Strong Second Order Necessary Optimality Conditions

LEONID MINCHENKO, ALEXEY LESCHOV
Computer Science Department
Belarusian State University of Informatics and Radioelectronics
Brovki 6, 220013, Minsk
BELARUS
leonidm@insoftgroup.com, http://www.bsuir.by

Abstract: In this paper the so-called strong second-order necessary conditions are considered and their validity are proved under critical regularity conditions.

Key words: Nonlinear programming, Necessary Optimality Conditions, Constraint Qualifications.

Библиотека БГУИР

1 Introduction

Necessary optimality conditions were studied in numerous works [1, 2, 4-18, 23, 24, 26]. They may be first-order or second-order, according to the use of derivatives in its formulation. First-order necessary optimality condition at a given feasible point are usually formulated as the Kuhn-Tucker necessary condition which requires the existence of Lagrange multipliers at a local minimizer. The second-order conditions in addition to the existence of Lagrange multipliers at a given point (such point is called by a stationary point) require the positive semidefiniteness of the Hessian of the Lagrangian function on some cone of critical directions.

Second-order necessary optimality conditions play an important role in the optimization theory. This is explained with that the most part of numerical optimization algorithms reduce to finding stationary points satisfying first-order necessary optimality conditions. As a rule the optimization problems, especially the high dimension problems, have a lot of stationary points and it is necessary to involve second-order necessary optimality conditions to delete not optimal points. The given problem is closely related with the existence of constraint qualifications which provide the validity of second-order necessary optimality conditions.

We consider a mathematical programming problem (*NLP*):

$$f(y) \to \inf ,$$

$$y \in C = \{y \in R^m \mid h_i(y) \le 0, \quad i \in I, \quad h_i(y) = 0, \quad i \in I_0\},$$

where $I = \{1,...,s\}$, $I_0 = \{s+1,...,p\}$, and all functions $f(y)$, $h_i(y)$ $i = 1,...,p$ are twice continuously differentiable.

Denote by $I(y) = \{i \in I \mid h_i(y) = 0\}$ the set of all active indices of inequality type constraints at a point $y \in C$ and introduce the Lagrange function $L(y,\lambda) = f(y) + \langle \lambda, h(y) \rangle$, where $\lambda = (\lambda_1,...,\lambda_p)$, $h = (h_1,...,h_p)$, and the set of Lagrange multipliers at a point $y \in C$

$$\Lambda(y) = \{\lambda \in \mathrm{R}^p \mid \nabla_y L(y,\lambda) = 0, \ \lambda_i \ge 0 \text{ and } \lambda_i h_i(y) = 0, \ i \in I\}.$$

At a point $y \in C$ introduce also the set of abnormal Lagrange multipliers

$$\Lambda_0(y) = \{\lambda \in \mathrm{R}^p \mid \sum_{i=1}^{p} \lambda_i \nabla h_i(y) = 0, \ \lambda_i \ge 0, \ i \in I(y), \ \lambda_i = 0, \ i \in I \setminus I(y)\},$$

the linearized tangent cone

$$\Gamma_C(y) = \{\bar{y} \in R^m \mid \langle \nabla h_i(y), \bar{y} \rangle \le 0, \quad i \in I(y), \quad \langle \nabla h_i(y), \bar{y} \rangle = 0, \quad i \in I_0\}$$

and the cones of critical directions

$$D_C(y) = \{\bar{y} \in \Gamma_C(y) \mid \langle \nabla f(y), \bar{y} \rangle \le 0\},$$

$$S_C(y) = \{\bar{y} \in R^m \mid \langle \nabla h_i(y), \bar{y} \rangle = 0, \quad i \in I_0 \cup I(y)\}.$$

2 Problem Formulation

There are three basic types of second-order necessary optimality conditions for the problem (*NLP*), see, e.g. [15].

ISBN: 978-1-61804-275-0

Definition 1. Let $y^0 \in C$ and $\Lambda(y^0) \neq \varnothing$.

1) We say that the refined second-order necessary optimality condition (RSONC) holds at a point y^0 iff, for every vector $\bar{y} \in D_C(y^0)$ there exists $\lambda \in \Lambda(y^0)$ such that $\langle \bar{y}, \nabla^2_{yy} L(y^0, \lambda)\bar{y} \rangle \geq 0$.

2) We say that the weak second-order necessary optimality condition (WSONC) holds at a point y^0 iff, there exists $\lambda \in \Lambda(y^0)$ such that $\langle \bar{y}, \nabla^2_{yy} L(y^0, \lambda)\bar{y} \rangle \geq 0$ for all vectors $\bar{y} \in S_C(y^0)$.

3) We say that the strong second- order necessary optimality condition (SSONC) holds at a point y^0 iff, for every $\lambda \in \Lambda(y^0)$, there holds $\langle \bar{y}, \nabla^2_{yy} L(y^0, \lambda)\bar{y} \rangle \geq 0$ for all $\bar{y} \in D_C(y^0)$.

The definitions of the strong second-order optimality condition is classical; see, e.g., [11, 12, 26]. Later this condition was studied in [2, 6, 24]. The refined second-order optimality condition was introduced in [17], and subsequently studied in [8, 21] and in other works. The weak second-order optimality condition was studied from theoretical and practical points of view in [1, 6, 14, 15, 23].

However, the necessary optimality conditions are valid only under some additional requirements to the structure of the set C which are called constraint qualifications. The most known constraint qualification for the problem (*NLP*) at a point $y^0 \in C$ is the linear independence of the gradients of active constraints $\nabla h_i(y^0)$, $i \in I(y^0) \cup I_0$ (LICQ). A weaker constraint qualification (MFCQ) was introduced by Mangasarian and Fromovitz [22]. MFCQ requires that at a given point $y^0 \in C$ the vectors $\nabla h_i(y^0)$, $i \in I_0$ are linearly independent and there exists a vector $\bar{y}^0$ such that

$\langle \nabla h_i(y^0),\ \bar{y}^0 \rangle = 0,\ i \in I_0,$

$\langle \nabla h_i(y^0),\ \bar{y}^0 \rangle < 0,\ i \in I(y^0).$

It is known that MFCQ is equivalent to the requirement $\Lambda_0(y^0) = \{0\}$.

The LICQ and MFCQ conditions are the first-order constraint qualifications which guarantee the validity of the Kuhn-Tucker condition at local minimizers in the problem (*NLP*). At the same time LICQ is also a second-order constraint qualification. Since the set of Lagrange multipliers consists of one multiplier λ, the conditions SSONC, RSONC and WSONC coincide in this case. The attraction of weaker constraint qualifications lead to not uniquely defined situation. It is known that the RSONC validity is provided with MFCQ. On the other hand, the counterexamples by Arutyunov [5], Anitescu [4], Baccari and Trad [6] show that MFCQ is not a second-order constraint qualification and can not guarantee the validity of SSONC and WSONC. At the same time, different additional conditions to MFCQ were introduced in [1, 6, 15] to provide the validity of SSONC and WSONC.

The goal of our paper is to generalize the results [2, 24] about strong second-order necessary optimality conditions.

3 Problem Solution

Set $K_C^{\lambda}(y^0) = \{\bar{y} \in R^m \mid \langle \nabla h_i(y^0), \bar{y} \rangle = 0, \quad i \in I_0,$

$\langle \nabla h_i(y^0), y \rangle = 0, i \in I_{\lambda}^{\oplus}(y^0),\ \langle \nabla h_i(y^0), y \rangle \leq 0, i \in I_{\lambda}\ (y^0)\},$

where

$I_{\lambda}^{\oplus}(y^0) = \{i \in I(y^0) \mid \lambda_i > 0\}, \quad I_{\lambda}\ (y^0) = \{i \in I(y^0) \mid \lambda_i = 0\}.$

Note that the cone $K_C^{\lambda}(y^0)$ depends of a multiplier λ and, therefore, depends of the goal function f.

Let $\widehat{D}_C(y^0) = \{\bar{y} \in \Gamma_C(y^0) \mid \langle \nabla f(y_0), \bar{y} \rangle = 0\}$.

Lemma 1. Let $\Lambda(y^0) \neq \varnothing$. Then $K_C^{\lambda}(y^0) = \widehat{D}_C(y^0) = D_C(y^0)$ for every $\lambda \in \Lambda(y^0)$.

Thus, if $\Lambda(y^0) \neq \varnothing$, then the SSONC at the point $y^0 \in C$ is equivalent to the following condition: for every $\lambda \in \Lambda(y^0)$, there holds $\langle \bar{y}, \nabla^2_{yy} L(y^0, \lambda)\bar{y} \rangle \geq 0$ for all $\bar{y} \in K_C^{\lambda}(y^0)$.

In [25] it has been proposed the relaxed Mangasarian-Fromovitz constraint qualification (RMFCQ) which was also studied in [20] (a bit later in [3] RMFCQ was introduced under the name the constant rank of the subspace component condition).

Let $I(y^0) = I^a(y^0) \cup I^+(y^0)$, where

$I^a(y^0) = \{i \in I(y^0) \mid \langle \nabla h_i(y^0), \bar{y} \rangle = 0, \quad \forall \bar{y} \in \Gamma_C(y^0)\},$

$I^+(y^0) = I(y^0) \setminus I^a(y^0).$

It is known [13] that in order to $i \in I(y)$ belongs to the set $I^a(y)$ it is necessary and sufficient that there exists $\lambda \in \Lambda_0(y)$ such as $\lambda_i > 0$.

Definition 2. The relaxed Mangasarian–Fromovitz constraint qualification (RMFCQ) is satisfied at $y^0 \in C$ if the system of vectors

Библиотека БГУИР

$\{\nabla h_i(y),\ \ i \in I_0 \cup I^a(y^0)\}$ has constant rank in a neighbourhood of y^0.

The relaxed Mangasarian-Fromovitz constraint qualification is implied with many constraint qualifications including MFCQ [22], the constant rank constraint qualification (CRCQ) [19] and the relaxed constant rank constraint qualification (RCRCQ) [24].

It is known [2, 24] that SSONC holds at the point of local minimum in the problem (*NLP*), if this point satisfies CRCQ or RCRCQ.

Set

$I_D(y^0) = \{i \in I(y^0) \mid \langle \nabla h_i(y_0), \bar{y}\rangle = 0, \quad \forall \bar{y} \in D_C(y^0)\}$, $I_{\#}(y^0) = I(y^0) \setminus I_D(y^0)$.

The following lemma follows immediately from the definition of $I_D(y^0)$.

Lemma 2. Let $y^0 \in C$. Then there exists a vector $\bar{y}^0 \in \widehat{D}_C(y^0)$ such that $\langle \nabla h_i(y^0), \bar{y}^0\rangle = 0 \quad i \in I_0 \cup I_D(y^0), \quad \langle \nabla h_i(y^0), \bar{y}^0\rangle < 0 \quad i \in I_{\#}(y^0).$

Lemma 3. Assume $\Lambda(y^0) \neq \varnothing$ at $y^0 \in C$. Then $\lambda_i = 0$ for all $i \in I_{\#}(y^0)$ and every $\lambda \in \Lambda(y^0)$.

Proof. If $I_{\#}(y^0) = \varnothing$, the assertion is trivial. Let $I_{\#}(y^0) \neq \varnothing$. Take any $\lambda \in \Lambda(y^0)$. In virtue by Lemma 2 there exists a vector $\bar{y}^0 \in \widehat{D}_C(y^0)$ such that

$$\begin{gathered}\langle \nabla h_i(y^0), \bar{y}^0\rangle = 0,\ i \in I_0 \cup I_D(y^0),\\ \langle \nabla h_i(y^0), \bar{y}^0\rangle < 0,\ i \in I_{\#}(y^0),\end{gathered}$$

and, since $\lambda \in \Lambda(y^0)$, obtain

$$\langle \nabla f(y^0) + \sum_{i \in I_0 \cup I(y^0)} \lambda_i \nabla h_i(y^0), \bar{y}^0\rangle = 0,$$

consequently,

$$\sum_{i \in I_{\#}(y^0)} \lambda_i \langle \nabla h_i(y^0), \bar{y}^0\rangle = 0.$$

This means $\lambda_i = 0$ for all $i \in I_{\#}(y^0)$. □

Corollary 1. Let $\Lambda(y^0) \neq \varnothing$ at $y^0 \in C$. Then $I_{\lambda}^{\oplus}(y^0) \subset I_D(y^0),\ I_{\#}(y^0) \subset I_{\lambda}^{\square}(y^0)$ for all $\lambda \in \Lambda(y^0)$.

Note that in general $I_{\lambda}^{\oplus}(y^0) \neq I_D(y^0)$ and $I_{\#}(y^0) \neq I_{\lambda}^{\square}(y^0)$.

Denote

$$I^{\oplus}(y^0) = \{i \in I(y^0) \mid \exists \lambda \in \Lambda(y^0) \ \textit{such that}\ \lambda_i > 0\} = = \bigcup_{\lambda \in \Lambda(y^0)} I_{\lambda}^{\oplus}(y^0)$$

Lemma 4. Assume $\Lambda(y^0) \neq \varnothing$ at $y^0 \in C$. Then the following assertions hold:

(a) $I^a(y^0) \subset I^{\oplus}(y^0) = I_D(y^0)$;

(b) an index $i \in I(y)$ belongs to the set $I_D(y^0)$ iff there exists $\lambda \in \Lambda(y^0)$ such that $\lambda_i > 0$;

(c) if $I_D(y^0) \neq \varnothing$, then there exists $\lambda \in \Lambda(y^0)$ such that $I_D(y^0) = I_{\lambda}^{\oplus}(y^0)$.

Proof. From Lemma 3 follows $I^{\oplus}(y^0) \subset I_D(y^0)$. Prove that $I^a(y^0) \subset I_D(y^0)$. Really, let $i \in I^a(y^0)$. Then $\langle \nabla h_i(y^0), \bar{y}\rangle = 0$ for all $\bar{y} \in \Gamma_C(y^0)$ and, consequently, for all $\bar{y} \in D_C(y^0)$. In this case $i \in I_D(y^0)$ and, hence, $I^a(y^0) \subset I_D(y^0)$. Thus, $I^{\oplus}(y^0) \cup I^a(y^0) \subset I_D(y^0)$.

On the other hand, let $k \in I_D(y^0)$. Then $\langle \nabla h_k(y^0), \bar{y}\rangle = 0$ and, therefore, $\langle \nabla h_k(y^0), \bar{y}\rangle \geq 0$ for every $\bar{y} \in D_C(y^0)$. In this case due to the Farkash lemma (see, e.g., [27]) there exist numbers $\lambda_0 \geq 0$ and $\lambda_i \ \ i \in I_0 \cup I(y^0)$ such that $\lambda_i \geq 0$ for all $i \in I(y^0)$, where $\lambda_k > 0$, and

$$\lambda_0 \nabla f(y^0) + \sum_{i \in I_0 \cup I(y^0)} \lambda_i \nabla h_i(y^0) = 0.$$

Then either $\lambda_0 = 0$ and, hence, $k \in I^a(y^0)$ and then there exists $\lambda \in \Lambda_0(y^0)$ with $\lambda_k > 0$, or $\lambda_0 > 0$ and for the index $k \in I_D(y^0)$ there exists $\lambda \in \Lambda(y^0)$ with $\lambda_k > 0$. That is, $k \in I^a(y^0) \cup I^{\oplus}(y^0)$ and $I_D(y^0) \subset (I^a(y^0) \cup I^{\oplus}(y^0))$. This means $I_D(y^0) = I^a(y^0) \cup I^{\oplus}(y^0)$ and the assertion (a) is true.

Thus, $i \in I(y)$ belongs to $I_D(y^0)$ iff there exists $\lambda \in \Lambda_0(y^0) \cup \Lambda(y^0)$ such that $\lambda_i > 0$. Suppose that for all $\lambda \in \Lambda(y^0)$ we have $\lambda_i = 0$ for any $i \in I(y^0)$ and at the same time there exists $\lambda \in \Lambda_0(y^0)$ with $\lambda_i > 0$ where $i \in I(y^0)$. However, $\Lambda_0(y^0)$ satisfies the inclusion $\Lambda(y^0) + \Lambda_0(y^0) \subset \Lambda(y^0)$. Then $\Lambda(y^0)$ has at least one element λ with $\lambda_i > 0$. This means that $I^a(y^0) \subset I^{\oplus}(y^0)$ and , consequently, the index $i \in I(y)$ belongs to $I_D(y^0)$ iff there exists

Библиотека БГУИР

ISBN: 978-1-61804-275-0

$\lambda \in \Lambda(y^0)$ such that $\lambda_i > 0$. That is, the assertion (b) holds

If $I_D(y^0) \neq \varnothing$, then due to (b) for every $k \in I_D(y^0)$ there exists $\lambda^{(k)} \in \Lambda(y^0)$ such that

$$\lambda_k^{(k)} > 0 \text{ and } \nabla f(y^0) + \sum_{i \in I_0 \cup I(y^0)} \lambda_i^{(k)} \nabla h_i(y^0) = 0,$$

where $\lambda_i^{(k)} = 0$ for all $i \in I_\#(y^0)$ due to Lemma 3. Denote $\left| I_D(y^0) \right| = q$. Then

$$0 = q\nabla f(y^0) + \sum_{k \in I_D(y^0)} \sum_{i \in I_0 \cup I(y^0)} \lambda_i^{(k)} \nabla h_i(y^0) =$$

$$= q\nabla f(y^0) + \sum_{i \in I_0 \cup I(y^0)} \Big(\sum_{k \in I_D(y^0)} \lambda_i^{(k)} \Big) \nabla h_i(y^0).$$

Set $\lambda = q^{-1} \sum_{k \in I_D(y^0)} \lambda_i^{(k)}$ and obtain from the last equality that

$$\nabla f(y^0) + \sum_{i \in I_0 \cup I(y^0)} \lambda_i^{(k)} \nabla h_i(y^0) = 0,$$

where $\lambda_i > 0$ for all $i \in I_D(y^0)$ and $\lambda_i = 0$ for all $i \in I_\#(y^0)$.

Consider the lower tangent cone to the set C at $y^0 \in C$:

$$T_C(y^0) = \{ \bar{y} \in R^m \mid \exists \text{ a number } t_0 > 0 \text{ such that } y^0 + t\bar{y} + o(t) \in C \quad \forall t \in [0, t_0] \}.$$

Note that $T_C(y^0)$ is a closed cone.

Denote

$$W_C(y^0) = \{ \bar{y} \in R^m \mid \langle \nabla h_i(y^0), \bar{y} \rangle = 0 \ \ i \in I_0 \cup I_D(y^0), \ \langle \nabla h_i(y^0), \bar{y} \rangle \le 0 \ \ i \in I_\#(y^0) \}.$$

From Lemma 2 follows that

$$aff W_C(y^0) = \{ \bar{y} \in R^m \mid \langle \nabla h_i(y^0), \bar{y} \rangle = 0 \ \ i \in I_0 \cup I_D(y^0) \},$$

$$ri W_C(y^0) = \{ \bar{y} \in R^m \mid \langle \nabla h_i(y^0), \bar{y} \rangle = 0 \ \ i \in I_0 \cup I_D(y^0), \ \langle \nabla h_i(y^0), \bar{y} \rangle < 0 \ \ i \in I_\#(y^0) \}.$$

Lemma 5. The following equalities are valid:

(a) $\hat{D}_C(y^0) = W_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$;

(b) $ri\hat{D}_C(y^0) = riW_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$;

(c) if $\Lambda(y^0) \neq \varnothing$, then $W_C(y^0) = \hat{D}_C(y^0)$.

Proof. (a) If $\bar{y} \in \hat{D}_C(y^0)$, then

$\langle \nabla f(y^0), \bar{y} \rangle = 0$,

$\langle \nabla h_i(y^0), \bar{y} \rangle \le 0, \ i \in I(y^0), \ \ \langle \nabla h_i(y^0), \bar{y} \rangle = 0, \ i \in I_0$

Therefore,

$\langle \nabla f(y^0), \bar{y} \rangle = 0$,

$\langle \nabla h_i(y^0), y \rangle = \bar{0} \ \ i \in I_0 \cup I_D(y^0), \ \langle \nabla h_i(y^0), y \rangle \le \bar{0} \ \ i \in I_\#(y^0)$

and, consequently,

$\bar{y} \in W_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$.

On the other hand, if $\bar{y} \in W_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$, then

$\langle \nabla h_i(y^0), y \rangle = \bar{0} \ \ i \in I_0 \cup I_D(y^0), \ \langle \nabla h_i(y^0), y \rangle \le \bar{0} \ \ i \in I_\#(y^0)$

and, hence, the vector $\bar{y}$ satisfies the following conditions

$\langle \nabla f(y^0), \bar{y} \rangle = 0$,

$\langle \nabla h_i(y^0), \bar{y} \rangle \le 0, \ i \in I(y^0), \ \ \langle \nabla h_i(y^0), \bar{y} \rangle = 0, \ i \in I_0$

This means that $\bar{y} \in \hat{D}_C(y^0)$.

(b) Due to Lemma 2 there exists a vector $\bar{y}^0 \in \hat{D}_C(y^0)$ such that

$\langle \nabla f(y^0), \bar{y} \rangle = 0$,

$\langle \nabla h_i(y^0), y^0 \rangle = \bar{0} \ \ i \in I_0 \cup I_D(y^0), \ \langle \nabla h_i(y^0), y^0 \rangle < \bar{0} \ \ i \in I_\#(y^0)$

Therefore, $\bar{y}^0 \in riW_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$.

Thus, $riW_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \} \neq \varnothing$.

Then in virtue by Theorem 6.5 [27] $ri\hat{D}_C(y^0) = riW_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$.

(c) Assume $\lambda \in \Lambda(y^0)$. Then for any $\bar{y} \in W_C(y^0)$. Since due to Lemma 3 $\lambda_i = 0$ for all $i \in I_\#(y^0)$, the last equality can be rewritten in the form

$$\langle \nabla f(y^0) + \sum_{i \in I_0 \cup I_D(y^0)} \lambda_i \nabla h_i(y^0), \bar{y} \rangle = 0.$$

Then, taking into account the definition of $W_C(y^0)$, obtain $\langle \nabla f(y^0), \bar{y} \rangle = 0$. Thus, $\bar{y} \in W_C(y^0) \cap \{ \bar{y} \mid \langle \nabla f(y^0), \bar{y} \rangle = 0 \}$, consequently, $\bar{y} \in \hat{D}_C(y^0)$ and $W_C(y^0) \subset \hat{D}_C(y^0)$. The inclusion $\hat{D}_C(y^0) \subset W_C(y^0)$ follows from (a). □

Definition 3. The critical regularity condition (CRC) holds at $y^0 \in C$ iff $\Lambda(y^0) \neq \varnothing$ and $rank\{ \nabla h_i(y), \ \ i \in I_0 \cup I_D(y^0) \} = const$ in a neighbourhood of y^0.

From Lemma 5 follows this definition can be formulated in the following equivalent form.

Definition 3a. The critical regularity condition (CRC) holds at $y^0 \in C$ iff $\Lambda(y^0) \neq \varnothing$ and

Библиотека БГУИР

$rank\left\{\nabla h_i(y),\quad i\in I_0\cup I^{\oplus}(y^0)\right\}=const$ in a neighbourhood of y^0.

Note that CRC always holds if CRCQ [19] or RCRCQ [24] holds.

Lemma 6. Suppose that critical regularity condition holds at $y^0\in C$. Then $W_C(y^0)\subset T_C(y^0)$ and for every $\bar y\in riW_C(y^0)$ there exist a twice continuously differentiable function $r(t)$ and a number $t_0>0$ such that $r(t)/t\to 0$ as $t\to 0$ and $y(t)=y^0+t\bar y+r(t)\in C$ for all $t\in[0,t_0]$, $h_i(y(t))=0$ $i\in J=I_0\cup I_D(y^0)$ for all $t\in(-t_0,t_0)$.

Proof. Let $\bar y\in riW_C(y^0)$. Denote $J=I^2(y^0,\bar y)\cup I_0$ where $I^2(y^0,\bar y)=\{i\in I(y^0)\mid \langle\nabla h_i(y^0),\bar y\rangle=0\}$. Then for every *m*-vector function *r(t)* such that $r(t)/t\to 0$ as $t\downarrow 0$ there exists a number $t_0>0$ such that $h_i(y^0+t\bar y+r(t))<0$ for all $i\in I\setminus I^2(y^0,\bar y)$ and all $t\in(0,t_0)$.

First of all note that that the rank of the Jacobi matrix for the system of functions $h_i(y^0+t\bar y+r)$ $i\in J$ with respect to (r,t) coincides with the rank of the Jacobi matrix of this system with respect to r. Suppose that the rank of the Jacobi matrix of the system $h_i(y^0+t\bar y+r)$ $i\in J$ with respect to r at the point $(r,t)=(0,0)$ is equel to l. Since for $\bar y\in riW_C(y^0)$ the set $I^2(y^0,\bar y)$ coincides with $I_D(y^0)$, then due to CRC the rank of this matrix is constant in some neighbourhood of the point $(0,0)$. Let $|J|=l+q$. Then (see, e.g. [28], p.504) without loss of generality one can assume that in this neighbourhood l functions of the system (suppose that these are $h_1,...,h_l$) are independent and the others depend of them, that is $h_{l+1}=\phi_1(h_1,...h_l),...,h_{l+q}=\phi_q(h_1,...h_l)$, where $\phi_i(0,...,0)=\phi_i(h_1(y^0),...,h_l(y^0))=0$ for $i=1,...,q$ and $\phi_1,...,\phi_q$ are twice continuously differentiable in some neighbourhood of $(h_1(y^0),...,h_l(y^0))$.

Then in a neighbourhood of $(0,0)$ the system of equations

$h_1(y_0+t\bar y+r)=0$,…., $h_{l+q}(y_0+t\bar y+r)=0$

is equivalent to the system

$h_1(y_0+t\bar y+r)=0$, …., $h_l(y_0+t\bar y+r)=0$.

Then due to the implicit function theorem (see [28], p.488) the given system defines in some neighbourhood of $(0,0)$ an implicit twice continuously differentiable function $r=r(t)$ such that $r(0)=0$, $r'(0)=\lim_{t\to 0}t^{-1}r(t)=0$.

Thus, for every $\bar y\in riW_C(y_0)$ there exist a number $t_0>0$ and twice continuously differentiable function $r=r(t)$ such that $r(t)/t\to 0$ as $t\to 0$, $y(t)=y^0+t\bar y+r(t)\in C$ for all $t\in[0,t_0]$ and $h_i(y(t))=$ ($i\in J=I_0\cup I_D(y^0)$ for all $t\in(-t_0,t_0)$. Then $riW_C(y^0)\subset T_C(y^0)$ and, consequently, $W_C(y^0)\subset T_C(y^0)$. □

The theorem below generalizes the results [2,24].

Theorem 1. Let a point $y^0\in C$ satisfy the critical regularity condition and be a local solution of the problem (*NLP*). Then SSONC holds at this point.

Proof. Consider any multiplier $\lambda\in\Lambda(y^0)$ and take any vector $\bar y\in ri\hat D_C(y^0)$. Due to Lemma 5 $ri\hat D_C(y^0)\subset riW_C(y^0)$. Therefore, according to Lemma 6 there exist a twice continuously differentiable function $r(t)$ and a number $t_0>0$ such that $r(\ t)\to$ as $t\to 0$, $y(t)=y^0+t\bar y+r(t)\in C$ for all $t\in[0,t_0]$ and $h_i(y(t))=0$ $i\in J=I_0\cup I_D(y^0)$ for all $t\in(-t_0,t_0)$. Since $y(t)\in C$ for all $t\in[0,t_0]$ and $y^0\in C$ is the point of local minimum, then

$$f(y(t))-f(y(0))=t\langle\nabla f(y^0),\bar y\rangle+t^2\{\frac{1}{2}\langle\bar y,\nabla^2 f(y^0)\bar y\rangle+$$

$$+\langle\nabla f(y^0),r''(0)\rangle\}+o(t^2)=$$

$$=t^2\{\frac{1}{2}\langle\bar y,\nabla^2 f(y^0)\bar y\rangle+\langle\nabla f(y^0),r''(0)\rangle\}+o(t^2)\geq 0,$$

consequently,

$$\frac{1}{2}\langle\bar y,\nabla^2 f(y^0)\bar y\rangle+\langle\nabla f(y^0),r''(0)\rangle\geq 0 \qquad (1)$$

On the other hand, $I_\lambda^{\oplus}(y^0)\subset I_D(y^0)$ due to Corollary 1 and, therefore, for all $t\in(-t_0,t_0)$ the following identity

$$\sum_{i\in I_0\cup I_D(y^0)}\lambda_i\, h_i(y(t))=\sum_{i=1}^{p}\lambda_i\, h_i(y(t))=0$$

holds. Then

$$0=\sum_{i=1}^{p}\lambda_i\, h_i(y(t))=\sum_{i=1}^{p}\{\lambda_i\, h_i(y^0)+t\langle\lambda_i\nabla h_i(y^0),\bar y\rangle+$$

Библиотека БГУИР

$$+\frac{1}{2}t^2[\langle \bar{y}, \lambda_i \nabla^2 h_i(y^0)\bar{y}\rangle + \langle \lambda_i \nabla h_i(y^0), r''(0)\rangle]\} + o(t^2),$$

consequently,

$$\frac{1}{2}\langle \bar{y}, \sum_{i=1}^{p} \lambda_i \nabla^2 h_i(y^0)\bar{y}\rangle + \langle \sum_{i=1}^{p} \lambda_i \nabla h_i(y^0), r''(0)\rangle = 0 \qquad (2)$$

Combining (1) and (2), obtain

$$\langle \bar{y}, \nabla^2 f(y^0)\bar{y}\rangle + \langle \bar{y}, \sum_{i=1}^{p} \lambda_i \nabla^2 h_i(y^0)\bar{y}\rangle \geq 0,$$

that is

$$\langle \bar{y}, \nabla^2_{yy} L(y^0, \lambda)\bar{y}\rangle \ \geq \ 0$$

for any $\bar{y} \in ri\widehat{D}_C(y^0)$ and, hence, for any $\bar{y} \in \widehat{D}_C(y^0)$.

Since $\widehat{D}_C(y^0) = D_C(y^0) = K_C^{\lambda}(y^0)$ due to Lemma 1, obtain the assertion of the theorem. □

Corollary 2. Let a point $y^0 \in C$ be a local solution of the problem (*NLP*) and satisfy *RCRCQ*. Then SSONC holds at this point.

Corollary 3. Assume that a point $y^0 \in C$ is a local solution of the problem (*NLP*), $\Lambda(y^0) \neq \varnothing$ and the vectors $\nabla h_i(y^0)$ $i \in I_0 \cup I^{\oplus}(y^0)$ are linearly independent. Then SSONC holds at this point.

4 Conclusions

In the paper the notion of the critical regularity has been introduced for the problems of nonlinear programming and the strong second-order necessary optimality conditions were proved. These necessary conditions generalize the second-order necessary conditions [1, 24].

References

[1] R. Andreani, J.M.Martinez, M.L Schuverdt, On second-order optimality conditions for nonlinear programming, Optimization 56 (2007), pp. 529–542.

[2] R. Andreani, C.E.Echagüe, M.L Schuverdt, Constant-rank condition and second-order constraint Qualification, J. Optim. Theory Appl. 146 (2010), pp. 255–266.

[3] R. Andreani, C. Haeser, M.L. Schuverdt, P.J.S. Silva, Two new weak constraint qualifications and applications, SIAM J. Optimization. 22 (2012), pp.1109-1125.

[4] M. Anitescu, Degenerate nonlinear programming with a quadratic growth condition, SIAM J. Optim. 10 (2000), pp. 1116–1135.

[5] A.V. Arutyunov, Perturbations of extremum problems with constraints and necessary optimality conditions, Journal of Soviet Mathematics 54 (1991), pp. 1342-1400.

[6] A. Baccari, A. Trad, On the classical necessary second-order optimality conditions in the presence of equality and inequality constraints, SIAM J. Optimization 15 (2004), pp. 394-408.

[7] D.P. Bertsekas, Nonlinear Programming, 2nd edition, Athena Scientific, Massachusetts, 1999.

[8] J.F. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.

[9] O. Brezhneva, A. Tretyakov, The p-th order optimality conditions for inequality constrained optimization problems, J. Pure Appl. Math., 1 (2010) , pp. 198-223.

[10] S. Dempe. A necessary and a sufficient optimality condition for bilevel programming Problems, Optimization 25(1992), pp. 341–354.

[11] A. V. Fiacco and G.P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques, John Wiley & Sons, New York, 1968.

[12] R. Fletcher, Practical Methods of Optimization, Academic Press, London, 1987.

[13] V.V. Gorokhovik, Finite-dimensional Optimization Problems, BSU Publ., Minsk, 2007.

[14] N.I.M. Gould, P.L. Toint, A note on the convergence of barrier algorithms for second-order necessary Points, Math. Program. 85 (1999), pp. 433–438.

[15] L. Guo, G.H. Lin and J.J. Ye, Second order optimality conditions for mathematical programs with equilibrium constraints. J. Optimiz. Theory. Appl., 158 (2013). 33-64.

[16] X.M. Hu, D. Ralph, Convergence of a penalty method for mathematical programming with equilibrium constraints, J. Optimiz. Theory and Appl. 123 (2004), pp. 365-390.

[17] A.D. Ioffe, Necessary and sufficient conditions for a local minimum III: second order conditions and augmented duality, SIAM J. Control Optimiz. 17 (1979), pp. 266–288.

[18] A.F. Izmailov, M.V. Solodov, An active-set Newton method for mathematical programs with complementarity constraints, SIAM J. Optimiz. 19 (2008), pp. 1003-1027.

Библиотека БГУИР

[19] R. Janin, Direction derivative of the marginal function in nonlinear programming, Mathematical Programming Study 21 (1984) pp. 127–138.
[20] A.Y. Kruger, L. Minchenko, J.V. Outrata, On relaxing the Mangasarain–Fromovitz constaint qualication. Positivity 18 (2013), pp. 171-189.
[21] B. Luderer, L. Minchenko, T. Satsura, Multivalued analysis and nonlinear programming problems with perturbations, Kluwer Acad. Publ., Dordrecht, 2002.
[22] O. L. Mangasarian and S. Fromovitz, The Fritz-John necessary optimality conditions in presence of equality and inequality constraints, J. Mathem. Anal. and Appl. 17 (1967), pp. 37–47.
[23] G.P. McCormick, Second order conditions for constrained minima, SIAM J. Appl.Math. 15 (1967), pp. 641–652.
[24] L. Minchenko, S. Stakhovski, Parametric nonlinear programming problems under relaxed constant rank regularity condition, SIAM J. Optimiz. 21 (2011), pp.314-332.
[25] L. Minchenko, S. Stakhovski, About generalizing the Mangasarian-Fromovitz regularity condition (in Russian). Doklady BGUIR, 8 (2010), pp. 104–109.
[26] J. Nocedal and S.J. Wright, Numerical Optimization, Springer, New York, 1999.
[27] R.T. Rockafellar, Convex Analysis, Prinston University Press, Prinston, 1970.
[28] V.A. Zorich, Mathematical Analysis, P.1. Nauka, Moscow, 1981.

Библиотека БГУИР