Spin 1/2 particle with anomalous magnetic moment in presence of external magnetic field, exact solutions

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Abstract

We examine a generalize Dirac equation for spin 1/2 particle with anomalous magnetic moment in presence of the external uniform magnetic field. After separation of the variables, the problem is reduced to a 4-order ordinary differential equation, which is solved exactly with the use of the factorization method. A generalized formula for Landau energy levels is found. Solutions are expressed in terms of confluent hypergeometric functions.

1 Ordinary Dirac equation in cylindrical coordinates, separation of the variables

We use the known representation for the uniform magnetic field: $\mathbf{A} = \frac{1}{2}\, c\mathbf{B} \times \mathbf{r},\ \mathbf{B} = (0, 0, B)$. After translating to cylindrical coordinates we get

$$A_t = 0 \,, \qquad A_r = 0 \,, \qquad A_z = 0 \,, \qquad A_\phi = -Br^2/2 \,. \tag{1}$$

a non-vanishing component of the electromagnetic tensor is $F_{\phi r} = Br$. We consider the Dirac equation in the magnetic field (1), using the tetrad formalism [1] for cylindrical coordinates $x^\alpha = (t, r, \phi, z)$:

$$dS^2 = dt^2 - dr^2 - r^2\, d\phi^2 - dz^2 \,, \qquad e^{\beta}_{(a)}(x) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{2}$$

Generally covariant tetrad Dirac equation [1] is

$$\left\{ \gamma^c [\, i\hbar\, (\, e^{\beta}_{(c)} \partial_\beta + \frac{1}{2} \sigma^{ab} \gamma_{abc}\,) \; - \; \frac{e}{c} A_c \,] \; - \; mc \right\} \Psi = 0 \,, \tag{3}$$

where γ_{abc} are the Ricci rotation symbols rotation: $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha}\, e^{\beta}_{(a)} e^{\alpha}_{(c)}$, $A_a = e^{\beta}_{(a)} A_\beta$ is the tetrad components of the 4-vector A_β; $\sigma^{ab} = 1/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$ are generators for bispinor representation of the Lorentz group.

We will use the shortening notation: $e/c\hbar \Longrightarrow e,\ mc/\hbar \Longrightarrow M$. The Dirac equation takes the form (let $\Psi = \varphi/\sqrt{r}$):

$$\left[i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \, \left(\frac{i\partial_\phi}{r} + \frac{eBr}{2} \right) + i\gamma^3 \frac{\partial}{\partial z} - M \right] \varphi = 0 \,. \tag{4}$$

We search solutions in the form

$$\varphi = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{vmatrix}, \qquad \left[+\epsilon\gamma^0 + i\gamma^1 \frac{\partial}{\partial r} - \gamma^2 \, \mu(r) - k\gamma^3 - M \right] \begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{vmatrix} = 0 \,,$$

where $\mu(r) = m/r - eBr/2$; further we will use the shortening notation $eB \Longrightarrow B$. When choosing Dirac matrices in the spinor basis, we find equations for the four functions $f_a(t,z)$:

$$(\frac{d}{dr} + \mu) f_4 + ikf_3 + i(\epsilon f_3 - Mf_1) = 0, \; (\frac{d}{dr} - \mu) f_3 - ikf_4 + i(\epsilon f_4 - Mf_2) = 0,$$
$$(\frac{d}{dr} + \mu) f_2 + ikf_1 - i(\epsilon f_1 - Mf_3) = 0, \; (\frac{d}{dr} - \mu) f_1 - ikf_2 - i(\epsilon f_2 - Mf_4) = 0. \qquad (5)$$

The equations are consistent with the linear constraint $f_3 = Af_1, \; f_4 = Af_2$, if the following condition is imposed

$$\epsilon - \frac{M}{A} = -\epsilon + MA \qquad \Longrightarrow \qquad A = A_{1,2} = \frac{\epsilon \pm \sqrt{\epsilon^2 - M^2}}{M} \,. \qquad (6)$$

As a result, the problem is reduced to the system of two equations

$$\left(\frac{d}{dr} + \mu \right) f_2 + i(k - \epsilon + MA) f_1 = 0, \; \left(\frac{d}{dr} - \mu \right) f_1 + i(-k - \epsilon + MA) f_2 = 0. \qquad (7)$$

In accordance with (6), we have two types of states:

$AM = \epsilon + \sqrt{\epsilon^2 - M^2}\,, \qquad (\sqrt{\epsilon^2 - M^2} = p \,)$

$$\left(\frac{d}{dr} + \; \mu \right) \; f_2 + i \; (k + p) \; f_1 = 0 \,, \; \left(\frac{d}{dr} - \; \mu \right) \; f_1 - i \; (k - p) \; f_2 = 0 \,; \qquad (8)$$

$AM = \epsilon - \sqrt{\epsilon^2 - M^2}\,, \qquad (\sqrt{\epsilon^2 - M^2} = p \,)$

$$\left(\frac{d}{dr} + \; \mu \right) \; f_2 + i \; (k - p) \; f_1 = 0 \,, \; \left(\frac{d}{dr} - \; \mu \right) \; f_1 - i \; (k + p) \; f_2 = 0 \,. \qquad (9)$$

For definiteness, we follow the variant (8).

2 Solving equations for r-variable

From (8) we obtain the second order equation for R_1

$$\frac{d^2 R_1}{dr^2} + \left[\frac{m}{r^2} + \frac{B}{2} - \left(\frac{m}{r} - \frac{Br}{2} \right)^2 + \lambda^2 \right] R_1 = 0 \,. \qquad (10)$$

where $\lambda^2 = \epsilon^2 - m^2 - k^2$. Parameter λ^2 describes the contribution of the electron transversal motion to the total energy, this part of the energy is quantized. Note that we diagonalize the operator

$$-i \frac{\partial}{\partial \phi} \Psi = m \; \Psi \;, \qquad (11)$$

which represents the third projection of the total angular momentum of the Dirac particle in cylindrical tetrad basis:

$$\hat{J}_3 \Psi_{Cart} = (-i\frac{\partial}{\partial \phi} + \Sigma_3)\ \Psi_{Cart} = m\ \Psi = m\ \Psi_{Cart} \tag{12}$$

therefore for m are permitted only half-integer values $m = \pm 1/2,\ \pm 3/2, ...$

We turn to eq. (10) and introduce variable $x = Br^2/2$, equation becomes[1]

$$4x\frac{d^2 R_1}{dx^2} + 2\frac{dR_1}{dx} + \left(\frac{m(1-m)}{x} - x + 1 + 2m + \frac{2\lambda^2}{B} \right) R_1 = 0\,. \tag{13}$$

We seek solutions in the form $R_1(x) = x^A e^{-Cx} R(x)$. If A, C are chosen according $A = m/2\,,\ (1-m)/2\,,\ C = \pm 1/2$, the equation for R reads

$$x\frac{d^2 R}{dx^2} + \left(2A + \frac{1}{2} - x\right)\frac{dR}{dx} - \left(A - \frac{m}{2} - \frac{\lambda^2}{2eB}\right) R = 0,$$

which is the confluent hypergeometric equation

$$x\,Y'' + (\gamma - x)Y' - \alpha Y = 0\,, \qquad \alpha = A - \frac{m}{2} - \frac{\lambda^2}{2B}, \qquad \gamma = 2A + \frac{1}{2}\,.$$

To obtain solutions that vanish at the origin $r \to 0$ and infinity $r \to \infty$, we must take positive values for A and C:

$$C = +1/2\,, \qquad A = \left\{ \begin{array}{ll} m = +1/2, +3/2, ..., & A = m/2\,, \\ m = -1/2, -3/2, ..., & A = (1-m)/2\,. \end{array} \right.$$

To obtain polynomials, we impose the know restriction $\alpha = -n,\ n = 0, 1, 2, ...$ This leads to the following rule for quantization of the parameter λ^2:

$$\frac{\lambda^2}{2eB} = A - \frac{m}{2} + n\,.$$

Depending on the sign of the quantum number m we get two formulas for $\lambda^2 = \epsilon^2 - M^2 - k^2$:

$$\begin{array}{lll} m > 0\,, & \lambda^2 = 2eB\ n\,, & n = 0, 1, 2, ...; \\ m < 0\,, & \lambda^2 = 2eB\ \left(n - m + \dfrac{1}{2}\right), & n = 0, 1, 2, ... \end{array} \tag{14}$$

3 Accounting of the anomalous magnetic moment

The Dirac equation for a particle with spin 1/2 with anomalous magnetic moment in the Riemannian space-time (using the tetrad formalism) can be represented in the form [2, 3]

$$\left\{ \gamma^c [\ i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2}\sigma^{ab}\gamma_{abc}\ )\ -\ \frac{e}{\hbar c}A_c\ ] - i\lambda\frac{2e}{Mc^2}\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) - \frac{Mc}{\hbar} \right\} \Psi = 0\,. \tag{15}$$

Dimensions of the parameters in the equation are

$$[\frac{Mc}{\hbar}] = l^{-1}, \quad [\frac{e}{\hbar c}A] = l^{-1}, \quad [\frac{e}{\hbar c}F] = l^{-2}, \quad [\frac{eF}{Mc^2}] = l^{-1}\ ;$$

[1]Without loss of generality, we assume that the parameter B positive.

free parameter λ is dimensionless. Consider this equation in the uniform magnetic field. In view of

$$\sigma^{\alpha\beta}(x)F_{\alpha\beta}(x) = 2\sigma^{\phi r}F_{\phi r} = i\gamma^2\gamma^1 B = iB\Sigma_3$$

instead of (4) we get a more general equation

$$\left[\, i\gamma^0\frac{\partial}{\partial t} + i\gamma^1\frac{\partial}{\partial r} + \gamma^2\ \left(\frac{i\partial_\phi}{r} + \frac{eBr}{2}\right) + i\gamma^3\frac{\partial}{\partial z} + \lambda\frac{2eB}{Mc^2}\Sigma_3 - \frac{Mc}{\hbar}\right]\varphi = 0\,. \tag{16}$$

Substitution for the wave function is the same as above

$$\left[\epsilon\gamma^0 + i\gamma^1\frac{\partial}{\partial r} - \gamma^2\,\mu(r) - k\gamma^3 + \Gamma\Sigma_3 - M\right]\left|\begin{array}{c} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{array}\right| = 0\,, \tag{17}$$

where we use the notation

$$\frac{m}{r} - \frac{eBr}{2} \Longrightarrow \mu(r)\,, \quad \lambda\frac{2eB}{Mc^2} \Longrightarrow \Gamma\,, \qquad \frac{Mc}{\hbar} \Longrightarrow M\,, \quad \frac{\epsilon}{\hbar c} \Longrightarrow \epsilon\,. \tag{18}$$

Further we get four radial equations

$$-i\left(\frac{d}{dr} + \mu\right) f_4 + (\epsilon + k) f_3 + (\Gamma - M) f_1 = 0,$$
$$-i\left(\frac{d}{dr} - \mu\right) f_3 + (\epsilon - k) f_4 - (\Gamma + M) f_2 = 0,$$
$$+i\left(\frac{d}{dr} + \mu\right) f_2 + (\epsilon - k) f_1 + (\Gamma - M) f_3 = 0,$$
$$+i\left(\frac{d}{dr} - \mu\right) f_1 + (\epsilon + k) f_2 - (\Gamma + M) f_4 = 0.$$

Let us try to impose a linear constraint (see the case of ordinary Dirac particle): $f_3 = Af_1,\ f_4 = Af_2$, equations take the form

$$-i\left(\frac{d}{dr} + \mu\right) f_2 + \left[\epsilon + k + \frac{(\Gamma - M)}{A}\right] f_1 = 0\,,$$
$$-i\left(\frac{d}{dr} - \mu\right) f_1 + \left[\epsilon - k - \frac{(\Gamma + M)}{A}\right] f_2 = 0\,,$$
$$+i\left(\frac{d}{dr} + \mu\right) f_2 + \left[\epsilon - k + (\Gamma - M)A\right] f_1 = 0\,,$$
$$+i\left(\frac{d}{dr} - \mu\right) f_1 + \left[\epsilon + k - (\Gamma + M)A\right] f_2 = 0\,. \tag{19}$$

In the system (19) equations 1 and 3, as well as equations 2 and 4 are the same, only if the ratio

$$\epsilon + k + \frac{(\Gamma - M)}{A} = -\left[\epsilon - k + (\Gamma - M)A\right],$$
$$\epsilon - k - \frac{(\Gamma + M)}{A} = -\left[\epsilon + k - (\Gamma + M)A\right];$$

they can be rewritten as:

$$\epsilon + \frac{(\Gamma - M)}{A} = -\epsilon - (\Gamma - M)A \quad \Longrightarrow \quad 2\epsilon = (M - \Gamma)(A + \frac{1}{A})\,,$$
$$\epsilon - \frac{(\Gamma + M)}{A} = -\epsilon + (\Gamma + M)A \quad \Longrightarrow \quad 2\epsilon = (M + \Gamma)(A + \frac{1}{A})\,. \tag{20}$$

Obviously, this system is not consistent. Thus, we are to examine the system of four equation in a different way.

4 Solving the radial equations

We turn again to the system of four equations. They can be represented in the form of two linear systems

$$(\Gamma - M)f_1 + (\epsilon + k)f_3 = i\left(\frac{d}{dr} + \mu\right) f_4 = iD_+ f_4 \, ,$$
$$+(\epsilon - k)f_1 + (\Gamma - M)f_3 = -i\left(\frac{d}{dr} + \mu\right) f_2 = -iD_+ f_2 \, ; \tag{21}$$
$$-(\Gamma + M)f_2 + (\epsilon - k)f_4 = i\left(\frac{d}{dr} - \mu\right) f_3 = iD_- f_3 \, ,$$
$$+(\epsilon + k)f_2 - (\Gamma + M)f_4 = -i\left(\frac{d}{dr} - \mu\right) f_1 = -iD_- f_1 \, . \tag{22}$$

Their solutions are as follows:

$$f_1 = +i\frac{(\epsilon + k)D_+ f_2 + (\Gamma - M)D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}, \quad f_3 = -i\frac{(\Gamma - M)D_+ f_2 + (\epsilon - k)D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}; \tag{23}$$

$$f_2 = +i\frac{(\epsilon - k)D_- f_1 - (\Gamma + M)D_- f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}, \quad f_4 = -i\frac{-(\Gamma + M)D_- f_1 + (\epsilon + k)D_- f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}. \tag{24}$$

We note two identities

$$D_+ D_- = (\frac{d}{dr} + \mu)(\frac{d}{dr} - \mu) = \frac{d^2}{dr^2} - \mu' - \mu^2 \, ,$$
$$D_- D_+ = (\frac{d}{dr} - \mu)(\frac{d}{dr} + \mu) = \frac{d^2}{dr^2} + \mu' - \mu^2 \, . \tag{25}$$

Substituting expression (23) into equation (22), we get

$$-(\Gamma + M)f_2 + (\epsilon - k)f_4 = \frac{(\Gamma - M)D_- D_+ f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\epsilon - k)D_- D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \, ,$$
$$+(\epsilon + k)f_2 - (\Gamma + M)f_4 = \frac{(\epsilon + k)D_- D_+ f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\Gamma - M)D_- D_+ f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \, . \tag{26}$$

By combining equations (26), we obtain

$$f_2 = \frac{1}{2\Gamma\,(\epsilon + k)}\left(\Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} + \mu' - \mu^2\right) f_4 \, , \tag{27}$$

and then we derive the 4-th order equation for the function f_4

$$-\frac{d^4 f_4}{dr^4} + \left[\frac{B^2}{2}\, r^2 - B\,(2\,m - 1) - 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2) + 2\,\frac{m\,(m+1)}{r^2}\right]\frac{d^2 f_4}{dr^2} +$$

$$+\left[B^2 r - 4\,\frac{m\,(m+1)}{r^3}\right]\frac{df_4}{dr} +$$

$$+\left[-\frac{B^4}{16}\, r^4 + \frac{B^2}{4}\Big(B\,(2m-1) + 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2)\Big)\, r^2 - \right.$$

$$-B\,(2m-1)\,(\Gamma^2 - M^2 - k^2 + \epsilon^2) - (\Gamma^2 + M^2 + k^2 - \epsilon^2)^2 + 4\,\Gamma^2 M^2 - \frac{B^2}{4}\,\Big(6\,m^2 - 2\,m - 1\Big) +$$

$$+\frac{m\,(m+1)\,(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2))}{r^2}-\frac{m\,(m-2)\,(m+3)\,(m+1)}{r^4}\Bigg]\,f_4=0\,. \tag{28}$$

Similarly, we can find the expression for f_4

$$f_4=\frac{1}{2\Gamma\,(\epsilon-k)}\left(\Gamma^2-M^2-k^2+\epsilon^2+\frac{d^2}{dr^2}+\mu'-\mu^2\right)f_2\,, \tag{29}$$

and then derive the 4-th order equation for the function f_2:

$$\frac{d^4 f_2}{dr^4}+\left[-\frac{B^2}{2}\,r^2+B\,(2\,m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2)-2\,\frac{m\,(m+1)}{r^2}\right]\frac{d^2 f_2}{dr^2}+$$
$$+\left[-B^2 r+4\,\frac{m\,(m+1)}{r^3}\right]\frac{df_2}{dr}+$$
$$+\left[\frac{B^4}{16}\,r^4-\frac{B^2}{4}\Big(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2)\Big)\,r^2+\right.$$
$$+B\,(2m-1)\,(\Gamma^2-M^2-k^2+\epsilon^2)+(\Gamma^2+M^2+k^2-\epsilon^2)^2-4\,\Gamma^2 M^2+\frac{e^2B^2}{4}\,\Big(6\,m^2-2\,m-1\Big)-$$
$$\left.-\frac{m\,(m+1)\,(B\,(2m-1)+2\,(\Gamma^2-M^2-k^2+\epsilon^2))}{r^2}+\frac{m\,(m-2)\,(m+3)\,(m+1)}{r^4}\right]f_2=0\,. \tag{30}$$

We note that the equation for f_2 and f_4 are the same. Therefore it is sufficient to consider one of them. To study the arising equations (28) and (30) we will use the factorization method:

$$\hat{F}_4(r)\;f(r)=\hat{f}_2(r)\hat{g}_2(r)\;f(r)=0\,,$$

$$\hat{f}_2(r)=\frac{d^2}{dr^2}+P_0r^2+P_1+\frac{P_2}{r^2}\,,\quad \hat{g}_2(r)=\frac{d^2}{dr^2}+Q_0r^2+Q_1+\frac{Q_2}{r^2}\,. \tag{31}$$

Calculating the operator $\hat{F}_4$ and comparing with (30), we find two sets of numerical coefficients:

1) $$P_0=-\frac{1}{4}\,B^2\,,\quad P_2=-m\,(m+1)\,,$$

$$P_1=B\,\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2+2\,\Gamma\sqrt{\epsilon^2-k^2}\,,$$

$$Q_0=-\frac{1}{4}\,B^2\,,\quad Q_2=-m\,(m+1)\,,$$

$$Q_1=B\,\left(m-\frac{1}{2}\right)+\Gamma^2-M^2-k^2+\epsilon^2-2\,\Gamma\sqrt{\epsilon^2-k^2}\,;$$

and

2) $$P_0=-\frac{1}{4}\,B^2\,,\quad P_2=-m\,(m+1)\,,$$

$$P_1 = B\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 - 2\,\Gamma\sqrt{\epsilon^2 - k^2}\,,$$

$$Q_0 = -\frac{1}{4}\,B^2\,, \quad Q_2 = -m\,(m+1)\,,$$

$$Q_1 = B\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 + 2\,\Gamma\sqrt{\epsilon^2 - k^2}\,.$$

Thus, we are to solve two equations (they differ only in the sign of Γ)

$$\left(\frac{d^2}{dr^2} - \frac{B^2r^2}{4} + B\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 + 2\,\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m\,(m+1)}{r^2}\right) f = 0\,, \tag{32}$$

$$\left(\frac{d^2}{dr^2} - \frac{B^2r^2}{4} + B\left(m - \frac{1}{2}\right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 - 2\,\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m\,(m+1)}{r^2}\right) g = 0\,. \tag{33}$$

Consider the first equation (32). We turn it to the variable $x = Br^2/2$:

$$x\,\frac{d^2 f}{dx^2} + \frac{1}{2}\,\frac{df}{dx} +$$

$$+\left[-\frac{x}{4} + \frac{4\,\Gamma\sqrt{\epsilon^2 - k^2} + (2\,m-1)\,B + 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B} - \frac{1}{4}\,\frac{m(m+1)}{x}\right] f = 0\,.$$

We will build solutions in the form:

$$f = x^a e^{bx} F\,,$$

$$x\,\frac{d^2 F}{dx^2} + \left(\frac{1}{2} + 2a + 2bx\right)\frac{dF}{dx} + \left[\left(b^2 - \frac{1}{4}\right)x + \right.$$

$$+\frac{2\,Bb(4\,a+1) + 4\,\Gamma\sqrt{\epsilon^2 - k^2} + (2\,m-1)\,B + 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B} +$$

$$\left. + \frac{1}{4}\,\frac{(2a+m)\,(2a-m-1)}{x}\right] F = 0\,.$$

If a, b are chosen as

$$a = -\frac{m}{2},\ \frac{1}{2} + \frac{m}{2}, \quad b = -\frac{1}{2}\,, \tag{34}$$

the equation is simplified

$$x\,\frac{d^2 F}{dx^2} + \left(\frac{1}{2} + 2a - x\right)\frac{dF}{dx} -$$

$$-\frac{B(4\,a+1) - 4\,\Gamma\sqrt{\epsilon^2 - k^2} - (2\,m-1)\,B - 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B}\,F = 0\,,$$

and it is the equation for a confluent hypergeometric function with parameters:

$$\alpha = \frac{B(4\,a+1) - 4\,\Gamma\sqrt{\epsilon^2 - k^2} - (2\,m-1)\,eB - 2\,(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B}\,,\ \gamma = \frac{1}{2} + 2a\,. \tag{35}$$

To get solutions that meet the bound states, we should use the positive value of the parameter a and the negative values of the parameter b (for definiteness assume that $B > 0$):

$$a = -\frac{m}{2}, \quad (m < 0)\;; \qquad a = \frac{m}{2} + \frac{1}{2} > 0 \quad (m \geq 0)\,. \tag{36}$$

Conditions of terminating the hypergeometric series to polynomials $\alpha = -n$ (introduce the notation $\epsilon^2 - k^2 = \lambda$):

$$\frac{B(4\,a+1) - 4\,\Gamma\sqrt{\lambda} - (2\,m-1)\,B - 2\,(\Gamma^2 - M^2) - 2\lambda}{4B} = -n \tag{37}$$

gives quantization rule for the energy values:

$$a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} + n = \frac{\Gamma\sqrt{\lambda}}{B} + \frac{\lambda}{2B}\,,$$

we obtain

$$(\sqrt{\lambda} + \Gamma)^2 = N,\; N = M^2 + 2B(a + \frac{1}{2} - \frac{m}{2} + n) \implies \lambda = (\sqrt{N} - \Gamma)^2 > 0\,. \tag{38}$$

From (38) we find the formula for the allowed values of λ

$$\epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(a + \frac{1}{2} - \frac{m}{2} + n)} - \Gamma\right)^2. \tag{39}$$

Depending on the sign of m, we obtain two formulas:

$$m < 0, \quad a = -\frac{m}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(\frac{1}{2} - m + n)} - \Gamma\right)^2; \tag{40}$$

$$m \geq 0, \quad a = \frac{m}{2} + \frac{1}{2}, \quad \epsilon^2 - k^2 = \left(\sqrt{M^2 + 2B(1+n)} - \Gamma\right)^2. \tag{41}$$

This, we have two possibilities for quantization

$$I, \quad \lambda = (\sqrt{N} - \Gamma)^2\,, \qquad II, \quad \lambda = (\sqrt{N} + \Gamma)^2\,. \tag{42}$$

that is the particle with anomalous magnetic moment has two series of energy levels, formally differing in sign of the parameter Γ.

5 Further analysis of solutions

Let us consider the function $f_2(r)$ as the primary. Obtained above ratios allow us to calculate other three functions. We start from the explicit form of the function f_2, the solution of equation (32):

$$f_2 = x^a e^{-x/2} F(\alpha, \gamma, x)\,, \tag{43}$$

where the parameters are given by

$$\alpha = a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = \frac{1}{2} + 2a\,. \tag{44}$$

The function f_4 can be found according to the following relationship:

$$f_4 = \frac{1}{2\Gamma\,(\epsilon - k)} \left(\Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} - \frac{m(m+1)}{r^2} + \frac{B(2m-1)}{2} - \frac{B^2}{4} r^2 \right) f_2 \,;$$

from where after translating to the variable x we find

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + \frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} - \frac{m(m+1)}{4x} - \frac{1}{4}x \right) f_2 \,. \qquad (45)$$

Given the identities

$$+\frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} = n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B}$$

the above relation can be written as

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{1}{4}x \right) f_2 \,. \qquad (46)$$

Depending on the values of the parameter m, we have two different cases:

$$A) \quad m < 0, \qquad a = -\frac{m}{2}, \quad f_2 = x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x)\,,$$

$$\alpha = m + \frac{1}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = -m + \frac{1}{2}\,,$$

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n - \frac{m}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2\,,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(\frac{1}{2} - m + n)} - \Gamma\,; \qquad (47)$$

$$B) \quad m \geq 0, \qquad a = \frac{m}{2} + \frac{1}{2}, \quad f_2 = x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x)\,,$$

$$\alpha = 1 + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \qquad \gamma = m + \frac{3}{2}\,,$$

$$f_4 = \frac{2B}{2\Gamma\,(\epsilon - k)} \left(x\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx} + n + \frac{m+1}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2\,,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(1+n)} - \Gamma\,. \qquad (48)$$

Using the relations (47) and (48) we find the explicit expression for the function f_4:
for the variant A,

$$m < 0, \qquad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k}\, x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k}\, f_2(x)\,, \qquad (49)$$

for the variant B,

$$m \geq 0, \qquad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k}\, x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k} f_2(x)\,. \qquad (50)$$

Changing in these formulas the parameter Γ on $-\Gamma$, we obtain the relations describing the second series of states. It is not difficult to calculate the explicit form of the other two functions f_1, f_3; corresponding calculated are made, we will not detail on this.

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