

# Spin 1/2 Particle with Anomalous Magnetic Moment in a Uniform Magnetic Field

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We examine a generalized Dirac equation for a spin 1/2 particle with anomalous magnetic moment in presence of an external uniform magnetic field. After separation of variables, the problem is reduced to a fourth order ordinary differential equation, which is solved exactly with the use of the factorization method. Generalized formula for the Landau energy levels is derived. Solutions are expressed in terms of confluent hypergeometric functions. Restriction to the case of an uncharged spin 1/2 particle with anomalous magnetic moment is performed.

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## 1. Introduction

Commonly, only simplest wave equations for fundamental particles of spin 0, 1/2, 1 are used. Meanwhile, it is known that other and more complicated equations can be proposed which are based on the use of the extended sets of the Lorentz group representations (see [1–16]).

Such generalized wave equations are used to describe more complex objects which have in addition to mass, spin, and electric charge other electromagnetic characteristics like polarizability or anomalous magnetic moment. These additional characteristics manifest themselves in the presence of external electromagnetic fields. In particular, within such an approach Petras proposed a 20-component theory for spin 1/2 particle which after elimination of 16 subsidiary components turns to be equivalent to the Dirac particle theory modified by the presence of the Pauli interaction term. In other words, this theory describes a spin 1/2 particle with

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anomalous magnetic moment.

In present paper, we investigate solutions of such a wave equation in the presence of an external uniform magnetic field. Generalized formulas for the Landau energy levels are derived, and corresponding wave functions are constructed. Restriction to the case of the neutron (uncharged spin 1/2 particle with anomalous magnetic moment) is performed.

## 2. Dirac equation in cylindrical coordinates, separation of variables

We use the known representation for the vector-potential of a uniform magnetic field:  $\mathbf{A} = \frac{1}{2} c\mathbf{B} \times \mathbf{r}$ ,  $\mathbf{B} = (0, 0, B)$ . After transformation to cylindrical coordinates we get

$$A_t = 0, A_r = 0, A_z = 0, A_\phi = -Br^2/2; \quad (2.1)$$

a non-vanishing component of the electromagnetic tensor is  $F_{\phi r} = Br$ . We consider the Dirac equation in the magnetic field (2.1), using the tetrad formalism [17, 18] for the cylindrical coordinates  $x^\alpha = (t, r, \phi, z)$ :

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad e_{(a)}^\beta(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.2)$$

The generally covariant tetrad Dirac equation [18] reads

$$\left\{ \gamma^c [i\hbar(e_{(c)}^\beta \partial_\beta + \frac{1}{2}\sigma^{ab}\gamma_{abc}) - \frac{e}{c}A_c] - mc \right\} \Psi = 0, \quad (2.3)$$

where  $\gamma_{abc}$  are the Ricci rotation coefficients:  $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha}e_{(a)}^\beta e_{(c)}^\alpha$ ,  $A_a = e_{(a)}^\beta A_\beta$  are the tetrad components of the 4-vector  $A_\beta$ ;  $\sigma^{ab} = 1/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$  are generators for the bispinor representation of the Lorentz group.

We use the shortening notation:  $e/c\hbar \Rightarrow e$ ,  $mc/\hbar \Rightarrow M$ . The Dirac equation takes the form (with  $\Psi = \varphi/\sqrt{r}$ ):

$$\left[ i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \left( \frac{i\partial_\phi}{r} + \frac{eBr}{2} \right) + i\gamma^3 \frac{\partial}{\partial z} - M \right] \varphi = 0. \quad (2.4)$$

We search solutions in the form

$$\varphi = e^{-iet} e^{im\phi} e^{ikz} \begin{pmatrix} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{pmatrix},$$

$$\left[ +\epsilon\gamma^0 + i\gamma^1 \frac{\partial}{\partial r} - \gamma^2 \mu(r) - k\gamma^3 - M \right] \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = 0$$

where  $\mu(r) = m/r - eBr/2$ ; further we use the shortening notation  $eB \Rightarrow B$ . When choosing the Dirac matrices in the spinor basis, we find the equations for four functions  $f_a(t, z)$ :

$$\begin{aligned} \left( \frac{d}{dr} + \mu \right) f_4 + ikf_3 + i(\epsilon f_3 - Mf_1) &= 0, \\ \left( \frac{d}{dr} - \mu \right) f_3 - ikf_4 + i(\epsilon f_4 - Mf_2) &= 0, \\ \left( \frac{d}{dr} + \mu \right) f_2 + ikf_1 - i(\epsilon f_1 - Mf_3) &= 0, \\ \left( \frac{d}{dr} - \mu \right) f_1 - ikf_2 - i(\epsilon f_2 - Mf_4) &= 0. \end{aligned} \quad (2.5)$$

These equations are consistent with the linear constraints  $f_3 = Af_1$ ,  $f_4 = Af_2$ , if the following condition is imposed

$$\begin{aligned} \epsilon - \frac{M}{A} &= -\epsilon + MA \Rightarrow \\ A = A_{1,2} &= \frac{\epsilon \pm \sqrt{\epsilon^2 - M^2}}{M}. \end{aligned} \quad (2.6)$$

As a result, the problem is reduced to the system of two equations

$$\begin{aligned} \left(\frac{d}{dr} + \mu\right) f_2 + i(k - \epsilon + MA)f_1 &= 0, \\ \left(\frac{d}{dr} - \mu\right) f_1 + i(-k - \epsilon + MA)f_2 &= 0. \end{aligned} \quad (2.7)$$

In accordance with (2.6), we have two types of states:

$$AM = \epsilon + \sqrt{\epsilon^2 - M^2}, \quad (\sqrt{\epsilon^2 - M^2} = p)$$

$$\begin{aligned} \left(\frac{d}{dr} + \mu\right) f_2 + i(k + p) f_1 &= 0, \\ \left(\frac{d}{dr} - \mu\right) f_1 - i(k - p) f_2 &= 0; \end{aligned} \quad (2.8)$$

$$AM = \epsilon - \sqrt{\epsilon^2 - M^2}, \quad (\sqrt{\epsilon^2 - M^2} = p)$$

$$\begin{aligned} \left(\frac{d}{dr} + \mu\right) f_2 + i(k - p) f_1 &= 0, \\ \left(\frac{d}{dr} - \mu\right) f_1 - i(k + p) f_2 &= 0. \end{aligned} \quad (2.9)$$

For definiteness, we follow the variant (2.8).

### 3. Solving the equation in $r$ -variable

From (2.8) we obtain the second order equation for  $R_1$

$$\frac{d^2 R_1}{dr^2} + \left[ \frac{m}{r^2} + \frac{B}{2} - \left( \frac{m}{r} - \frac{Br}{2} \right)^2 + \lambda^2 \right] R_1 = 0 \quad (3.1)$$

where  $\lambda^2 = \epsilon^2 - m^2 - k^2$ . Parameter  $\lambda^2$  describes the contribution of the electron transversal motion to the total energy, this part of the energy is quantized. Note that we have diagonalized the operator

$$-i \frac{\partial}{\partial \phi} \Psi = m \Psi, \quad (3.2)$$

which represents the third projection of the total angular momentum of the Dirac particle in cylindrical tetrad basis:

$$\left(-i \frac{\partial}{\partial \phi} + \Sigma_3\right) \Psi_{Cart} = m \Psi_{Cart} \quad (3.3)$$

$\Sigma_3$  is the third projection of the spin; for  $m$  only half-integer values  $m = \pm 1/2, \pm 3/2, \dots$  are permitted.

We turn to eq. (3.1) and introduce the variable  $x = Br^2/2$ , the equation becomes (without loss of generality, we assume that the parameter  $B$  is positive)

$$\begin{aligned} 4x \frac{d^2 R_1}{dx^2} + 2 \frac{dR_1}{dx} \\ + \left( \frac{m(1-m)}{x} - x + 1 + 2m + \frac{2\lambda^2}{B} \right) R_1 = 0. \end{aligned} \quad (3.4)$$

We seek solutions in the form  $R_1(x) = x^A e^{-Cx} R(x)$ . If  $A, C$  are chosen according to  $A = m/2, (1-m)/2, C = \pm 1/2$ , the equation for  $R$  reads

$$\begin{aligned} x \frac{d^2 R}{dx^2} + \left( 2A + \frac{1}{2} - x \right) \frac{dR}{dx} \\ - \left( A - \frac{m}{2} - \frac{\lambda^2}{2eB} \right) R = 0, \end{aligned}$$

which is the confluent hypergeometric equation

$$\begin{aligned} x Y'' + (\gamma - x) Y' - \alpha Y = 0, \\ \alpha = A - \frac{m}{2} - \frac{\lambda^2}{2B}, \quad \gamma = 2A + \frac{1}{2}. \end{aligned}$$

To obtain solutions which vanish at the origin  $r \rightarrow 0$  and in the infinity  $r \rightarrow \infty$ , we must take  $C = +1/2$  and positive values for  $A$

$$A = \begin{cases} m = +1/2, +3/2, \dots, & A = m/2, \\ m = -1/2, -3/2, \dots, & A = (1-m)/2. \end{cases}$$

To obtain polynomials, we impose the known restriction  $\alpha = -n, n = 0, 1, 2, \dots$ . This

leads to the following quantization rule for the parameter  $\lambda^2$ :

$$\frac{\lambda^2}{2eB} = A - \frac{m}{2} + n.$$

Depending on the sign of the quantum number  $m$  we get two formulas for  $\lambda^2 = \epsilon^2 - M^2 - k^2$ :

$$\begin{aligned} m > 0, \quad \lambda^2 &= 2eB n, \quad n = 0, 1, 2, \dots; \\ m < 0, \quad \lambda^2 &= 2eB \left( n - m + \frac{1}{2} \right). \end{aligned} \quad (3.5)$$

#### 4. Accounting of the anomalous magnetic moment

The Dirac equation for a particle with spin 1/2 and anomalous magnetic moment in the Riemannian space-time (using the tetrad formalism) can be presented in the form [12, 13]

$$\left\{ \gamma^c \left[ i(e_{(c)}^\beta \partial_\beta + \frac{1}{2} \sigma^{ab} \gamma_{abc}) - \frac{e}{\hbar c} A_c \right] - i\lambda \frac{2e}{Mc^2} \sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) - \frac{Mc}{\hbar} \right\} \Psi = 0. \quad (4.1)$$

Physical dimensions of the parameters in the equation are as follows

$$\left[ \frac{Mc}{\hbar} \right] = l^{-1}, \quad \left[ \frac{e}{\hbar c} A \right] = l^{-1},$$

$$\left[ \frac{e}{\hbar c} F \right] = l^{-2}, \quad \left[ \frac{eF}{Mc^2} \right] = l^{-1};$$

so that the free parameter  $\lambda$  is dimensionless. Consider this equation in the above used magnetic field. In view of identities

$$\sigma^{\alpha\beta}(x) F_{\alpha\beta}(x) = 2\sigma^{\phi r} F_{\phi r} = i\gamma^2 \gamma^1 B = iB\Sigma_3$$

instead of (2.4) we get more general equation

$$\left[ i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \left( \frac{i\partial_\phi}{r} + \frac{eBr}{2} \right) + i\gamma^3 \frac{\partial}{\partial z} + \lambda \frac{2eB}{Mc^2} \Sigma_3 - \frac{Mc}{\hbar} \right] \varphi = 0. \quad (4.2)$$

Substitution for the wave function is the same as above

$$\left[ \epsilon\gamma^0 + i\gamma^1 \frac{\partial}{\partial r} - \gamma^2 \mu(r) - k\gamma^3 + \Gamma\Sigma_3 - M \right] \begin{pmatrix} f_1(r) \\ f_2(r) \\ f_3(r) \\ f_4(r) \end{pmatrix} = 0 \quad (4.3)$$

where we use the notation

$$\begin{aligned} \frac{m}{r} - \frac{eBr}{2} &\Rightarrow \mu(r), \quad \lambda \frac{2eB}{Mc^2} \Rightarrow \Gamma, \\ \frac{Mc}{\hbar} &\Rightarrow M, \quad \frac{\epsilon}{\hbar c} \Rightarrow \epsilon. \end{aligned} \quad (4.4)$$

Further we get four radial equations

$$\begin{aligned} -i \left( \frac{d}{dr} + \mu \right) f_4 + (\epsilon + k) f_3 + (\Gamma - M) f_1 &= 0, \\ -i \left( \frac{d}{dr} - \mu \right) f_3 + (\epsilon - k) f_4 - (\Gamma + M) f_2 &= 0, \\ +i \left( \frac{d}{dr} + \mu \right) f_2 + (\epsilon - k) f_1 + (\Gamma - M) f_3 &= 0, \\ +i \left( \frac{d}{dr} - \mu \right) f_1 + (\epsilon + k) f_2 - (\Gamma + M) f_4 &= 0. \end{aligned}$$

Let us try to impose linear constraints (see the case of the ordinary Dirac particle):  $f_3 = Af_1$ ,  $f_4 = Af_2$ , then the equations take the form

$$\begin{aligned} -i \left( \frac{d}{dr} + \mu \right) f_2 + \left[ \epsilon + k + \frac{(\Gamma - M)}{A} \right] f_1 &= 0, \\ -i \left( \frac{d}{dr} - \mu \right) f_1 + \left[ \epsilon - k - \frac{(\Gamma + M)}{A} \right] f_2 &= 0, \\ +i \left( \frac{d}{dr} + \mu \right) f_2 + [\epsilon - k + (\Gamma - M)A] f_1 &= 0, \\ +i \left( \frac{d}{dr} - \mu \right) f_1 + [\epsilon + k - (\Gamma + M)A] f_2 &= 0. \end{aligned}$$

In the system (4.5), the equations 1 and 3, as well as equations 2 and 4 will be the same, only if the following identities are valid

$$\epsilon + k + \frac{(\Gamma - M)}{A} = -[\epsilon - k + (\Gamma - M)A],$$

$$\epsilon - k - \frac{(\Gamma + M)}{A} = -[\epsilon + k - (\Gamma + M)A];$$

Thus, we need to examine the system of four equations in a different way.

so that they can be rewritten as:

$$\begin{aligned} 2\epsilon &= (M - \Gamma)\left(A + \frac{1}{A}\right), \\ 2\epsilon &= (M + \Gamma)\left(A + \frac{1}{A}\right). \end{aligned} \quad (4.5)$$

Obviously, this system is not consistent.

## 5. Solving the radial equations

These equations can be represented in the form of two linear sub-systems

$$\begin{aligned} (\Gamma - M)f_1 + (\epsilon + k)f_3 &= i\left(\frac{d}{dr} + \mu\right)f_4 = iD_+f_4, \\ (\epsilon - k)f_1 + (\Gamma - M)f_3 &= -i\left(\frac{d}{dr} + \mu\right)f_2 = -iD_+f_2; \end{aligned} \quad (5.1)$$

$$\begin{aligned} -(\Gamma + M)f_2 + (\epsilon - k)f_4 &= i\left(\frac{d}{dr} - \mu\right)f_3 = iD_-f_3, \\ (\epsilon + k)f_2 - (\Gamma + M)f_4 &= -i\left(\frac{d}{dr} - \mu\right)f_1 = -iD_-f_1. \end{aligned} \quad (5.2)$$

Their solutions are as follows:

$$f_1 = +i\frac{(\epsilon + k)D_+f_2 + (\Gamma - M)D_+f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}, \quad f_3 = -i\frac{(\Gamma - M)D_+f_2 + (\epsilon - k)D_+f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}; \quad (5.3)$$

$$f_2 = +i\frac{(\epsilon - k)D_-f_1 - (\Gamma + M)D_-f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}, \quad f_4 = -i\frac{-(\Gamma + M)D_-f_1 + (\epsilon + k)D_-f_3}{(\Gamma + M)^2 - (\epsilon^2 - k^2)}. \quad (5.4)$$

We note two identities

$$\begin{aligned} D_+D_- &= \left(\frac{d}{dr} + \mu\right)\left(\frac{d}{dr} - \mu\right) = \frac{d^2}{dr^2} - \mu' - \mu^2, \\ D_-D_+ &= \left(\frac{d}{dr} - \mu\right)\left(\frac{d}{dr} + \mu\right) = \frac{d^2}{dr^2} + \mu' - \mu^2. \end{aligned} \quad (5.5)$$

Substituting the expression (5.3) into equation (5.2), we get

$$\begin{aligned} -(\Gamma + M)f_2 + (\epsilon - k)f_4 &= \frac{(\Gamma - M)D_-D_+f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\epsilon - k)D_-D_+f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}, \\ +(\epsilon + k)f_2 - (\Gamma + M)f_4 &= \frac{(\epsilon + k)D_-D_+f_2}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} + \frac{(\Gamma - M)D_-D_+f_4}{(\Gamma - M)^2 - (\epsilon^2 - k^2)}. \end{aligned} \quad (5.6)$$

By combining equations (5.6), we obtain

$$f_2 = \frac{1}{2\Gamma(\epsilon + k)} \left( \Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} + \mu' - \mu^2 \right) f_4, \quad (5.7)$$

and then we derive the 4-th order equation for the function  $f_4$

$$\begin{aligned}
 & -\frac{d^4 f_4}{dr^4} + \left[ \frac{B^2}{2} r^2 - B(2m-1) - 2(\Gamma^2 - M^2 - k^2 + \epsilon^2) + 2 \frac{m(m+1)}{r^2} \right] \frac{d^2 f_4}{dr^2} \\
 & \quad + \left[ B^2 r - 4 \frac{m(m+1)}{r^3} \right] \frac{df_4}{dr} \\
 & \quad + \left[ -\frac{B^4}{16} r^4 + \frac{B^2}{4} (B(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)) r^2 \right. \\
 & \quad - B(2m-1)(\Gamma^2 - M^2 - k^2 + \epsilon^2) - (\Gamma^2 + M^2 + k^2 - \epsilon^2)^2 + 4\Gamma^2 M^2 - \frac{B^2}{4} (6m^2 - 2m - 1) \\
 & \quad \left. + \frac{m(m+1)(B(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2))}{r^2} - \frac{m(m-2)(m+3)(m+1)}{r^4} \right] f_4 = 0.
 \end{aligned} \tag{5.8}$$

Similarly, we can find the expression for  $f_4$

$$f_4 = \frac{1}{2\Gamma(\epsilon - k)} \left( \Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} + \mu' - \mu^2 \right) f_2, \tag{5.9}$$

and then derive 4-th order equation for the function  $f_2$ :

$$\begin{aligned}
 & \frac{d^4 f_2}{dr^4} + \left[ -\frac{B^2}{2} r^2 + B(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2) - 2 \frac{m(m+1)}{r^2} \right] \frac{d^2 f_2}{dr^2} + \\
 & \quad + \left[ -B^2 r + 4 \frac{m(m+1)}{r^3} \right] \frac{df_2}{dr} + \left[ \frac{B^4}{16} r^4 - \frac{B^2}{4} (B(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)) r^2 + \right. \\
 & \quad + B(2m-1)(\Gamma^2 - M^2 - k^2 + \epsilon^2) + (\Gamma^2 + M^2 + k^2 - \epsilon^2)^2 - 4\Gamma^2 M^2 + \frac{\epsilon^2 B^2}{4} (6m^2 - 2m - 1) - \\
 & \quad \left. - \frac{m(m+1)(B(2m-1) + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2))}{r^2} + \frac{m(m-2)(m+3)(m+1)}{r^4} \right] f_2 = 0.
 \end{aligned} \tag{5.10}$$

We note that the equation for  $f_2$  and  $f_4$  are the same. Therefore it is sufficient to consider one of them. To study the arising equations (5.8) and (5.10), we use the factorization method:

$$\hat{F}_4(r) f(r) = \hat{f}_2(r) \hat{g}_2(r) f(r) = 0,$$

$$\hat{f}_2(r) = \frac{d^2}{dr^2} + P_0 r^2 + P_1 + \frac{P_2}{r^2}, \quad \hat{g}_2(r) = \frac{d^2}{dr^2} + Q_0 r^2 + Q_1 + \frac{Q_2}{r^2}. \quad (5.11)$$

Calculating the operator  $\hat{F}_4$  and comparing with (5.10), we find two sets of coefficients:

$$\begin{aligned} 1) \quad & P_0 = -\frac{1}{4} B^2, \quad P_2 = -m(m+1), \\ & P_1 = B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 + 2\Gamma\sqrt{\epsilon^2 - k^2}, \\ & Q_0 = -\frac{1}{4} B^2, \quad Q_2 = -m(m+1), \\ & Q_1 = B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}; \end{aligned}$$

and

$$\begin{aligned} 2) \quad & P_0 = -\frac{1}{4} B^2, \quad P_2 = -m(m+1), \\ & P_1 = B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 - 2\Gamma\sqrt{\epsilon^2 - k^2}, \\ & Q_0 = -\frac{1}{4} B^2, \quad Q_2 = -m(m+1), \\ & Q_1 = B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 + 2\Gamma\sqrt{\epsilon^2 - k^2}. \end{aligned}$$

Thus, we are to solve two equations (they differ only in the sign of  $\Gamma$ )

$$\left( \frac{d^2}{dr^2} - \frac{B^2 r^2}{4} + B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 + 2\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m(m+1)}{r^2} \right) f = 0, \quad (5.12)$$

$$\left( \frac{d^2}{dr^2} - \frac{B^2 r^2}{4} + B \left( m - \frac{1}{2} \right) + \Gamma^2 - M^2 - k^2 + \epsilon^2 - 2\Gamma\sqrt{\epsilon^2 - k^2} - \frac{m(m+1)}{r^2} \right) g = 0. \quad (5.13)$$

Consider the first equation (5.12). We turn it to the variable  $x = Br^2/2$ :

$$x \frac{d^2 f}{dx^2} + \frac{1}{2} \frac{df}{dx} + \left[ -\frac{x}{4} + \frac{4\Gamma\sqrt{\epsilon^2 - k^2} + (2m-1)B + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B} - \frac{1}{4} \frac{m(m+1)}{x} \right] f = 0.$$

We build the solutions in the form:

$$f = x^a e^{bx} F,$$

$$x \frac{d^2 F}{dx^2} + \left( \frac{1}{2} + 2a + 2bx \right) \frac{dF}{dx} \left[ \left( b^2 - \frac{1}{4} \right) x + \right. \\ \left. + \frac{2Bb(4a+1) + 4\Gamma\sqrt{\epsilon^2 - k^2} + (2m-1)B + 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B} \right. \\ \left. + \frac{1}{4} \frac{(2a+m)(2a-m-1)}{x} \right] F = 0.$$

If  $a, b$  are chosen as

$$a = -\frac{m}{2}, \quad \frac{1}{2} + \frac{m}{2}, \quad b = -\frac{1}{2}, \quad (5.14)$$

the equation is simplified

$$x \frac{d^2 F}{dx^2} + \left( \frac{1}{2} + 2a - x \right) \frac{dF}{dx} \\ - \frac{B(4a+1) - 4\Gamma\sqrt{\epsilon^2 - k^2} - (2m-1)B - 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B} F = 0,$$

and it is the equation for the confluent hypergeometric functions with parameters:

$$\alpha = \frac{B(4a+1) - 4\Gamma\sqrt{\epsilon^2 - k^2} - (2m-1)B - 2(\Gamma^2 - M^2 - k^2 + \epsilon^2)}{4B}, \quad \gamma = \frac{1}{2} + 2a. \quad (5.15)$$

To get solutions that correspond to bound states, we should use the positive values of the parameter  $a$  and the negative values of the parameter  $b$  (for definiteness we assume that  $B > 0$ ):

$$a = -\frac{m}{2}, \quad (m < 0); \quad a = \frac{m}{2} + \frac{1}{2} > 0 \quad (m \geq 0). \quad (5.16)$$

Conditions of terminating the hypergeometric series to polynomials  $\alpha = -n$  (introducing the notation  $\epsilon^2 - k^2 = \lambda$ ):

$$\frac{B(4a+1) - 4\Gamma\sqrt{\lambda} - (2m-1)B - 2(\Gamma^2 - M^2) - 2\lambda}{4B} = -n \quad (5.17)$$

gives the quantization rule for the energy values:

$$a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} + n = \frac{\Gamma\sqrt{\lambda}}{B} + \frac{\lambda}{2B},$$

so we obtain

$$(\sqrt{\lambda} + \Gamma)^2 = N, \quad N = M^2 + 2B\left(a + \frac{1}{2} - \frac{m}{2} + n\right) \implies \lambda = (\sqrt{N} - \Gamma)^2 > 0. \quad (5.18)$$



From (5.18) we find the formula for the allowed values of  $\lambda$

$$\epsilon^2 - k^2 = \left( \sqrt{M^2 + 2B\left(a + \frac{1}{2} - \frac{m}{2} + n\right)} - \Gamma \right)^2. \quad (5.19)$$

Depending on the sign of  $m$ , we obtain two formulas:

$$m < 0, \quad a = -\frac{m}{2}, \quad \epsilon^2 - k^2 = \left( \sqrt{M^2 + 2B\left(\frac{1}{2} - m + n\right)} - \Gamma \right)^2; \quad (5.20)$$

$$m \geq 0, \quad a = \frac{m}{2} + \frac{1}{2}, \quad \epsilon^2 - k^2 = \left( \sqrt{M^2 + 2B(1 + n)} - \Gamma \right)^2. \quad (5.21)$$

This, we have two possibilities for quantization

$$I, \quad \lambda = (\sqrt{N} - \Gamma)^2, \quad II, \quad \lambda = (\sqrt{N} + \Gamma)^2. \quad (5.22)$$

So, the particle with anomalous magnetic moment has two series of the energy levels, formally differing in sign of the parameter  $\Gamma$ .

## 6. Further analysis of the solutions

Let us consider the function  $f_2(r)$  as the primary one. Obtained above ratios allow us to calculate other three functions. We start from the explicit form of the function  $f_2$ , which reads

$$f_2 = x^\alpha e^{-x/2} F(\alpha, \gamma, x), \quad (6.1)$$

where the parameters are given by

$$\alpha = a + \frac{1}{2} - \frac{m}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \quad \gamma = \frac{1}{2} + 2a. \quad (6.2)$$

The function  $f_4$  can be found according to the following relationship:

$$f_4 = \frac{1}{2\Gamma(\epsilon - k)} \left( \Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} - \frac{m(m+1)}{r^2} + \frac{B(2m-1)}{2} - \frac{B^2}{4} r^2 \right) f_2;$$

so after transformation to the variable  $x$  we get

$$f_4 = \frac{2B}{2\Gamma(\epsilon - k)} \left( x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} + \frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} - \frac{m(m+1)}{4x} - \frac{1}{4} x \right) f_2. \quad (6.3)$$

Given the identities

$$+\frac{\lambda}{2B} - \frac{M^2 - \Gamma^2}{2B} + \frac{m}{2} - \frac{1}{4} = n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B}$$

the above relation can be written as

$$f_4 = \frac{2B}{2\Gamma(\epsilon - k)} \left( x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} + n + a + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{1}{4}x \right) f_2. \quad (6.4)$$

Depending on the values of the parameter  $m$ , we have two different cases:

$$A) \quad m < 0, \quad a = -\frac{m}{2}, \quad f_2 = x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x),$$

$$\alpha = m + \frac{1}{2} + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \quad \gamma = -m + \frac{1}{2},$$

$$f_4 = \frac{2B}{2\Gamma(\epsilon - k)} \left( x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} + n - \frac{m}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(\frac{1}{2} - m + n)} - \Gamma; \quad (6.5)$$

$$B) \quad m \geq 0, \quad a = \frac{m}{2} + \frac{1}{2}, \quad f_2 = x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x),$$

$$\alpha = 1 + \frac{M^2 - \Gamma^2}{2B} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{\lambda}{2B} = -n, \quad \gamma = m + \frac{3}{2},$$

$$f_4 = \frac{2B}{2\Gamma(\epsilon - k)} \left( x \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} + n + \frac{m+1}{2} + \frac{1}{4} - \frac{\Gamma\sqrt{\lambda}}{B} - \frac{m(m+1)}{4x} - \frac{x}{4} \right) f_2,$$

$$\sqrt{\lambda} = \sqrt{M^2 + 2B(1 + n)} - \Gamma. \quad (6.6)$$

Using the relations (6.5) and (6.6) we find the expression for function  $f_4$ :  
variant A,

$$m < 0, \quad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k} x^{-m/2} e^{-x/2} F(-n, -m + \frac{1}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k} f_2(x), \quad (6.7)$$

variant B,

$$m \geq 0, \quad f_4 = -\frac{\sqrt{\lambda}}{\epsilon - k} x^{(m+1)/2} e^{-x/2} F(-n, m + \frac{3}{2}, x) = -\frac{\sqrt{\lambda}}{\epsilon - k} f_2(x). \quad (6.8)$$

Changing in these formulas the parameter  $\Gamma$  on  $-\Gamma$ , we obtain the relations describing the second series of states. It is not difficult to calculate the explicit form of the other two functions  $f_1, f_3$ ; here we omit the details.

## 7. The case of an uncharged particle

Now let us consider in detail the special case of an uncharged particle with anomalous magnetic moment (neutron). After applying the above factorization method we get the following 2-nd order equations:

$$\begin{aligned} \left( \frac{d^2}{dr^2} + (\sqrt{\epsilon^2 - k^2} + \Gamma)^2 - M^2 - \frac{m(m+1)}{r^2} \right) f &= 0, \\ \left( \frac{d^2}{dr^2} + (\sqrt{\epsilon^2 - k^2} - \Gamma)^2 - M^2 - \frac{m(m+1)}{r^2} \right) g &= 0. \end{aligned} \quad (7.1)$$

The general solutions have the form

$$\begin{aligned} f(r) &= \sqrt{r} (J_{m+1/2}(x) + Y_{m+1/2}(x)), \quad x = \sqrt{(\sqrt{\epsilon^2 - k^2} + \Gamma)^2 - M^2} r; \\ g(r) &= \sqrt{r} (J_{m+1/2}(y) + Y_{m+1/2}(y)), \quad y = \sqrt{(\sqrt{\epsilon^2 - k^2} - \Gamma)^2 - M^2} r. \end{aligned} \quad (7.2)$$

So we obtain the solutions of the Dirac equation with cylindrical symmetry which are similar to ordinary cylindric waves but with one change in parameters:

$$\epsilon^2 - M^2 \implies (\sqrt{\epsilon^2 - k^2} \pm \Gamma)^2 - M^2; \quad (7.3)$$

We can consider the function  $f_2(r) = f(r)$  as the main one (transition to  $f_2(r) = g(r)$  is reached by the formal change  $\Gamma \implies -\Gamma$ ). Then the above written relations permit to find three remaining ones

$$\begin{aligned} f_4 &= \frac{1}{2\Gamma(\epsilon - k)} \left( \Gamma^2 - M^2 - k^2 + \epsilon^2 + \frac{d^2}{dr^2} - \frac{m}{r^2} - \frac{m^2}{r^2} \right) f_2, \\ f_1 &= \frac{i}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \left( \frac{d}{dr} + \frac{m}{r} \right) [(\epsilon + k)f_2 + (\Gamma - M)f_4], \\ f_3 &= \frac{-i}{(\Gamma - M)^2 - (\epsilon^2 - k^2)} \left( \frac{d}{dr} + \frac{m}{r} \right) [(\Gamma - M)f_2 + (\epsilon - k)f_4]. \end{aligned}$$

These result in

$$\begin{aligned} f_4 &= \sqrt{\frac{\epsilon + k}{\epsilon - k}} \sqrt{r} [J_{m+1/2}(x) + Y_{m+1/2}(x)] = \sqrt{\frac{\epsilon + k}{\epsilon - k}} f(r), \\ f_1 &= -i \sqrt{\frac{\epsilon + k}{\epsilon - k}} \sqrt{\frac{\sqrt{\epsilon^2 - k^2} + \Gamma + M}{\sqrt{\epsilon^2 - k^2} + \Gamma - M}} \sqrt{r} [J_{m-1/2}(x) + Y_{m-1/2}(x)], \\ f_3 &= -i \sqrt{\frac{\sqrt{\epsilon^2 - k^2} + \Gamma + M}{\sqrt{\epsilon^2 - k^2} + \Gamma - M}} \sqrt{r} [J_{m-1/2}(x) + Y_{m-1/2}(x)]. \end{aligned} \quad (7.4)$$

As one can see, the only qualitative manifestation of the neutron anomalous magnetic moment is the deformation (space scaling) of its wave function in comparison with a particle without such a moment.

## 8. Conclusion

We examine a generalized Dirac equation for spin 1/2 particle with anomalous magnetic moment in the presence of an external uniform magnetic field. After separation of variables, the problem is reduced to the 4-order ordinary

differential equation, which is solved exactly with the use of the factorization method. A generalized formula for the Landau energy levels is found. Solutions are expressed in terms of the confluent hypergeometric functions. Restriction to the case of neutron is performed.

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