb{R}^3 \;:\; \bar{y}_3 = \bar{y}_1 = \bar{x}\right\}$$

and, therefore, $I^a(z^0, \bar{x}) = \{1, 2\}$. Then, for the functions $h_1(z) = y_1^2 - y_3 + x$, $h_2(z) = -y_1^2 + y_3 - x$, $h_3(z) = y_1 - y_3$ one has

$$rank\left\{\begin{pmatrix} \nabla_y h_i(z) \\ \langle \nabla_x h_i(z),\; \bar{x}\rangle \end{pmatrix}\; i \in I_0 \cup I^a(z^0, \bar{x})\right\} = 2 = const.$$

Thus, $RMF_{\bar{x}}$ holds at z^0 in the direction $\bar{x}$. At the same time it is easy to verify that $MF_{\bar{x}}$ does not hold at the point z^0.

Example 2.4 Let $F(x) = \{y \in \mathbb{R}^3 \;:\; y_1 + y_2^2 - x \le 0,\; -y_1 + \sqrt{3}\, y_2 + x \le 0,\; -y_1 - \sqrt{3}\, y_2 + x \le 0,\; y_1^2 + y_2^2 - y_3 \le 0\}$ and $x^0 = 0,\;\; y^0 = (0, 0, 0)^T,\;\; \bar{x} = 1$. Since $\lambda = (2, 1, 1, 0)^T \in \Lambda_0(z^0)$ and $\sum_{i=1}^{3} \lambda_i \langle \nabla_x h_i(z^0), \bar{x}\rangle = 0$, and there is no vector $\lambda \in \Lambda_0(z^0)$ with positive λ_4, we obtain $I^a(z^0, \bar{x}) = \{1, 2, 3\}$ due to Lemma 2.4. It is easy to check that $RMF_{\bar{x}}$ holds at z^0. At the same time the classical Mangasarian–Fromovitz constraint qualification does not hold at the point z^0 since $\Lambda_0(z^0) \neq \{0\}$. Moreover, $MF_{\bar{x}}$ does not hold too.

Lemma 2.5 $I^a(z^0, \bar{x}) \subset I^a(z^0)$.

Proof The inclusion is valid if $I^a(z^0, \bar{x}) = \varnothing$. Let $I^a(z^0, \bar{x}) \neq \varnothing$ and choose any $i \in I^a(z^0, \bar{x})$. Then, due to Lemma 2.4, there is a vector $\lambda \in \Lambda_0(z^0)$ such that $\lambda_i > 0$, $\lambda_j = 0$ for all $j \in I \backslash I^a(z^0, \bar{x})$ and $\sum\limits_{i \in I_0 \cup I(z^0)} \lambda_i \langle \nabla_x h_i(z^0), \bar{x} \rangle = 0$.

On the other hand, according to Lemma 2.1 it follows from the inequality $\lambda_i > 0$ with $i \in I^a(z^0, \bar{x})$ that the constraint with the index $i \in I^a(z^0, \bar{x})$ is essentially active for the set $\Gamma_{F(x^0)}(y^0)$. Thus $i \in I^a(z^0)$. □

Example 2.5 Under the hypotheses of Example 2.3, $I^a(z^0, \bar{x}) = \{1\}$, $I^a(z^0) = \{1, 2\}$, that is, $I^a(z^0, \bar{x}) \subset I^a(z^0)$.

Theorem 2.1 *Let RMF_x hold at the point z^0. Then,*

$$DF(z^0; \overline{x}) = \Gamma(z^0; \overline{x}) \neq \varnothing.$$

Proof We have $\Gamma(z^0; \overline{x}) \neq \varnothing$ by the very definition of RMF_x and, as mentioned in Remark 2.1, $DF(z^0; \overline{x}) \subset \Gamma(z^0; \overline{x})$. Therefore, we only need to prove $\Gamma(z^0; \overline{x}) \subset DF(z^0; \overline{x})$. Due to the definition of $I^a(z^0; \overline{x})$ and Lemma 2.3, we have

$$\begin{aligned} aff\,\Gamma(z^0; \overline{x}) = \{\overline{y} \in \mathbb{R}^m : \langle \nabla_y h_i(z^0), \overline{y} \rangle + \langle \nabla_x h_i(z^0), \overline{x} \rangle = 0 \\ i \in I_0 \cup I^a(z^0, \overline{x})\}. \end{aligned}$$

Consider the convex function

$$g(\overline{y}) = \begin{cases} \max\limits_{i \in I^-(z^0, \overline{x})} [\langle \nabla_y h_i(z^0), \overline{y} \rangle + \langle \nabla_x h_i(z^0), \overline{x} \rangle], & \overline{y} \in aff\,\Gamma(z^0; \overline{x}) \\ +\infty, & \overline{y} \notin aff\,\Gamma(z^0; \overline{x}). \end{cases}$$

Then, $\Gamma(z^0; \overline{x}) = \{\overline{y} \in \mathbb{R}^m : g(\overline{y}) \leq 0\}$ and there is a point $\overline{y}^0$ such that $g(\overline{y}^0) < 0$. It follows from Corollary 7.6.1 of Rockafellar [26] that

$$\begin{aligned} ri\,\Gamma(z^0; \overline{x}) &= ri\{\overline{y} : g(\overline{y}) \leq 0\} = \{\overline{y} : g(\overline{y}) < 0\} \\ &= \left\{\overline{y} : \langle \nabla h_i(z^0), \overline{z} \rangle = 0 \;\; i \in I_0 \cup I^a(z^0, \overline{x}), \langle \nabla_y h_i(z^0), \overline{z} \rangle < 0 \;\; i \in I^-(z^0, \overline{x})\right\}. \end{aligned}$$

Let $\overline{y} \in ri\,\Gamma(z^0; \overline{x})$ and let $J = I^a(z^0; \overline{x}) \cup I_0$. Then, for any m-vector function $r(t)$ such that $r(t)/t \to 0$ as $t \downarrow 0$, there exists a number $t_0 > 0$ such that $h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r(t)) < 0$ for all $i \in I \backslash J$ and all $t \in (0; t_0)$. Indeed, if $i \in I \backslash I(z^0)$, then $h_i(x^0, y^0) < 0$ and $h_i(x^0, y^0 + t\overline{y} + r(t)) < 0$ for all sufficiently small $t > 0$. If $i \in I(z^0)$ but $i \notin I^a(z^0, \overline{x})$ (i.e., $i \in I^-(z^0, \overline{x})$), then $h_i(x + t\overline{x}^0, y^0 + t\overline{y} + r(t)) = t\langle \nabla h_i(z^0), \overline{z} \rangle + o(t)$ and, therefore, $h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r(t)) < 0$ for all sufficiently small positive t since $\langle \nabla h_i(x^0, y^0), \overline{z} \rangle < 0$.

Let N denote the number of elements of the set J. Since

$$\begin{aligned} &\frac{\partial h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r)}{\partial t} \\ &= \langle \nabla_x h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r), \overline{x} \rangle + \langle \nabla_y h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r), \overline{y} \rangle, \end{aligned}$$

$$\langle \nabla h_i(z^0), (\overline{x}, \overline{y}) \rangle = 0 \;\; i \in I_0 \cup I^0(z^0, \overline{x}),$$

the rank of the Jacobi matrix

$$J(r,t) = \begin{bmatrix} \frac{\partial h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} & \frac{\partial h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial t} \\ \frac{\partial h_2(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_2(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} & \frac{\partial h_2(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial t} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_N(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_N(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} & \frac{\partial h_N(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial t} \end{bmatrix}$$

for the functions $h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r)$, $i \in J$, coincides with the rank of the system (1). Moreover, the last column of the matrix $J(r,t)$ is null at the point $(r,t) = (0,0)$. Hence, at the point $(r,t) = (0,0)$ its rank coincides with the rank of the Jacobi matrix

$$A(r,t) = \begin{bmatrix} \frac{\partial h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} \\ \frac{\partial h_2(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_2(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_N(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_1} & \cdots & \frac{\partial h_N(x^0+t\overline{x}, y^0+t\overline{y}+r)}{\partial r_m} \end{bmatrix}$$

for $h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r)$, $i \in J$, regarded as functions of the variable r.

Let the rank of $A(r,t)$ at the point $(r,t) = (0,0)$ be equal to l. Then, from the condition RMF_x it follows that

$$rank\, J(r,t) = rank\, J(0,0) = rank\, A(0,0) = const = l$$

for all (r,t) near $(0,0)$. Consequently, $rank\, A(r,t) = l$. This means the system $h_i(x^0 + t\overline{x}, y^0 + t\overline{y} + r)$, $i \in J$, keeps the rank l with respect to r in some neighbourhood of the point $(0,0)$. Then, (see p. 505 of Zorich [27]) in this neighbourhood l functions of the system (let them be $h_1, \dots, h_l$) are independent and others depend on them, that is, $h_{l+1} = \varphi_1(h_1, \dots h_l), \dots, h_{l+q} = \varphi_q(h_1, \dots h_l)$, where $q = N - l$ and $\varphi_1, \dots, \varphi_q$ are twice continuously differentiable functions in some neighbourhood of $(h_1(y^0), \dots, h_l(y^0))$.

In the neighbourhood of the point $(0,0)$ consider the systems of equations

$$\begin{aligned} &h_1(x^0 + t\overline{x}, y^0 + t\overline{y} + r) = 0, \\ &\dots \\ &h_l(x^0 + t\overline{x}, y^0 + t\overline{y} + r) = 0, \\ &\dots \\ &h_{l+q}(x^0 + t\overline{x}, y^0 + t\overline{y} + r) = 0, \end{aligned} \tag{2}$$

with $l+q$ being equal to N, according to the notation introduced above. In some neighbourhood of the point $(0,0)$, this system is equivalent to the system

$$
\begin{aligned}
&h_1(x^0+t\overline{x}, y^0+t\overline{y}+r)=0.\\
&\dots\\
&h_l(x^0+t\overline{x}, y^0+t\overline{y}+r)=0,
\end{aligned}
\tag{3}
$$

with the additional condition

$$
\begin{aligned}
&h_{l+1}(x^0+t\overline{x}, y^0+t\overline{y}+r)\\
&=\phi_1(h_1(x^0+t\overline{x}, y^0+t\overline{y}+r),\dots, h_l(x^0+t\overline{x}, y^0+t\overline{y}+r))=0.\\
&\dots\\
&h_{l+q}(x^0+t\overline{x}, y^0+t\overline{y}+r)\\
&=\phi_q(h_1(x^0+t\overline{x}, y^0+t\overline{y}+r),\dots, h_l(x^0+t\overline{x}, y^0+t\overline{y}+r))=0.
\end{aligned}
$$

Note that

$$\phi_1(h_1(z^0),\dots,h_l(z^0))=0,\dots,\phi_q(h_1(z^0),\dots,h_l(z^0))=0$$

and, consequently,

$$\phi_1(0,\dots,0)=0,\dots,\phi_q(0,\dots,0)=0.$$

If $l=m$, then, due to the implicit function theorem (see p. 488 of Zorich [27]), in some neighbourhood of $(0,0)$ the system (3) defines a continuously differentiable function $r=r(t)$ such that $r(0)=0$ and $\frac{dr}{dt}(0)=\lim_{t\to 0}\frac{r(t)}{t}=0$.

Let $l<m$. Without any loss of generality, we can assume that the rank of the Jacobian of the system (2) is equal to l with respect to the first l coordinates of the vector r. Denote $r:=(\overline{r},\overline{\overline{r}})$, where $\overline{r}=(r_1,\dots,r_l)$, $\overline{\overline{r}}=(r_{l+1},\dots,r_m)$. Due to the implicit function theorem, the system (3) defines a continuously differentiable function $\overline{r}=\overline{r}(t,\overline{\overline{r}})$ near the point $(0,0)$ such that $\overline{r}(0,0)=0$, $\frac{\partial\overline{r}}{\partial t}(0,0)=0$. Let $\overline{\overline{r}}=0$. Denote $\overline{r}:=\overline{r}(t)=\overline{r}(t,0)$. Then, the function $r=r(t)=(\overline{r}(t),0)$ satisfies the system (3) and the additional condition to (3). Therefore, this function satisfies (2). Moreover $r(t)/t\to 0$ as $t\downarrow 0$. Thus, for any $\overline{y}\in ri\Gamma(z^0,\overline{x})$ there exists a function $r(t)$ such that

$$
\begin{aligned}
h_i(x^0+t\overline{x}, y^0+t\overline{y}+r(t))=0\ \ i\in J.\\
h_i(x^0+t\overline{x}, y^0+t\overline{y}+r(t))<0\ \ i\in I\backslash J.
\end{aligned}
$$

for all $t\in[0,t_0]$ where t_0 is a sufficiently small positive number. In addition, $r(t)t^{-1}\to 0$ as $t\downarrow 0$. This means that $y^0+t\overline{y}+r(t)\in F(x^0+t\overline{x})$ for all $t\in[0,t_0]$ and, consequently, $\overline{y}\in DF(z^0;\overline{x})$. Thus, $ri\Gamma(z^0;\overline{x})\subset DF(z^0;\overline{x})$. This means that $\Gamma(z^0;\overline{x})=DF(z^0;\overline{x})$. □

3 Second-Order Derivatives of Multivalued Mapping F

Consider a vector $\bar{y} \in \Gamma\left(z^0; \bar{x}\right)$ and introduce, following [14], the lower and upper second-order Dini derivatives of the multivalued mapping F at the point z^0 along the vector $\bar{z} = (\bar{x}, \bar{y})$ in the direction $\bar{x}$:

$$\hat{D}^2 F(z^0, \bar{z}; \bar{x}) := \{\bar{v} \in \mathbb{R}^m : \exists t_k \downarrow 0 \; such\ that$$
$$y^0 + t_k \bar{y} + t_k^2 \bar{v} + o(t_k^2) \in F(x^0 + t_k \bar{x}) \;\; k = 1, 2, \ldots\},$$
$$D^2 F\left(z^0, \bar{z}; \bar{x}\right) := \left\{\bar{v} \in \mathbb{R}^m \; : \; y^0 + t\bar{y} + t^2 \bar{v} + o\left(t^2\right) \in F\left(x^0 + t\bar{x}\right) \; \forall t > 0\right\}.$$

Denote

$$\Gamma^2(z^0, \bar{z}; \bar{x}) := \{\bar{v} \in \mathbb{R}^m \; : \; \left\langle \nabla_y h_i(z^0), \bar{v} \right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z} \right\rangle \leq 0 \;\; i \in I^2(z^0, \bar{z}),$$
$$\left\langle \nabla_y h_i\left(z^0\right), \bar{v} \right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i\left(z^0\right)\bar{z} \right\rangle = 0 \;\; i \in I_0 \},$$

where $I^2\left(z^0, \bar{z}\right) := \left\{i \in I\left(z^0\right) : \left\langle \nabla h_i\left(z^0\right), \bar{z}\right\rangle = 0\right\}$.

Denote also

$$I^a(z^0, \bar{z}) := \{i \in I^2(z^0, \bar{z}) : \left\langle \nabla_y h_i(z^0), \bar{v} \right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z} \right\rangle = 0$$
$$\forall \bar{v} \in \Gamma^2(z^0, \bar{z}; \bar{x})\},$$
$$I^-(z^0, \bar{z}) := I^2(z^0, \bar{z}) \setminus I^a(z^0, \bar{z}).$$

Definition 3.1 Let $\bar{y} \in \Gamma(z^0; \bar{x})$. We say that the *relaxed second-order Mangasarian–Fromovitz condition at the point* z^0 *along the vector* $\bar{z} = (\bar{x}, \bar{y})$ *in the direction* $\bar{x}$ (briefly , $RMF^2_{\bar{x}}(\bar{z})$) holds iff $\Gamma^2(z^0, \bar{z}; \bar{x}) \neq \varnothing$ and the system

$$\begin{pmatrix} \nabla_y h_i(z) \\ \langle \nabla_x h_i(z), \bar{x} \rangle \end{pmatrix} \;\; i \in I_0 \cup I^a(z^0, \bar{z})$$

has constant rank for all z in some neighbourhood of the point z^0.

Lemma 3.1 $I^a(z^0, \bar{z}) \subset I^a(z^0)$.

The proof is similar to the proof of Lemma 2.5.

By employing the sets $\Gamma^2\left(z^0, \bar{z}; \bar{x}\right)$ and $\hat{D}^2 F(z^0, \bar{z}; \bar{x})$, we can now derive the following additional conclusion from the assumptions used in Theorem 2.1.

Lemma 3.2 *Let* $RMF_{\bar{x}}$ *hold at* z^0. *Then,* $\Gamma^2(z^0, \bar{z}; \bar{x}) = \hat{D}^2 F(z^0, \bar{z}; \bar{x}) \neq \varnothing$ *for any* $\bar{y} \in \Gamma(z^0; \bar{x})$.

Proof Let $\bar{y} \in \Gamma(z^0; \bar{x})$. Then, due to Theorem 2.1, $\bar{y} \in DF(z^0; \bar{x})$.
Set $C := \left\{(t, y) \in \mathbb{R} \times \mathbb{R}^m \ : \ y \in F(x^0 + t\bar{x}), \ t \geq 0\right\}$. That is,

$$C = \{(t, y) \in \mathbb{R} \times \mathbb{R}^m \ : \ t \geq 0, \ h_i(x^0 + t\bar{x}, y) \leq 0 \ \ i \in I,$$
$$h_i(x^0 + t\bar{x}, y) = 0 \ \ i \in I_0\}$$

and, therefore, the linearized cone $\Gamma_C(0, y^0)$ to the set C at the point $(0, y^0)$ is given by

$$\Gamma_C(0, y^0) = \{(\tilde{t}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^m : \tilde{t} \geq 0,$$
$$\left\langle \nabla_y h_i(z^0), \tilde{y} \right\rangle + \tilde{t} \left\langle \nabla_x h_i(z^0), \bar{x} \right\rangle \leq 0, \quad i \in I(z^0),$$
$$\left\langle \nabla_y h_i(z^0), \tilde{y} \right\rangle + \tilde{t} \left\langle \nabla_x h_i(z^0), \bar{x} \right\rangle = 0, \quad i \in I_0\}.$$

By using the notation *cone* S for the cone spanned by a set S, we obtain from this expression that

$$\Gamma_C(0, y^0) = \left\{(0, \tilde{y}) : \tilde{y} \in \Gamma_{F(x^0)}(y^0)\right\} \cup cone \left\{(1, \tilde{y}) : \tilde{y} \in \Gamma(z^0; \bar{x})\right\}.$$

Let's check the validity of the condition *RMFCQ* for the set C at the point $(0, y^0) \in C$. The *RMFCQ* at this point means (see [19] and Definitions 6 and 6′ of Kruger et al. [21]) that

$$rank \left\{ \begin{pmatrix} \nabla_y h_i(x^0 + t\bar{x}, y) \\ \nabla_x h_i(x^0 + t\bar{x}, y), \bar{x} \end{pmatrix} i \in I_0 \cup I_C^a(0, y^0) \right\} = const$$

for all (t, y) near $(0, y^0)$, where

$$I_C^a(0, y^0) = \{i \in I(z^0) : \left\langle \nabla_y h_i(z^0), \tilde{y} \right\rangle + \tilde{t} \left\langle \nabla_x h_i(z^0), \bar{x} \right\rangle = 0 \ \forall (\tilde{t}, \tilde{y}) \in \Gamma_C(0, y^0)\}.$$

Let $k \in I_C^a(0, y^0)$, that is, $\langle \nabla_y h_k(z^0), \tilde{y} \rangle + \tilde{t} \langle \nabla_x h_k(z^0), \bar{x} \rangle = 0$ for all

$$(\tilde{t}, \tilde{y}) \in \left\{(0, \tilde{y}) \ : \ \tilde{y} \in \Gamma_{F(x^0)}(y^0)\right\} \cup cone \left\{(1, \tilde{y}) \ : \ \tilde{y} \in \Gamma(z^0; \bar{x})\right\}.$$

For $\tilde{t} = 1$ (and for all $\tilde{t} > 0$) the condition above is valid iff $k \in I^a(z^0, \bar{x})$, and for $\tilde{t} = 0$ it is valid iff $k \in I^a(z^0)$. Since $I^a(z^0, \bar{x}) \subset I^a(z^0)$ due to Lemma 2.5, we obtain $I_C^a(0, y^0) = I^a(z^0, \bar{x})$. Thus, the $RMF_{\bar{x}}$ at z^0 implies the *RMFCQ* for the set C at the point $(0, y^0) \in C$. Then, due to Kruger et al. [21], the local error bound property holds for the set C at $(0, y^0) \in C$. This means there exist a number $\alpha > 0$ and neighbourhoods $V(0)$ and $V(y^0)$ such that the Euclidean distance from (t, y) to C can be estimated as

$$d((t, y), C) \leq \alpha \ \max\{0, -t, h_i(x^0 + t\bar{x}, y) \ \ i \in I(z^0), |\ h_i(x^0 + t\bar{x}, y)\ | \ \ i \in I_0\}$$

for all $t \in V(0)$, $y \in V(y^0)$. It follows from the last inequality that

$$d((t, y^0 + t\bar{y}), C) \le \alpha \ \max\{0, -t, h_i(x^0 + t\bar{x}, y^0 + t\bar{y}) \ \ i \in I(z^0), \\ \mid h_i(x^0 + t\bar{x}, y^0 + t\bar{y}) \mid i \in I_0\} \le Mt^2,$$

for all sufficiently small positive t, where $M = const > 0$. This means that for any sequence $t_k \downarrow 0$ there exist bounded sequences $\{\alpha_k\}$, $\{p^k\}$ such that $y^0 + t_k\bar{y} + t_k^2 p^k \in F(x^0 + t_k\bar{x} + \alpha_k t_k^2 \bar{x})$ for all $k = 1, 2, \dots$.

Denoting $\delta_k := t_k + \alpha_k t_k^2$ and assuming without any loss of generality that $\alpha_k \neq 0$ for all $k = 1, 2, \dots$, we obtain

$$t_k = \frac{-1 + \sqrt{1 + 4\alpha_k\delta_k}}{2\alpha_k} = \delta_k + O(\delta_k^2),$$

and, consequently,

$$y^0 + \delta_k\bar{y} + \delta_k^2 p^k + O(\delta_k^2)\bar{y} + o(\delta_k^2) \in F(x^0 + \delta_k\bar{x}).$$

This means that there exists a bounded sequence $\{\bar{v}^k\}$ such that $y^0 + \delta_k\bar{y} + \delta_k^2\bar{v}^k \in F(x^0 + \delta_k\bar{x})$ for all $k = 1, 2, \dots$ and without any loss of generality one can suppose that $\bar{v}^k \to \bar{v}$ and $y^0 + \delta_k\bar{y} + \delta_k^2\bar{v} + o(\delta_k^2) \in F(x^0 + \delta_k\bar{x})$. Then, from the equalities and inequalities below

$$h_i(x^0 + \delta_k\bar{x}, y^0 + \delta_k\bar{y} + \delta_k^2\bar{v} + o(\delta_k^2)) \le 0 \ \ i \in I^2(z^0, \bar{z}), \\ h_i(x^0 + \delta_k\bar{x}, y^0 + \delta_k\bar{y} + \delta_k^2\bar{v} + o(\delta_k^2)) = 0 \ \ i \in I_0,$$

it follows that

$$\left\langle \nabla_y h_i(z^0), \bar{v} \right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z} \right\rangle \le 0 \ \ i \in I^2(z^0, \bar{z}), \\ \left\langle \nabla_y h_i(z^0), \bar{v} \right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z} \right\rangle = 0 \ \ i \in I_0,$$

that is $\bar{v} \in \Gamma^2(z^0, \bar{z}; \bar{x})$ and $\Gamma^2(z^0, \bar{z}; \bar{x}) \neq \varnothing$.

Take an arbitrary vector $\bar{v} \in \Gamma^2(z^0, \bar{z}; \bar{x})$. Then, for all sufficiently small $t > 0$ we have

$$d((t, y^0 + t\bar{y} + t^2\bar{v}), C) \le \alpha \ \max\{0, -t, h_i(x^0 + t\bar{x}, y^0 + t\bar{y} + t^2\bar{v}) \ \ i \in I(z^0), \\ \mid h_i(x^0 + t\bar{x}, y^0 + t\bar{y} + t^2\bar{v}) \mid \ \ i \in I_0\} = o(t^2)$$

and, therefore,

$$y^0 + t\bar{y} + t^2\bar{v} + o_{\bar{v}}(t^2) \in F(x^0 + t\bar{x} + o(t^2)\bar{x}).$$

Set $\tau = t + o(t^2)$ and obtain $y^0 + \tau\bar{y} + \tau^2\bar{v} + o_{\bar{v}}(\tau^2) \in F(x^0 + \tau\bar{x})$. The function $\tau = \tau(t) = t + o(t^2)$ may not be monotone, but it follows from the last inclusion that there exists a sequence $\delta_k \downarrow 0$ and a convergent sequence $\bar{v}^k \to \bar{v}$ such that $y^0 + \bar{y} + \delta_k^2\bar{v}^k \in F(x^0 + \delta_k\bar{x})$ for all $k = 1, 2, \ldots$. Thus, $\bar{v} \in \hat{D}^2F(z^0, \bar{z}; \bar{x})$ and $\Gamma^2(z^0, \bar{z}; \bar{x}) \subset \hat{D}^2F(z^0, \bar{z}; \bar{x})$ and, therefore, $\Gamma^2(z^0, \bar{z}; \bar{x}) = \hat{D}^2F(z^0, \bar{z}; \bar{x})$. □

Corollary 3.1 *If, in addition to the conditions of Lemma 3.2, the multivalued mapping F satisfies the Aubin property [28,29] at the point z^0, then $\Gamma(z^0; \bar{x}) = DF(z^0; \bar{x})$ and $\Gamma^2(z^0, \bar{z}; \bar{x}) = D^2F(z^0, \bar{z}; \bar{x}) \neq \varnothing$ for any $\bar{y} \in \Gamma(z^0; \bar{x})$.*

Corollary 3.2 *Let $\bar{y} \in \Gamma(z^0; \bar{x})$. Then, under conditions of Lemma 3.2, for any $\bar{v} \in \Gamma^2(z^0, \bar{z}; \bar{x})$ there exist an m-vector function $o_v(t^2)$ and a function $o(t^2)$ such that*

$$y^0 + t\bar{y} + t^2\bar{v} + o_v(t^2) \in F(x^0 + (t + o(t^2))\bar{x})$$

for all sufficiently small positive t.

The relaxed second-order Mangasarian–Fromovitz condition at the point z^0 along the vector $\bar{z} = (\bar{x}, \bar{y})$ in the direction $\bar{x}$ implies that the multivalued mapping F has second-order directional derivative. Specifically, the following theorem holds.

Theorem 3.1 *Let $\bar{y} \in \Gamma(z^0; \bar{x})$. If the condition $RMF^2_{\bar{x}}(\bar{z})$ holds at the point z^0 along the vector $\bar{z} = (\bar{x}, \bar{y})$, then $D^2F(z^0, \bar{z}; \bar{x}) = \Gamma^2(z^0, \bar{z}; \bar{x}) \neq \varnothing$.*

Proof The proof of Theorem 3.1 follows the scheme which was accepted in the proof of Theorem 2.1. Namely, we can show that under the assumptions of Theorem 3.1,

$$ri\,\Gamma^2(z^0, \bar{z}; \bar{x}) = \{\bar{v} : \langle \nabla_y h_i(z^0), \bar{v}\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z}\right\rangle = 0 \;\; i \in I_0 \cup I^a(z^0, \bar{z}),$$
$$\left\langle \nabla_y h_i(z^0), \bar{v}\right\rangle + \frac{1}{2}\left\langle \bar{z}, \nabla^2 h_i(z^0)\bar{z}\right\rangle < 0 \;\; i \in I^-(z^0, \bar{z})\}.$$

Moreover, the implicit function theorem along with the argument similar to the one used in the proof of Theorem 2.1 yields that the inclusion $\bar{v} \in ri\,\Gamma^2(z^0, \bar{z}; \bar{x})$ implies $\bar{v} \in D^2F(z^0, \bar{z}; \bar{x})$. □

Definition 3.2 We say that the *uniform relaxed Mangasarian–Fromovitz condition* ($URMF_{\bar{x}}$) *holds at the point z^0 in the direction $\bar{x}$* iff $\Gamma(z^0; \bar{x}) \neq \varnothing$ and any system $\begin{pmatrix} \nabla_y h_i(z) \\ \langle \nabla_x h_i(z), \bar{x}\rangle \end{pmatrix}$, $i \in I_0 \cup K$, where $K \subset I^a(z^0)$, has constant rank near the point z^0.

Obviously $URMF_{\bar{x}}$ is implied by $MF_{\bar{x}}$. On the other hand, $URMF_{\bar{x}}$ implies $RMF_{\bar{x}}$.

Corollary 3.3 *Let $URMF_{\bar{x}}$ hold at z^0. Then, $DF(z^0; \bar{x}) = \Gamma(z^0; \bar{x}) \neq \varnothing$ and $D^2F(z^0, \bar{z}; \bar{x}) = \Gamma^2(z^0, \bar{z}; \bar{x})$ for any $\bar{z} = (\bar{x}, \bar{y})$ such that $\bar{y} \in \Gamma(z^0; \bar{x})$ and $\Gamma^2(z^0, \bar{z}; \bar{x}) \neq \varnothing$.*

Proof By virtue of Lemma 2.4, $RMF_{\bar{x}}$ follows from $URMF_{\bar{x}}$, therefore all requirements of Theorem 2.1 are fulfilled. Taking into account Lemma 2.5 and $\Gamma^2(z^0, \bar{z}; \bar{x}) \neq \varnothing$, it follows from $URMF_{\bar{x}}$ that $RMF^2_{\bar{x}}(\bar{z})$ is valid at z^0. Thus, all requirements of Theorem 3.1 are satisfied and, therefore, $DF(z^0; \bar{x}) = \Gamma(z^0; \bar{x}) \neq \varnothing$ and $D^2F(z^0, \bar{z}; \bar{x}) = \Gamma^2(z^0, \bar{z}; \bar{x})$. □

4 Directional Derivatives of the Value Function

Introduce the cone of critical directions at the point z^0:

$$D(z^0) := \left\{ \bar{y} \in \Gamma(z^0; 0) : \langle \nabla_y f(z^0), \bar{y} \rangle \leq 0 \right\}.$$

The following lemma is a slight modification of some results of Shapiro [8].

Lemma 4.1 *Let $x^k \to x^0$, $y^k \in \omega(x^k)$ and $y^k \to y^0 \in \omega(x^0)$, $\varphi(x^k) - \varphi(x^0) \leq M \left| x^k - x^0 \right|$, $k = 1, 2, \dots$, where $M = const > 0$. If $\lim_{k \to \infty} \left| x^k - x^0 \right|^{-1} \left| y^k - y^0 \right| = \infty$, then all limit points of the sequence $\{ \left| y^k - y^0 \right|^{-1} (y^k - y^0) \}$ belong to $D(z^0)$.*

Denote $\Phi(z^0, \bar{z}, \bar{v}) := \langle \nabla_y f(z^0), \bar{v} \rangle + \frac{1}{2} \langle \bar{z}, \nabla^2 f(z^0) \bar{z} \rangle$, where $\bar{z} = (\bar{x}, \bar{y})$.
Introduce the sets

$$\Gamma^*(z; \bar{x}) := \left\{ \bar{y}^* \in \Gamma(z; \bar{x}) : \langle \nabla f(z), (\bar{x}, \bar{y}^*) \rangle = \min_{\bar{y} \in \Gamma(z; \bar{x})} \langle \nabla f(z), (\bar{x}, \bar{y}) \rangle \right\}$$

and

$$\Lambda^2(z; \bar{x}) := \{ \lambda \in \Lambda(z) : \langle \nabla_x L(z, \lambda), \bar{x} \rangle = \max_{\lambda \in \Lambda(z)} \langle \nabla_x L(z, \lambda), \bar{x} \rangle \}$$

at a point $z = (x, y) \in gr\,F$.

Lemma 4.2 ([9]) *Let $z = (x, y) \in gr\,F$. The following assertions are valid.*

1. *If at least one of the sets $\Lambda(z)$ or $\Gamma(z; \bar{x})$ is non-empty, then*

$$\inf_{\bar{y} \in \Gamma(z; \bar{x})} \langle \nabla f(z), \bar{z} \rangle = \sup_{\lambda \in \Lambda(z)} \langle \nabla_x L(z, \lambda), \bar{x} \rangle.$$

In addition, if both sets are non-empty, then the extrema on both sides of this equality are attained.

2. *If $\Gamma^2(z, \bar{z}; \bar{x}) \neq \varnothing$ for some $\bar{y} \in \Gamma^*(z; \bar{x})$, then*

$$\inf_{\bar{v} \in \Gamma^2(z, \bar{z}; \bar{x})} 2\Phi(z, \bar{z}, \bar{v}) = \sup_{\lambda \in \Lambda^2(z, \bar{x})} \langle \bar{z}, \nabla^2 L(z^0, \lambda) \bar{z} \rangle.$$

In addition, if $DF(z; \bar{x}) = \Gamma(z; \bar{x}) \neq \varnothing$ and $D^2F(z, \bar{z}; \bar{x}) = \Gamma^2(z, \bar{z}; \bar{x}) \neq \varnothing$ for all $\bar{y} \in \Gamma(z; \bar{x})$, then the extrema on both sides of the equality are attained.

Note that it follows from Lemma 4.2 that both sets $\Gamma^*(z;\bar{x})$ and $\Lambda^2(z;\bar{x})$ are non-empty if $\Gamma(z;\bar{x}) \neq \varnothing$ and $\Lambda(z) \neq \varnothing$.

Following Shapiro [8] we say that the *strong second-order sufficient condition in the direction* $\bar{x}$ ($SSOSC_{\bar{x}}$) holds at the point z^0 if $\Lambda(z^0) \neq \varnothing$ and

$$\sup_{\lambda \in \Lambda^2(z^0;\bar{x})} \langle \bar{y}, \nabla^2_{yy} L(z^0,\lambda)\bar{y}\rangle > 0 \text{ for all nonzero vectors } \bar{y} \in D(z^0).$$

Lemma 4.3 *Let the conditions* $RMF_{\bar{x}}$ *and* $SSOSC_{\bar{x}}$ *hold at the point* $z^0 = (x^0, y^0)$ *such that* $y^0 \in \omega(x^0)$. *Let also* $\bar{y} \in \Gamma^*(z^0;\bar{x})$ *and* $\liminf_{t\downarrow 0} t^{-2}(\varphi(x^0+t\bar{x}) - \varphi(x^0) - t\langle \nabla f(z^0), \bar{z}\rangle)$ *be attained on the sequence* $t_k \downarrow 0$, $y^k \in \omega(x^0 + t_k\bar{x})$ *and* $y^k \to y^0 \in \omega(x^0)$. *Then,*

1. $\limsup_{k\to\infty} t_k^{-1}\left|y^k - y^0\right| < \infty$ *and all limit points* $\bar{y}^0$ *of the sequence* $\left\{t_k^{-1}(y^k - y^0)\right\}$ *belong to* $\Gamma^*(z^0;\bar{x})$;
2. *there exists derivative* $\varphi'(x^0;\bar{x}) = \langle \nabla f(z^0), \bar{z}\rangle$;
3. *the following equality*

$$D^2_+\varphi(x^0;\bar{x}) = \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})} \inf_{\bar{v}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{v})$$
$$= \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})} \sup_{\lambda\in\Lambda^2(z^0,\bar{x})} \langle(\bar{x},\bar{y}^0), \nabla^2_{zz}L(z^0,\lambda)(\bar{x},\bar{y})\rangle$$

is valid.

Proof Note first of all, that due to Theorem 2.1 and Lemma 4.2, it follows from the hypotheses that $DF(z^0;\bar{x}) = \Gamma(z^0;\bar{x}) \neq \varnothing$ and both sets $\Lambda^2(z^0,\bar{x})$ and $\Gamma^*(z^0;\bar{x})$ are non-empty. Let's prove the first assertion of the lemma. Suppose to the contrary that $\limsup_{k\to\infty} t_k^{-1}\left|y^k - y^0\right| = \infty$. Denote $x^k := x^0 + t_k\bar{x}$. Since $\Gamma(z^0;\bar{x}) = DF(z^0;\bar{x})$, for $\bar{y} \in \Gamma^*(z^0;\bar{x})$ there exists a function $o(t)$ such that $t^{-1}o(t) \to 0$ as $t \downarrow 0$ and $y^0 + t\bar{y} + o(t) \in F(x^0 + t\bar{x})$ for sufficiently small $t \geq 0$. Hence,

$$\begin{aligned}&\varphi(x^0+t\bar{x}) - \varphi(x^0)\\ &\quad \leq f(x^0+t\bar{x}, y^0+t\bar{y}+o(t)) - f(x^0,y^0) \leq Mt, \qquad M = const, \end{aligned} \tag{4}$$

and, consequently, $\varphi(x^k) - \varphi(x^0) \leq M\left|x^k - x^0\right|$ for $k = 1, 2, \ldots$ Then, due to Lemma 4.1, one can assume without any loss of generality that

$$(y^k - y^0)\left|y^k - y^0\right|^{-1} \to \widehat{y} \in D(z^0).$$

Take a vector $\tilde{\lambda} \in \Lambda^2(z^0,\bar{x})$ such that $\langle \widehat{y}, \nabla^2_{yy}L(x^0,y^0,\tilde{\lambda})\widehat{y}\rangle > 0$.

Since $\Gamma^2(z^0,\bar{z};\bar{x}) = \widehat{D}^2 F(z^0,\bar{z};\bar{x}) \neq \varnothing$ due to Lemma 3.2, we can take a vector $\bar{v} \in \Gamma^2(z^0,\bar{z};\bar{x})$. Then, by virtue of Corollary 3.2, there exist functions $o(t)$ and $o_v(t)$ such that $t^{-1}o(t) \to 0$, and $t^{-1}o_v(t) \to 0$ as $t \downarrow 0$, and $y^0 + t\bar{y} + t^2\bar{v} + o_v(t^2)$

$\in F(x^0 + (t + o(t^2))\bar{x})$ for all sufficiently small $t \geq 0$. Thus, we can assume that $y^0 + t_k\bar{y} + t_k^2\bar{\nu} + o_\nu(t_k^2) \in F(x^k + o(t_k^2)\bar{x})$ for all $k = 1, 2, \ldots$ Denote $\tilde{y}^k := y^0 + t_k\bar{y} + t_k^2\bar{\nu} + o_\nu(t_k^2)$ and consider the equality

$$
\begin{aligned}
&\liminf_{t\downarrow 0} t^{-2}(\varphi(x^0 + t\bar{x}) - \varphi(x^0) - t\langle\nabla f(z^0), \bar{z}\rangle)\\
&\quad = \lim_{k\to\infty} t_k^{-2}(\varphi(x^0 + t_k\bar{x}) - \varphi(x^0) - t_k\langle\nabla f(z^0), \bar{z}\rangle)\\
&\quad \leq \liminf_{k\to\infty} \frac{\varphi(x^0 + (t_k + o(t_k^2))\bar{x}) - \varphi(x^0) - (t_k + o(t_k^2))\langle\nabla f(z^0), \bar{z}\rangle}{(t_k + o(t_k^2))^2}\\
&\quad \leq \lim_{k\to\infty} \frac{f(x^0 + (t_k + o(t_k^2))\bar{x}, \tilde{y}^k) - f(x^0, y^0) - (t_k + o(t_k^2))\langle\nabla f(z^0), \bar{z}\rangle}{(t_k + o(t_k^2))^2}\\
&\quad = \lim_{k\to\infty} \frac{t_k^2}{(t_k + o(t_k^2))^2}\\
&\qquad \times \lim_{k\to\infty} \frac{f(x^0 + (t_k + o(t_k^2))\bar{x}, \tilde{y}^k) - f(x^0, y^0) - (t_k + o(t_k^2))\langle\nabla f(z^0), \bar{z}\rangle}{t_k^2}\\
&\quad \leq \Phi(z^0, \bar{z}, \bar{\nu}).
\end{aligned} \tag{5}
$$

It follows from the inequality (5) that, for any $\lambda \in \Lambda^2(z^0, \bar{x})$, one has

$$
\begin{aligned}
&L(x^k, y^k, \lambda) - L(x^0, y^0, \lambda) - t_k\langle\nabla f(z^0), (\bar{x}, \bar{y})\rangle\\
&\quad \leq f(x^k, y^k) - f(x^0, y^0) - t_k\langle\nabla f(z^0), (\bar{x}, \bar{y})\rangle\\
&\quad \leq \varphi(x^k) - \varphi(x^0) - t_k\langle\nabla f(z^0), (\bar{x}, \bar{y})\rangle \leq M_0 t_k^2,
\end{aligned} \tag{6}
$$

where $M_0 = const > 0$.

Taking into account Lemma 4.2, we obtain from (6) for $\lambda = \tilde{\lambda}$

$$
\frac{L(x^k, y^k, \tilde{\lambda}) - L(x^0, y^0, \tilde{\lambda}) - t_k\langle\nabla_x L(x^0, y^0, \tilde{\lambda}), \bar{x}\rangle}{|y^k - y^0|^2} \leq \frac{M_0 t_k^2}{|y^k - y^0|^2},
$$

which yields, after passing to the limit, that

$$
\langle\widehat{y}, \nabla_{yy}^2 L(x^0, y^0, \tilde{\lambda})\widehat{y}\rangle \leq 0.
$$

The last inequality contradicts the definition of $\tilde{\lambda}$. Thus, the sequence $\{t_k^{-1}(y^k - y^0)\}$ is bounded, and without any loss of generality one can assume that $t_k^{-1}(y^k - y^0) \to \bar{y}^0$ and, hence, $y^k = y^0 + t_k\bar{y}^0 + o(t_k)$. Furthermore,

$$
h_i(x^k, y^k) - h_i(x^0, y^0) \leq 0,\ i \in I(z^0), h_i(x^k, y^k) - h_i(x^0, y^0) = 0,\ i \in I_0,
$$

and, therefore, after passing to the limit,

$$\langle \nabla h_i(z^0), (\bar{x}, \bar{y}^0)\rangle \leq 0,\ i \in I(z^0), \langle \nabla h_i(z^0), (\bar{x}, \bar{y}^0)\rangle = 0, \quad i \in I_0.$$

Thus, $\bar{y}^0 \in \Gamma(z^0; \bar{x})$. Moreover, by virtue of Theorem 2.1, for any $\hat{y} \in \Gamma(z^0; \bar{x})$ there exists a function $o(t)$ such that

$$\begin{aligned} f(x^0 + t_k\bar{x}, y^k) - f(x^0, y^0) &= \varphi(x^0 + t_k\bar{x}) - \varphi(x^0) \\ &\leq f(x^0 + t_k\bar{x}, y^0 + t_k\hat{y} + o(t_k)) - f(x^0, y^0), \end{aligned}$$

and, therefore, $\langle \nabla f(z^0), (\bar{x}, \bar{y}^0)\rangle \leq \langle \nabla f(z^0), (\bar{x}, \hat{y})\rangle$, that is, $\bar{y}^0 \in \Gamma^*(z^0; \bar{x})$.

Now let's prove the second assertion of the lemma. From the inequalities (5) and (6), it immediately follows that

$$\liminf_{t\downarrow 0} t^{-2}(\varphi(x^0 + t\bar{x}) - \varphi(x^0) - t\langle \nabla f(z^0), \bar{z}\rangle) = \alpha \neq \infty$$

and

$$\sup_{\lambda \in \Lambda^2(z^0, \bar{x})} \langle (\bar{x}, \bar{y}^0), \nabla^2_{zz} L(z^0, \lambda)(\bar{x}, \bar{y}^0)\rangle \leq 2\alpha \leq \inf_{\bar{\nu} \in \Gamma^2(z^0, \bar{z}; \bar{x})} 2\Phi(z^0, \bar{z}, \bar{\nu}). \tag{7}$$

Let the limit

$$D_+\varphi(x^0; \bar{x}) = \liminf_{s\downarrow 0} s^{-1}(\varphi(x^0 + s\bar{x}) - \varphi(x^0))$$

be attained on a sequence $s_k \downarrow 0$. Then,

$$\alpha \leq \liminf_{k\to\infty} s_k^{-2}(\varphi(x^0 + s_k\bar{x}) - \varphi(x^0) - s_k\langle \nabla f(z^0), \bar{z}\rangle).$$

Therefore, for any $\varepsilon > 0$ there exists a positive integer number $k_0 = k_0(\varepsilon)$ such that

$$\alpha - \varepsilon \leq s_k^{-2}(\varphi(x^0 + s_k\bar{x}) - \varphi(x^0) - s_k\langle \nabla f(z^0), \bar{z}\rangle)$$

and, consequently,

$$s_k(\alpha - \varepsilon) \leq s_k^{-1}(\varphi(x^0 + s_k\bar{x}) - \varphi(x^0) - s_k\langle \nabla f(z^0), \bar{z}\rangle)$$

for all $k > k_0$. Then,

$$D_+\varphi(x^0; \bar{x}) - \langle \nabla f(z^0), \bar{z}\rangle \geq 0,$$

that is, $D_+\varphi(x^0; \bar{x}) \geq \langle \nabla f(z^0), \bar{z}\rangle$. On the other hand, it follows from (4) that $D^+\varphi(x^0; \bar{x}) \leq \langle \nabla f(z^0), \bar{z}\rangle$. This means that $\varphi'(x^0; \bar{x}) = \langle \nabla f(z^0), \bar{z}\rangle$.

Finally, let's prove the third assertion of the lemma. Due to the second assertion, we have $2\alpha = D^2_+\varphi(x^0;\bar{x})$ in (7). Then, it follows from (7) that

$$\inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\sup_{\lambda\in\Lambda^2(z^0,\bar{x})}\langle(\bar{x},\bar{y}^0),\nabla^2_{zz}L(z^0,\lambda)(\bar{x},\bar{y})\rangle \le D^2_+\varphi(x^0;\bar{x})$$
$$\le \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\inf_{\bar{\nu}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{\nu}).$$

Taking into consideration Lemma 4.2, we obtain from the last inequality that

$$D^2_+\varphi(x^0;\bar{x}) = \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\inf_{\bar{\nu}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{\nu})$$
$$= \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\sup_{\lambda\in\Lambda^2(z^0,\bar{x})}\langle(\bar{x},\bar{y}^0),\nabla^2_{zz}L(z^0,\lambda)(\bar{x},\bar{y})\rangle.$$

□

Denote $\omega(x^0,\bar{x}) := \left\{ y^0\in\omega(x^0) : \varphi'(x^0;\bar{x}) = \min_{y^0\in\omega(x^0)}\min_{\bar{y}\in\Gamma(z^0;\bar{x})}\langle\nabla f(z^0),\bar{z}\rangle \right\}.$

Theorem 4.1 *Let $RMF_{\bar{x}}$ and $SSOSC_{\bar{x}}$ hold at all points $z^0 = (x^0, y^0)$, where $y^0 \in \omega(x^0)$. Then,*

1. *the function φ is differentiable at the point x^0 in the direction $\bar{x}$ and*

$$\varphi'(x^0;\bar{x}) = \min_{y^0\in\omega(x^0)}\min_{\bar{y}\in\Gamma(z^0;\bar{x})}\langle\nabla f(z^0),\bar{z}\rangle$$
$$= \min_{y^0\in\omega(x^0)}\max_{\lambda\in\Lambda(z^0)}\langle\nabla_x L(z^0,\lambda),\bar{x}\rangle; \qquad (8)$$

2. *the following formula is valid*

$$D^2_+\varphi(x^0;\bar{x}) = \inf_{y^0\in\omega(x^0,\bar{x})}\inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\inf_{\bar{\nu}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{\nu})$$
$$= \inf_{y^0\in\omega(x^0,\bar{x})}\inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\sup_{\lambda\in\Lambda^2(z^0,\bar{x})}\langle(\bar{z}),\nabla^2_{zz}L(z^0,\lambda)(\bar{z})\rangle. \qquad (9)$$

Proof (1) Let $y^0 \in \omega(x^0)$. By virtue of Theorem 2.1,

$\Gamma(z^0;\bar{x}) = DF(z^0;\bar{x}) \neq \varnothing$. Then, for any vector $\bar{y} \in \Gamma(z^0;\bar{x})$ there exists a function $o(t)$ such that $o(t)/t \to 0$ as $t \downarrow 0$, and $y^0 + t\bar{y} + o(t) \in F(x^0 + t\bar{x})$ for all $t \ge 0$. Therefore,

$$\varphi(x^0 + t\bar{x}) - \varphi(x^0) \le f(x^0 + t\bar{x}, y^0 + t\bar{y} + o(t)) - f(x^0, y^0)$$

and we obtain

$$D^+\varphi(x^0;\bar{x}) = \lim_{t\downarrow 0}\sup t^{-1}\left[\varphi(x^0 + t\bar{x}) - \varphi(x^0)\right] \le \langle\nabla f(z),\bar{z}\rangle$$

for all $\bar{z} = (\bar{x}, \bar{y})$ such that $\bar{y} \in \Gamma(z^0; \bar{x})$. It follows from this inequality that

$$D^+\varphi(x^0; \bar{x}) \leq \inf_{y^0 \in \omega(x^0)} \inf_{\bar{y} \in \Gamma(z^0; \bar{x})} \langle \nabla f(z^0), \bar{z} \rangle. \tag{10}$$

Under the hypotheses of the theorem, $\Gamma^*(z^0; \bar{x}) \neq \varnothing$. Let $\bar{y} \in \Gamma^*(z^0; \bar{x})$ and consider the sequence $t_k \downarrow 0$ such that

$$\lim_{k \to \infty} t_k^{-2}(\varphi(x^0 + t_k \bar{x}) - \varphi(x^0) - t_k \langle \nabla f(z^0), \bar{z} \rangle)$$
$$\liminf_{t \downarrow 0} t^{-2}(\varphi(x^0 + t\bar{x}) - \varphi(x^0) - t\langle \nabla f(z^0), \bar{z} \rangle).$$

Denote $x^k := x^0 + t_k \bar{x}$, $y^k \in \omega(x^k)$, $k = 1, 2, \ldots$ Without any loss of generality one can suppose that the sequence $\{y^k\}$ is convergent. Then, $y^k \to y^0 \in F(x^0)$ due to the closedness of the graph of the multivalued mapping F.

Since $\varphi(x^k) \leq \varphi(x^0) + t_k D^+\varphi(x^0; \bar{x}) + o(t_k)$, the passage to the limit in the equality $f(x^k, y^k) = \varphi(x^k)$ gives $f(x^0, y^0) = \limsup_{k \to \infty} \varphi(x^k) \leq \varphi(x^0)$, and, therefore, $y^0 \in \omega(x^0)$.

Thus all requirements of Lemma 4.3 are satisfied, so due to this lemma we have

$$D_+\varphi(x^0; \bar{x}) = \langle \nabla f(z^0), \bar{z} \rangle) \geq \inf_{y^0 \in \omega(x^0)} \inf_{\bar{y} \in \Gamma(z^0; \bar{x})} \langle \nabla f(z^0), \bar{z} \rangle. \tag{11}$$

Comparing the estimates (10) and (11), we obtain that there exists a finite directional derivative

$$\varphi'(x^0; \bar{x}) = \min_{y^0 \in \omega(x^0)} \min_{\bar{y} \in \Gamma(z^0; \bar{x})} \langle \nabla f(z^0), \bar{z} \rangle.$$

Application of Lemma 4.2 now yields (8).

(2) From the hypotheses and the first assertion it follows that $\Gamma^*(z^0; \bar{x}) \neq \varnothing$ and $\Gamma^2(z^0, \bar{z}; \bar{x}) = \overset{\frown}{D}{}^2 F(z^0, \bar{z}; \bar{x}) \neq \varnothing$ for all $y^0 \in \omega(x^0)$. Moreover, $\omega(x^0, \bar{x}) \neq \varnothing$ for any $\bar{y} \in \Gamma^*(z^0; \bar{x})$. Take arbitrary $y^0 \in \omega(x^0, \bar{x})$, $\bar{y} \in \Gamma^*(z^0; \bar{x})$ and $\bar{v} \in \Gamma^2(z^0, \bar{z}; \bar{x})$. Then, there exists a sequence $s_k \downarrow 0$ such that

$$y^k = y^0 + s_k \bar{y} + s_k^2 \bar{v} + o_v(s_k^2) \in F(x^0 + s_k \bar{x}), \; k = 1, 2, \ldots$$

Hence,

$$\begin{aligned} &\varphi(x^0 + s_k \bar{x}) - \varphi(x^0) - s_k \varphi'(x^0; \bar{x}) \\ &\leq f(x^0 + s_k \bar{x}, y^k) - f(x^0, y^0) - s_k \langle \nabla f(z^0), \bar{z} \rangle) \\ &\leq s_k \langle \nabla f(z^0), \bar{z} \rangle + s_k^2 \Phi(z^0, \bar{z}, \bar{v}) + o(s_k^2). \end{aligned}$$

Therefore,

$$D^2_+\varphi(x^0;\bar{x}) \le \liminf_{k\to\infty} 2s_k^{-2}(\varphi(x^0+s_k\bar{x})-\varphi(x^0)-s_k\varphi'(x^0;\bar{x})) \le 2\Phi(z^0,\bar{z},\bar{v})$$

for all $y^0 \in \omega(x^0,\bar{x})$, $\bar{y} \in \Gamma^*(z^0;\bar{x})$ and $\bar{v} \in \Gamma^2(z^0,\bar{z};\bar{x})$. Thus,

$$D^2_+\varphi(x^0;\bar{x}) \le \inf_{y^0\in\omega(x^0,\bar{x})}\ \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \inf_{\bar{v}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{v}\rangle. \tag{12}$$

On the other hand, due to the third assertion of Lemma 4.3

$$\begin{aligned} D^2_+\varphi(x^0;\bar{x}) &= \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \inf_{\bar{v}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{v}) \\ &= \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \sup_{\lambda\in\Lambda^2(z^0,\bar{x})} \langle(\bar{x},\bar{y}^0),\nabla^2_{zz}L(z^0,\lambda)(\bar{x},\bar{y})\rangle, \end{aligned}$$

where $y^0 \in \omega(x^0,\bar{x})$. Therefore,

$$D^2_+\varphi(x^0;\bar{x}) \ge \inf_{y^0\in\omega(x^0,\bar{x})}\ \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \inf_{\bar{v}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{v}).$$

From the last inequality, Lemma 4.2 and (12), the validity of (9) follows. □

Theorem 4.2 *Let conditions $URMF_{\bar{x}}$ and $SSOSC_{\bar{x}}$ hold at all points $z^0=(x^0,y^0)$ such that $y^0\in\omega(x^0)$. Then, the value function φ has directional derivatives $\varphi'(x^0;\bar{x})$ and $\varphi''(x^0;\bar{x})$ at the point x^0 in the direction $\bar{x}$ and*

$$\begin{aligned} \varphi'(x^0;\bar{x}) &= \min_{y^0\in\omega(x^0)}\ \min_{\bar{y}\in\Gamma(z^0;\bar{x})} \langle\nabla f(z^0),\bar{z}\rangle \\ &= \min_{y^0\in\omega(x^0)}\ \max_{\lambda\in\Lambda(z^0)} \langle\nabla_x L(z^0,\lambda),\bar{x}\rangle, \end{aligned} \tag{13}$$

$$\begin{aligned} \varphi''(x^0;\bar{x}) &= \inf_{y^0\in\omega(x^0,\bar{x})}\ \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \inf_{\bar{v}\in\Gamma^2(z^0,\bar{z};\bar{x})} 2\Phi(z^0,\bar{z},\bar{v}\rangle \\ &= \inf_{y^0\in\omega(x^0,\bar{x})}\ \inf_{\bar{y}\in\Gamma^*(z^0;\bar{x})}\ \sup_{\lambda\in\Lambda^2(z^0,\bar{x})} \langle\bar{z},\nabla^2 L(z^0,\lambda)\bar{z}\rangle. \end{aligned} \tag{14}$$

Proof The equality (13) is valid due to Theorem 4.1. Moreover, from the hypotheses of the theorem, it follows that $\Gamma^*(z^0;\bar{x}) \ne \varnothing$ for all $y^0 \in \omega(x^0)$. Furthermore, $D^2F(z^0,\bar{z};\bar{x}) = \Gamma^2(z^0,\bar{z};\bar{x}) \ne \varnothing$ for any $\bar{y} \in \Gamma^*(z^0;\bar{x})$ by virtue of Theorem 3.1. Take arbitrary $y^0 \in \omega(x^0,\bar{x})$, $\bar{y} \in \Gamma^*(z^0;\bar{x})$ and $\bar{v} \in \Gamma^2(z^0,\bar{z};\bar{x})$. Since $\Gamma^2(z^0,\bar{z};\bar{x}) = D^2F(z^0,\bar{z};\bar{x})$, the inclusion $y^0+t\bar{y}+t^2\bar{v}+o(t^2) \in F(x^0+t\bar{x})$ is valid for all $t \ge 0$ and, hence,

$$\begin{aligned} &\varphi(x^0+t\bar{x})-\varphi(x^0) \\ &\quad \le f(x^0+t\bar{x},y^0+t\bar{y}+t^2\bar{v}+o(t^2))-f(x^0,y^0) \\ &\quad = t\langle\nabla f(z^0),\bar{z}\rangle + t^2\Phi(z^0,\bar{z},\bar{v}\rangle + o(t^2). \end{aligned}$$

Taking into consideration that $y^0 \in \omega(x^0, \bar{x})$ and $\bar{y} \in \Gamma^*(z^0; \bar{x})$, for the derivative $D^{+2}\varphi(x^0; \bar{x}) = \limsup_{t \downarrow 0} \frac{2}{t^2}[\varphi(x^0 + t\bar{x}) - \varphi(x^0) - t\varphi'(x^0; \bar{x})]$ one can obtain the following upper estimate: $D^{+2}\varphi(x^0; \bar{x}) \leq 2\Phi(z^0, \bar{z}, \bar{\nu})$. Then,

$$D^{+2}\varphi(x^0; \bar{x}) \leq \inf_{y^0 \in \omega(x^0, \bar{x})} \inf_{\bar{y} \in \Gamma^*(z^0; \bar{x})} \inf_{\bar{\nu} \in \Gamma^2(z^0, \bar{z}; \bar{x})} 2\Phi(z^0, \bar{z}, \bar{\nu}). \tag{15}$$

Combining (15) and the estimate for $D^2_+\varphi(x^0; \bar{x})$ from Theorem 4.1, we obtain

$$\varphi''(x^0; \bar{x}) = \inf_{y^0 \in \omega(x^0, \bar{x})} \inf_{\bar{y} \in \Gamma^*(z^0; \bar{x})} \inf_{\bar{\nu} \in \Gamma^2(z^0, \bar{z}; \bar{x})} 2\Phi(z^0, \bar{z}, \bar{\nu}).$$

By applying Lemma 4.2 now, we obtain (13). □

Example 4.1 Let $F(x) = \{y \in \mathbb{R}^2 : y_1 + y_2 - x \leq 0, -y_1 - y_2 + x \leq 0, -y_1 \leq 0, -y_2 \leq 0\}$, $f(y) = y_1^2 + y_2^2, x \in \mathbb{R}$, $x^0 = 2, y^0 = (1, 1)^T$. Thus, functions h_i $i \in I = \{1, 2, 3, 4\}$, are given by $h_1(x, y) = y_1 + y_2 - x$, $h_2(x, y) = -y_1 - y_2 + x$, $h_3(x, y) = -y_1$, $h_4(x, y) = -y_2$, and it is not difficult to check that neither $MFCQ$ nor $MF_{\bar{x}}$ hold at the point z^0. On the other hand, for any $\bar{x} \in \mathbb{R}^n$, we have $\Gamma(z^0; \bar{x}) = \{\bar{y} \in \mathbb{R}^2 : \bar{y}_1 + \bar{y}_2 \leq \bar{x}, \bar{y}_1 + \bar{y}_2 \geq \bar{x}\}$, $I^0(z^0; \bar{x}) = \{1, 2\}$. Furthermore, $RMF_{\bar{x}}$ at the point z^0 in the direction $\bar{x}$ holds because

$$rank\{\nabla_y h_i(x^0, y^0),\ i = 1, 2\} = rank \left\{ \begin{pmatrix} 1 \\ 1 \\ -\bar{x} \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ \bar{x} \end{pmatrix} \right\} = const.$$

The condition $SSOSC_{\bar{x}}$ is equivalent to $2(\bar{y}_1^2 + \bar{y}_2^2) > 0$ for all nonzero vectors $\bar{y}$ and, therefore, holds. Thus, due to Theorem 4.1, there exists the derivative $\varphi'(x^0; \bar{x})$. Let's calculate it for the direction $\bar{x} = 1$ using Theorem 4.1. It is easy to see that $\omega(x) = \{(2^{-1}x, 2^{-1}x)^T\}$, hence, $\varphi'(x^0; \bar{x}) = \min\{2(\bar{y}_1 + \bar{y}_2) \,|\, \bar{y}_1 + \bar{y}_2 = 1\} = 2$. The requirements of Theorem 4.2 are also satisfied in this example. So the second order derivative $\varphi''(x^0; \bar{x})$ exists. Let's calculate $\varphi''(x^0; \bar{x})$. Obviously, $\Gamma^*(z^0; \bar{x}) = \Gamma(z^0; \bar{x})$. It is easy to obtain that $\Gamma^2(z^0, \bar{z}; \bar{x}) = \{\bar{\nu} \,|\, \bar{\nu}_1 + \bar{\nu}_2 = 0\}$. Then, due to (14) we have

$$\begin{aligned} \varphi''(x^0; \bar{x}) &= \min_{\bar{y} \in \Gamma^*(z^0; \bar{x})} \min_{\bar{\nu} \in \Gamma^2(z^0, \bar{z}; \bar{x})} \{2(\nu_1 + \nu_2) + 2(\bar{y}_1^2 + \bar{y}_2^2)\} \\ &= \min_{\bar{y} \in \Gamma^*(z^0; \bar{x})} 2(\bar{y}_1^2 + \bar{y}_2^2) = 1. \end{aligned}$$

5 Conclusions

In the present paper, problems of parametric nonlinear programming have been studied. A new *relaxed Mangasarian–Fromovitz condition in the direction* ($RMF_{\bar{x}}$) has been introduced. This condition is a weaker requirement than the well-known

Mangasarian–Fromovitz condition in the direction ($MF_{\bar{x}}$). It has also been proven that, just like $MF_{\bar{x}}$, the new condition ensures directional differentiability of the multivalued mapping F defined by the set of feasible points of the mathematical programming problem, and it permits to calculate derivatives of the multivalued mapping. This allowed us to establish sufficient conditions for directional differentiability of the value functions of nonlinear mathematical problems under weaker requirements than those used traditionally (see [8, 9, 12]) as well as to obtain explicit formulas for calculating directional derivatives of the value functions.

Furthermore, a new *relaxed second-order Mangasarian–Fromovitz condition* ($RMF^2_{\bar{x}}(\bar{z})$) has been introduced, which allows to calculate second-order directional derivatives of the multivalued mapping F, establish new sufficient conditions for second-order differentiability of the value functions and obtain formulas for calculating the second directional derivatives.

The obtained results generalize the known results [8, 9, 12] on directional differentiability of value functions and stability analysis in problems of nonlinear programming.

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