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Invariant Affine Connections on Three-Dimensional Homogeneous Spaces with Nonsolvable Transformation Group

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Abstract—The aim of this paper is to describe all invariant affine connections on three-dimensional homogeneous spaces with nonsolvable transformation group. We present complete local classification of homogeneous spaces, it is equivalent to the description of effective pairs of Lie algebras. We describe all invariant affine connections together with their curvature and torsion tensors, holonomy algebras.

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1. INTRODUCTION

Classification of homogeneous spaces in low dimensions is a classical problem which goes back to Sophus Lie, who provided local classification of homogeneous spaces in dimensions 1 and 2 and some three-dimensional spaces. A large class of the homogeneous spaces is spaces with nonsolvable transformation group. The preprint [1] gives the local classification of three-dimensional isotropically-faithful homogeneous spaces.

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, and let the Lie group $\overline{G}$ be nonsolvable. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. Since we are interested only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, that is two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$, a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective.

An *isotropic* $\mathfrak{g}$*-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y + \mathfrak{g}) = [x, y] + \mathfrak{g}$. The corresponding representation $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [2] with linear mappings $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [3, 4]. All of this pairs are described in [1]. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas:

$R : \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}),\ (x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)] - \Lambda([x_1, x_2]);$

$T : \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m},\ (x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto \Lambda(x_1)(x_2 + \mathfrak{g}) - \Lambda(x_2)(x_1 + \mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}.$

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To refer to the pair we use the notation $d.n.m$, where d is the dimension of the subalgebra, n is the number of the subalgebra of $\mathfrak{gl}(3,\mathbb{R})$, m is the number of $(\bar{\mathfrak{g}},\mathfrak{g})$ in [1]. We define $(\bar{\mathfrak{g}},\mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$. Here by $\{e_1,...,e_n\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n=\dim\bar{\mathfrak{g}}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1,...,e_{n-3}$. Let $\{u_1=e_{n-2},u_2=e_{n-1},u_3=e_n\}$ be a basis of $\mathfrak{m}$. We describe affine connection by $\Lambda(e_{n-2})$, $\Lambda(e_{n-1})$, $\Lambda(e_n)$, curvature tensor R by $R(e_{n-2},e_{n-1})$, $R(e_{n-2},e_n)$, $R(e_{n-1},e_n)$ and torsion tensor T by $T(e_{n-2},e_{n-1})$, $T(e_{n-2},e_n)$, $T(e_{n-1},e_n)$.

2. PAIRS OF LIE ALGEBRAS DON'T HAVE AFFINE CONNECTIONS

Find isotropy-faithful pairs, such that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ don't have invariant connections.

Theorem 1. *If the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ don't have affine connections, $\bar{\mathfrak{g}}$ is semisimple then $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to one of the pairs* 3.19.2, 3.19.12, 3.21.2.

Proof. Let

$$\Lambda(u_1)=\begin{pmatrix}p_{1,1}&p_{1,2}&p_{1,3}\\p_{2,1}&p_{2,2}&p_{2,3}\\p_{3,1}&p_{3,2}&p_{3,3}\end{pmatrix},\quad \Lambda(u_2)=\begin{pmatrix}q_{1,1}&q_{1,2}&q_{1,3}\\q_{2,1}&q_{2,2}&q_{2,3}\\q_{3,1}&q_{3,2}&q_{3,3}\end{pmatrix},\quad \Lambda(u_3)=\begin{pmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\r_{2,1}&r_{2,2}&r_{2,3}\\r_{3,1}&r_{3,2}&r_{3,3}\end{pmatrix}$$

for $p_{i,j},q_{i,j},r_{i,j}\in\mathbb{R}\,(i,j=\overline{1,3})$. If $(\bar{\mathfrak{g}},\mathfrak{g})$ is the three-dimensional homogeneous space 3.19.2 [1] then

$$\Lambda(e_1)=\begin{pmatrix}0&0&0\\0&1&0\\0&0&\lambda\end{pmatrix},\quad \Lambda(e_2)=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad \Lambda(e_3)=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$(\Lambda|_{\mathfrak{g}}=\lambda)$. Λ is $\mathfrak{g}$-invariant $\Rightarrow$ $[\Lambda(e_3),\Lambda(u_1)]=\Lambda([e_3,u_1])$ $\Rightarrow$ $[\Lambda(e_3),\Lambda(u_1)]=0$, $\begin{pmatrix}p_{3,1}&p_{3,2}&p_{3,3}-p_{1,1}\\0&0&-p_{2,1}\\0&0&-p_{3,1}\end{pmatrix}=0$, $p_{3,1}=p_{3,2}=p_{2,1}=0$, $p_{3,3}=p_{1,1}$. $[\Lambda(e_2),\Lambda(u_1)]=\Lambda([e_2,u_1])\Rightarrow$ $[\Lambda(e_2),\Lambda(u_1)]=-\Lambda(e_2)$, $\begin{pmatrix}0&p_{2,2}-p_{1,1}-1&p_{2,3}\\0&0&0\\0&0&0\end{pmatrix}=0$, we have $p_{2,2}=p_{1,1}+1, p_{2,3}=0$. $[\Lambda(e_1),\Lambda(u_1)]=\Lambda([e_1,u_1])\Rightarrow[\Lambda(e_1),\Lambda(u_1)]=0$ then $p_{1,2}=0$. $[\Lambda(e_3),\Lambda(u_2)]=0$, $q_{3,1}=q_{3,2}=q_{2,1}=0$, $q_{3,3}=q_{1,1}$; if $[\Lambda(e_2),\Lambda(u_2)]=\Lambda(u_1)$ then $\begin{pmatrix}-p_{1,1}&q_{2,2}-q_{1,1}&q_{2,3}-p_{1,3}\\0&-p_{1,1}-1&0\\0&0&-p_{1,1}\end{pmatrix}=0$ and the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ don't have affine connections. In other cases are the same. □

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Theorem 2. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ don't have affine connections, $\bar{\mathfrak{g}}$ is not semisimple (or solvable), radical of $\bar{\mathfrak{g}}$ is commutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the pairs*

Pair	Levi decomposition
4.10.3	$\{\{u_1\}, \{e_1, e_2 + u_1, u_2, e_3, e_4 + \alpha u_1, u_3\}\}$
4.14.2	$\{\{u_1\}, \{e_1 + u_1, e_2, e_3, -e_4, u_2, u_3\}\}$
4.14.3	$\{\{u_1\}, \{e_2 + \alpha e_1 + u_1, -e_1 + \alpha e_2, -e_4 + \alpha e_3, -e_3 - \alpha e_4, u_3 + \alpha u_2, -u_2 + \alpha u_3\}\}$
4.20.11	$\{\{e_4, -e_3, -u_1, -u_2\}, \{-e_1 - e_3, -e_2, -u_1 - u_3\}\}$
4.20.12	$\{\{-u_2, -e_3, -u_1, e_4\}, \{-e_1 - e_3 - e_4 + u_1, -e_2 - e_3, -u_1 + 2u_2 - u_3\}\}$
3.19.11	$\{\{u_1, u_2, -e_3\}, \{-e_2, u_3 - u_1, e_1 - e_3\}\}$
3.19.15	$\{\{u_1, e_3, -e_2\}, \{u_2 - u_1 - 2e_3, e_2 - u_3, e_1 - e_2 - u_1\}\}$
3.21.4	$\{\{u_1, e_2, -e_3\}, \{-u_3 - (1/2)e_2, u_2 - u_1, e_1 - e_3 - (1/2)u_1\}\}$
3.21.5	$\{\{u_1, e_2, -e_3\}, \{-(1/2)e_2 - u_3, u_1 + u_2, -e_1 - e_3 - (1/2)u_1\}\}$
3.25.31	$\{\{e_1, 2u_1, e_2\}, \{2e_1 + 4e_3 + (2/3)u_1, e_2 + 4u_2, u_1 + 4u_3\}\}$
3.26.2	$\{\{e_3, u_2, u_1\}, \{e_1 + e_3, e_2 + e_3 - u_1, u_1 - 2u_2 + u_3\}\}$
2.9.17	$\{\{u_1, -u_3\}, \{2e_2, -2u_2 - 2u_3, e_1 + u_1\}\}$
2.13.9	$\{\{u_1, e_2\}, \{2e_1 + (2/3)u_1, e_2 + u_2, 2u_1 + 2u_3\}\}$

Proof. If $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 2.9.17 [1] then

$$\Lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Lambda(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$(\Lambda|_{\mathfrak{g}} = \lambda)$. Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_2), \Lambda(u_1)] = \Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)] = 0$, $p_{3,1} = p_{3,2} = p_{2,1} = 0$, $p_{3,3} = p_{1,1}$. If $[\Lambda(e_1), \Lambda(u_1)] = \Lambda([e_1, u_1]) \Rightarrow [\Lambda(e_1), \Lambda(u_1)] = \Lambda(u_1)$ then $p_{1,1} = p_{1,2} = p_{1,3} = p_{2,2} = p_{2,3} = 0$. $[\Lambda(e_2), \Lambda(u_2)] = \Lambda(e_1) \Rightarrow \begin{pmatrix} q_{3,1} - 1 & q_{3,2} & q_{3,3} - q_{1,1} \\ 0 & 2 & -q_{2,1} \\ 0 & 0 & -q_{3,1} + 1 \end{pmatrix} = 0$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ don't have affine connections. In other cases are the same. □

Theorem 3. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ don't have affine connections, $\bar{\mathfrak{g}}$ is not semisimple (or solvable), radical of $\bar{\mathfrak{g}}$ is noncommutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the pairs*

Pair	Levi decomposition
5.10.3	$\{\{e_5, u_2, e_1 + e_2, u_1, -e_4\}, \{e_3, u_1 - u_3, e_2 - e_4\}\}$
4.6.2	$\{\{u_2 - e_2, e_1, e_4, u_1\}, \{-e_2 + (1/2)e_4 - u_2, -2e_3, u_1 - 2u_3\}\}$
4.12.2	$\{\{e_1, u_3, e_3, u_1\}, \{e_3 - e_2, -e_4, u_1 - u_2\}\}$
4.21.8	$\{\{e_4, -2e_2 - 2u_1, -2e_1 + u_2, u_1\}, \{-(4/3)e_2 + 4e_3 + (2/3)u_1, e_2 + u_1 - 4u_3, e_4 + 4u_2\}\}$
3.8.4	$\{\{e_1, u_1, u_3\}, \{e_1 - 2e_2 + (1/2)u_1, -2e_3, u_3 - 2u_2\}\}$
3.19.6	$\{\{u_1 - e_1, u_3, e_3\}, \{-e_2, u_2, u_1\}\}$
3.19.9	$\{\{e_3, u_1 - e_1, u_3\}, \{-e_2, u_2, u_1\}\}$
3.19.10	$\{\{u_3, u_1 - e_1, e_3\}, \{-e_2, u_2, u_1\}\}$
3.23.7	$\{\{-e_3, -2u_1, u_2 - 2e_1\}, \{-(1/2)u_1 - u_2, -2e_2 - e_3 + (2/3)u_1, -2u_3\}\}$
3.24.2	$\{\{u_1, u_2, -e_2\}, \{-2e_3, u_1 - 2u_2 + 2u_3, e_1 - e_2 + u_1\}\}$

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The proof just as above.

3. PAIRS OF LIE ALGEBRAS WITH ONLY TRIVIAL AFFINE CONNECTIONS

That means $\Lambda(e_{n-2}) = \Lambda(e_{n-1}) = \Lambda(e_n) = 0$. The torsion tensor is zero for all connections here, the holonomy algebra is zero too. In this case semisimple transformation groups are not exist.

3.1. The Curvature Tensor is Zero

Theorem 4. If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ have only trivial affine connection, the curvature tensor is zero, $\mathfrak{g}$ is semisimple then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to 8.1.1 ($\mathfrak{sl}(3, \mathbb{R})$), Levi decomposition $\{\{-u_3, -u_2, u_1\}, \{2e_3, e_4, -2e_5 - 2u_2, -e_6, -e_7 - u_3, e_8, e_1 + u_1, e_1 + e_2 + u_1\}\}$.

Proof. If $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 8.1.1 [1], $\Lambda|_{\mathfrak{g}} = \lambda$ then Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_1), \Lambda(u_1)] = \Lambda([e_1, u_1]) \Rightarrow [\Lambda(e_1), \Lambda(u_1)] = \Lambda(u_1)$ and $p_{1,1} = p_{1,2} = p_{2,1} = p_{2,2} = p_{2,3} = p_{3,1} = p_{3,3} = 0$. If $[\Lambda(e_2), \Lambda(u_1)] = \Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)] = 0$ then $p_{1,3} = p_{3,2} = 0$. $[\Lambda(e_5), \Lambda(u_1)] = \Lambda([e_5, u_1])$, то $[\Lambda(e_5), \Lambda(u_1)] = \Lambda(u_2)$, $q_{1,1} = q_{1,2} = q_{1,3} = q_{2,1} = q_{2,2} = q_{2,3} = q_{3,1} = q_{3,2} = q_{3,3} = 0$. If $[\Lambda(e_7), \Lambda(u_1)] = \Lambda(u_3)$ then $r_{1,1} = r_{1,2} = r_{1,3} = r_{2,1} = r_{2,2} = r_{2,3} = r_{3,1} = r_{3,2} = r_{3,3} = 0$. We have $\Lambda(u_1) = \Lambda(u_2) = \Lambda(u_3) = 0$. In other cases $\mathfrak{g}$ is not semisimple. □

Theorem 5. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ have only trivial affine connection, the curvature tensor is zero, $\mathfrak{g}$ is not semisimple, commutant of the radical of $\bar{\mathfrak{g}}$ is noncommutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
7.2.1	$\{\{u_3, u_2, u_1, 2e_4 + e_5, e_3, e_2, e_1\}, \{(1/2)e_2 - e_5, -e_3 + 2e_6, 2e_7\}\}$
6.2.1	$\{\{u_3, e_1, (8\lambda - 8)e_4, (-8\lambda + 8)e_6, (4\lambda - 4)u_1, (4\lambda - 4)u_2\},$
	$\{32e_3, -32e_5 + (8 - 8\lambda)u_2, 16e_2 + (4\lambda - 4)u_1\}\}$
6.3.1	$\{\{u_3, u_2, u_1, e_6, e_5, e_4\}, \{-4e_1 + 2e_5, -4e_2 + 2e_6, -4e_3\}\}$
6.4.1, $\lambda \neq 1/2$	$\{\{u_3, u_2, u_1, e_6, e_5, e_1\}, \{-4e_2 + (2 - 2\lambda)e_5, -4e_3 + (2 - 2\lambda)e_6, -4e_4\}\}$
Pair	Levi decomposition $\mathfrak{g}$
7.2.1	$\{\{e_2, -e_3, e_1, 2e_4 + e_5\}, \{(1/2)e_2 - e_5, -e_3 + 2e_6, 2e_7\}\}$
6.2.1	$\{\{e_1, (-\lambda + 1)e_4, (2\lambda - 2)e_6\}, \{-4e_2 + (2\lambda - 2)e_4, -4e_3, -4e_5 + (2\lambda - 2)e_6\}\}$
6.3.1	$\{\{2e_5, 2e_6, e_4\}, \{-4e_1 + 2e_5, -4e_2 + 2e_6, -4e_3\}\}$
6.4.1, $\lambda \neq 1/2$	$\{\{e_5, e_6, e_1\}, \{-4e_2 + (2 - 2\lambda)e_5, -4e_3 + (2 - 2\lambda)e_6, -4e_4\}\}$

The proof just as above.

Theorem 6. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ have only trivial affine connection, the curvature tensor is zero, $\mathfrak{g}$ is not semisimple, commutant of the radical of $\bar{\mathfrak{g}}$ is commutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs*

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Pair	Levi decomposition $\bar{\mathfrak{g}}$
9.1.1	$\{\{e_9, -u_3, -2u_2, -2u_1\}, \{-4e_1 - 8u_1, -2e_4, 2e_8, -4e_3, 2e_6, -e_1 - e_2 - 2u_1, 2e_7 + 4u_3, -4e_5 - 8u_2\}\}$
7.1.1	$\{\{-8e_7, u_3, e_3, e_1, 4u_2, 4u_1, 8e_6\}, \{32e_4 + 16e_6, -32e_5, 16e_2 - 8e_7\}\}$
5.1.1	$\{\{u_1, e_3, e_1 + e_2, -u_2, -u_1\}, \{4e_4, -4e_5 + u_2, 4e_1 - 4e_2 - u_1\}\}$
4.2.1, $\lambda \neq 1/2$	$\{\{u_3, e_1, 8\lambda u_1, -8\lambda u_2\}, \{32e_3 + 16\lambda u_1, -32e_4, 16e_2 - 8\lambda u_2\}\}$
4.3.1	$\{\{-u_1, u_3, u_2, e_1\}, \{-e_2 + u_1, -e_3, -e_4 + u_1\}\}$
4.5.1	$\{\{u_1, u_3, -u_2, e_1\}, \{e_2 - u_1, e_3, e_4 - u_2\}\}$
Pair	**Levi decomposition $\mathfrak{g}$**
9.1.1	$\{\{u_3\}, \{-4e_1, -2e_4, 2e_8, -4e_3, 2e_6, -e_1 - e_2, 2e_7, -4e_5\}\}$
7.1.1	$\{\{e_1, e_3, -e_6, 2e_7\}, \{-4e_2 + 2e_6, -4e_4, -4e_5 + 2e_7\}\}$
5.1.1	$\{\{e_3, e_1 + e_2\}, \{-e_1 + e_2, -2e_4, -2e_5\}\}$
4.2.1, $\lambda \neq 1/2$	$\{\{e_1\}, \{-4e_2, -4e_3, -4e_4\}\}$
4.3.1	$\{\{e_1\}, \{-e_2, -e_3, -e_4\}\}$
4.5.1	$\{\{e_1\}, \{e_2, e_3, e_4\}\}$

The proof just as above.

3.2. The Curvature Tensor is Not Zero

Theorem 7. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *have only trivial affine connection, the curvature tensor is not zero and the radical of* $\bar{\mathfrak{g}}$ *is commutative then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to* 2.9.12 *(Levi decomposition* $\{\{u_1, -e_2\}, \{-2u_2, 2u_1 + 2u_3, -e_1 - e_2\}\}$*). Then curvature tensor*

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

holonomy algebra $\begin{pmatrix} p_2 & 0 & p_1 \\ 0 & -2p_2 & 0 \\ 0 & 0 & 2p_2 \end{pmatrix}.$

Proof. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 2.9.12 [1], Λ is $\mathfrak{g}$-invariant $\Rightarrow$ $[\Lambda(e_2), \Lambda(u_1)] = \Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)] = 0$, we have $p_{3,1} = p_{3,2} = p_{2,1} = p_{3,1} = 0, p_{3,3} = p_{1,1}$. $[\Lambda(e_1), \Lambda(u_1)] = \Lambda([e_1, u_1]) \Rightarrow [\Lambda(e_1), \Lambda(u_1)] = \Lambda(u_1)$, $p_{1,1} = p_{1,2} = p_{1,3} = p_{2,2} = p_{2,3} = 0$. If $[\Lambda(e_2), \Lambda(u_2)] = \Lambda([e_2, u_2])$ then $[\Lambda(e_2), \Lambda(u_2)] = 0$, $q_{3,1} = q_{3,2} = q_{2,1} = 0$, $q_{3,3} = q_{1,1}$. If $[\Lambda(e_1), \Lambda(u_2)] = \lambda\Lambda(u_2)$ then $q_{1,1} = q_{1,2} = q_{1,3} = q_{2,2} = q_{2,3} = 0$. $[\Lambda(e_1), \Lambda(u_3)] = \mu\Lambda(u_3)$, $r_{1,1} = r_{1,2} = r_{1,3} = r_{2,1} = r_{2,2} = r_{2,3} = r_{3,1} = r_{3,2} = r_{3,3} = 0$. $[\Lambda(e_2), \Lambda(u_3)] = \Lambda(u_1)$, $\Lambda(u_1) = \Lambda(u_2) = \Lambda(u_3) = 0$. $V = \{[\Lambda(x), \Lambda(y)] - \Lambda([x, y]) | x, y \in \bar{\mathfrak{g}}\}$ is the holonomy algebra $\mathfrak{h}^*$: $\Lambda(\bar{\mathfrak{g}}) = \Lambda(\mathfrak{g})$ and $[\Lambda(\bar{\mathfrak{g}}), V] = [\Lambda(\mathfrak{g}), V] = V$, $\Lambda(\mathfrak{g})$ is equal to V. In this case $\mathfrak{a}_{\bar{\mathfrak{g}}} = \Lambda(\mathfrak{g})$ and $\mathfrak{h}^* = \mathfrak{a}_{\bar{\mathfrak{g}}}$. In other cases $\mathfrak{g}$ is solvable. □

Theorem 8. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *have only trivial affine connection, the curvature tensor is not zero and the radical of* $\bar{\mathfrak{g}}$ *is not commutative then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the pairs*

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Pair	Levi decomposition
4.11.2	$\{\{-e_4, e_1, e_3, u_1\}, \{-e_2 - e_4, -u_2, u_1 - u_3\}\}$
4.13.2	$\{\{e_1, -e_2, e_3, u_1\}, \{e_4 - e_2, u_2, u_1 + u_3\}\}$
4.13.3	$\{\{e_1, e_2, -e_3, u_1\}, \{-e_2 - e_4, -u_2, u_1 - u_3\}\}$
3.8.8	$\{\{-e_3, -u_1, e_1\}, \{e_1 - 2e_2 + (1/2)e_3, -2u_2, -u_1 - 2u_3\}\}$
2.1.2	$\{\{u_3, e_2\}, \{u_1, -u_2, e_1\}\}$
2.3.2	$\{\{u_3, e_2\}, \{-u_2, u_1, e_1\}\}$
2.3.3	$\{\{u_3, e_2\}, \{-u_2, u_1, -e_1\}\}$

Pair	Curvature tensor
4.11.2	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4.13.2	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
4.13.3	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
3.8.8	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
2.1.2	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.3.2 2.3.3	$\begin{pmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Pair	Holonomy algebra	Pair	Holonomy algebra
4.11.2	$\begin{pmatrix} 0 & p_2 & p_1 \\ 0 & -p_3 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$	4.13.2 4.13.3	$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & -p_3 \\ 0 & p_3 & 0 \end{pmatrix}$
3.8.8	$\begin{pmatrix} p_2 & 0 & -p_1 \\ 0 & -2p_2 & 0 \\ 0 & 0 & 2p_2 \end{pmatrix}$	2.1.2	$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & -p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2.3.2	$\begin{pmatrix} 0 & -p_1 & 0 \\ p_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2.3.3	$\begin{pmatrix} 0 & -p_1 & 0 \\ p_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4. PAIRS OF LIE ALGEBRAS ALLOWS NONTRIVIAL AFFINE CONNECTIONS

4.1. The Curvature and Torsion Tensors are Zero for All Connections

In this case semisimple transformation groups are not exist.

Theorem 9. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature and torsion tensors are zero for all connections and* $\mathfrak{g}$ *is nonsolvable then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to* 6.3.2 *(Levi*

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decomposition $\bar{\mathfrak{g}}$ $\{\{u_1\},\{-4e_1,2e_6,-2u_2,-4e_2,2e_5,-2u_3,-4e_3,-e_1-3e_4-u_1\}\}$*, Levi decomposition* $\mathfrak{g}$ $\{\{e_4,2e_5,2e_6\},\{-4e_1+2e_5,-4e_2+2e_6,-4e_3\}\}$*), the connection*

$$\begin{pmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{pmatrix},\quad \begin{pmatrix}0&0&0\\-2&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\-2&0&0\end{pmatrix}.$$

Proof. If $(\bar{\mathfrak{g}},\mathfrak{g})$ is 6.3.2 [1] then $[\Lambda(e_2),\Lambda(u_1)]=\Lambda([e_2,u_1])\Rightarrow[\Lambda(e_2),\Lambda(u_1)]=0$, $p_{3,1}=p_{3,2}=p_{1,2}=0$, $p_{3,3}=p_{2,2}$. $[\Lambda(e_1),\Lambda(u_1)]=\Lambda([e_1,u_1])\Rightarrow p_{1,3}=p_{2,1}=p_{2,3}=0$. $[\Lambda(e_5),\Lambda(u_1)]=\Lambda([e_5,u_1])\Rightarrow p_{2,2}=p_{1,1}$. If $[\Lambda(e_2),\Lambda(u_2)]=0$ then $q_{3,1}=q_{3,2}=q_{1,2}=0$, $q_{3,3}=q_{2,2}$. $[\Lambda(e_1),\Lambda(u_2)]=\Lambda(u_2)$, $q_{1,1}=q_{2,2}=q_{2,3}=0$. $[\Lambda(e_3),\Lambda(u_2)]=\Lambda(u_3)$, $r_{1,1}=r_{1,3}=r_{2,1}=r_{2,2}=r_{2,3}=r_{3,2}=r_{3,3}=0$, $r_{3,1}=q_{2,1}$, $r_{1,2}=-q_{1,3}$. If $[\Lambda(e_4),\Lambda(u_2)]=\Lambda(u_2)$ then $r_{1,2}=0$. $[\Lambda(e_5),\Lambda(u_2)]=\Lambda(u_1)+\Lambda(e_1)+3\Lambda(e_4)$, $p_{1,1}=r_{3,1}=-2$. □

In this case if $\mathfrak{g}$ is nonsolvable then the radical of $\bar{\mathfrak{g}}$ is commutative.

Theorem 10. *If the pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *allows nontrivial affine connections, the curvature and torsion tensors are zero for all connections and* $\mathfrak{g}$ *is solvable then* $(\bar{\mathfrak{g}},\mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
5.9.2	$\{\{2u_1,e_5,-2e_2+u_2,e_3,e_1\},\{-4e_4,u_1+4u_3,e_5+4u_2\}\}$
4.19.2	$\{\{2e_1+u_2,e_2,-u_1,e_4\},\{-(1/2)u_1-u_2,2e_3+e_4,2u_3\}\}$
4.21.11, $\mu\neq0,1,1/2$	$\{\{-\mu^3e_3,-2\mu^2e_2-2\mu^2u_1,u_1,(21/\mu)e_1+u_2\},$ $\{-4\mu^3e_3,\mu^4e_2+\mu^4u_1+4\mu^3u_3,-\mu^3e_4+4\mu^2u_2\}\}$
3.6.2	$\{\{u_2,-e_3+u_1,e_2\},\{e_1,u_1,u_3\}\}$
3.12.2	$\{\{-u_1,-e_3,-2e_1+u_2\},\{-(1/2)u_1-u_2,-2e_2-e_3,-2u_3\}\}$
3.13.6, $\mu\neq0,1$	$\{\{-(21/\mu)e_1+u_2,(1-\mu)\mu u_1,-\mu e_3\},$ $\{(1-\mu)\mu^2/(2(-1+\mu))u_1-\mu^2u_2,-2\mu e_2-\mu e_3,-2\mu u_3)\}\}$
3.28.2	$\{\{u_2,-e_3,-u_1\},\{-2e_1-u_2,-2e_2,-2u_3\}\}$
2.8.7, $\lambda\neq0$	$\{\{u_2,-(1/\lambda)e_2+u_1\},\{\lambda^3e_1,\lambda^2u_1,\lambda^3u_3\}\}$
Pair	**Connection**
5.9.2 3.12.2 3.13.6, $\mu\neq0,1,-1,1/2$	$\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix}$
4.19.2.	$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$
4.21.11, $\mu\neq0,1,1/2$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$
3.6.2 2.8.7, $\lambda\neq0,1,-1,1/2$	$\begin{pmatrix}-1/2&0&0\\0&0&0\\0&0&1/2\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\-1/2&0&0\end{pmatrix}$
3.28.2	$\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&-2&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix}$

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In this case if $\mathfrak{g}$ is solvable then the radical of $\bar{\mathfrak{g}}$ is noncommutative and commutant of the radical of $\bar{\mathfrak{g}}$ is commutative.

4.2. The Curvature Tensor is Zero for All Connections, Torsion Tensor is Not Zero for Some Connections

In this case the transformation group is not semisimple.

Theorem 11. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature tensor is zero for all connections, torsion tensor is not zero for some connections,* $\mathfrak{g}$ *is nonsolvable, the radical of* $\bar{\mathfrak{g}}$ *is noncommutative, commutant of the radical of* $\bar{\mathfrak{g}}$ *is noncommutative too then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one of the following pairs:* 6.4.1 *(*$\lambda = 1/2$*, radical* $\bar{\mathfrak{g}}$ $\{u_3, u_2, u_1, e_6, e_5, e_1\}$*, Levi decomposition* $\mathfrak{g}$ $\{\{(1/2)e_5, e_6, e_1\}, \{-4e_2 + e_5, -4e_3 + e_6, -4e_4\}\}$*),* 4.2.2 *(Levi decomposition* $\bar{\mathfrak{g}}$ $\{\{e_1, u_2, -(1/2)u_1, (1/4)u_3\}, \{-4e_2 + u_1, -4e_4 + u_2 + (1/8)u_3, -4e_3\}\}$*,* $\mathfrak{g}$ *–* $\{\{e_1\}, \{-4e_2, -4e_3, -4e_4\}\}$*).*

Pair	Connection
6.4.1, $\lambda = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -r_{1,2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4.2.2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -p_{3,2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Pair	**Torsion tensor**
6.4.1, $\lambda = 1/2$	$(0,0,0), (0,0,0), (-2r_{1,2}, 0, 0)$
4.2.2	$(0, 0, 2p_{3,2} - 1), (0,0,0), (0,0,0)$

Theorem 12. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature tensor is zero for all connections, torsion tensor is not zero for some connections,* $\mathfrak{g}$ *is nonsolvable, radical of* $\bar{\mathfrak{g}}$ *is noncommutative, commutant of the radical of* $\bar{\mathfrak{g}}$ *is commutative then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one of the following pairs: 5.3.1 (radical* $\bar{\mathfrak{g}}$ $\{u_3, u_2, u_1, e_2, e_1\}$*, Levi decomposition* $\mathfrak{g}$ $\{\{e_1, e_2\}, \{-(1/2)e_1 + e_5, e_2 - 2e_3, 2e_4\}\}$*), 4.2.1 (*$\lambda = 1/2$*, Levi decomposition* $\bar{\mathfrak{g}}$ $\{\{u_3, e_1, 4u_1, -4u_2\}, \{32e_3 + 8u_1, -32e_4, 16e_2 - 4u_2\}\}$*, Levi decomposition* $\mathfrak{g}$ $\{\{e_1\}, \{-4e_2, -4e_3, -4e_4\}\}$*).*

Pair	Connection
5.3.1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -r_{1,2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4.2.1, $\lambda = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -p_{3,2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Pair	**Torsion tensor**
5.3.1	$(0,0,0), (0,0,0), (-2r_{1,2}, 0, 0)$
4.2.1, $\lambda = 1/2$	$(0, 0, 2p_{3,2}), (0,0,0), (0,0,0)$

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Theorem 13. *If $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature tensor is zero for all connections, torsion tensor is not zero for some connections, $\mathfrak{g}$ is solvable then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs:*

Pair	Levi decomposition
4.21.11, $\mu = 1$	$\{\{-e_4, 2e_1 + u_2, u_1, -2e_2 - 2u_1\}, \{-4e_3, e_2 + u_1 + 4u_3, -e_4 + 4u_2\}\}$
3.13.6, $\mu = 1$	$\{\{-u_1, -2e_1 + u_2, -e_3\}, \{-(1/2)u_1 - u_2, -2e_2 - e_3, -2u_3\}\}$
2.8.7, $\lambda = 1$	$\{\{u_2, -e_2 + u_1\}, \{e_1, u_1, u_3\}\}$
Pair	**Connection**
4.21.11, $\mu = 1$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & q_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -q_{1,3} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
3.13.6, $\mu = 1$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & q_{1,3} \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -q_{1,3} & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
2.8.7, $\lambda = 1$	$\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & p_{2,3} \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -p_{2,3} & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}$
Pair	**Torsion tensor**
4.21.11, $\mu = 1$	$(0,0,0), (0,0,0), (2q_{1,3}, 0, 0)$
3.13.6, $\mu = 1$	$(0,0,0), (0,0,0), (2q_{1,3}, 0, 0)$
2.8.7, $\lambda = 1$	$(0,0,0), (0, 2p_{2,3}, 0), (0,0,0)$

If $\mathfrak{g}$ is solvable then the radical of $\bar{\mathfrak{g}}$ is noncommutative, commutant of the radical of $\bar{\mathfrak{g}}$ is commutative.

4.3. The Curvature Tensor is Not Zero for Some Connections, Torsion Tensor is Zero for All Connections

In this case $\bar{\mathfrak{g}}$ is not semisimple and $\mathfrak{g}$ is solvable.

Theorem 14. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature tensor is not zero for some connections, torsion tensor is zero for all connections then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
4.21.11, $\mu = 1/2$	$\{\{(-1/2)e_2, (-1/2)u_1, (-1/8)e_4, 4e_1 + u_2, u_1\},$ $\{(-1/2)e_3, (1/16)e_2 + (1/16)u_1 + (1/2)u_3, (-1/8)e_4 + u_2\}\}$
3.13.6, $\mu = 1/2$	$\{-42e_1 + u_2, u_1, e_3\}, \{u_1 + 2u_2, 2e_2 + e_3, u_3\}\}$
2.8.7, $\lambda \neq 0$	$\{\{u_2, (-1/\lambda)e_2 + u_1\}, \{\lambda^3 e_1, \lambda^2 u_1, \lambda^3 u_3\}\}$
Pair	**Connection**
4.21.11, $\mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
3.13.6, $\mu = 1/2$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & r_{1,3} \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
2.8.7, $\lambda = 1/2$	$\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r_{2,3} \\ -1/2 & 0 & 0 \end{pmatrix}$

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Pair	Curvature tensor
$4.21.11, \mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$3.13.6, \mu = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -3r_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$2.8.7, \lambda = 1/2$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3r_{2,3}/2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Pair		Holonomy algebra $\bar{\mathfrak{g}}$
$4.21.11, \mu = 1/2$ $3.13.6, \mu = 1/2$	$r_{1,3} \neq 0$	$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{1,3} = 0$	is equal to zero
$2.8.7, \lambda = 1/2$	$r_{2,3} \neq 0$	$\begin{pmatrix} 0 & 0 & 0 \\ p_1 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix}$
	$r_{2,3} = 0$	is equal to zero

In this case if the radical of $\bar{\mathfrak{g}}$ is not commutative then commutant of the radical of $\bar{\mathfrak{g}}$ is commutative.

4.4. The Curvature and Torsion Tensors are Not Zero for Some Connections

Theorem 15. *If $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections, $\bar{\mathfrak{g}}$ is semisimple and $\mathfrak{g}$ is semisimple too then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs:*

Pair	Connection
3.4.2 3.4.3	$\begin{pmatrix} 0 & p_{1,2} & 0 \\ 0 & 0 & p_{1,2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -p_{1,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{1,2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -p_{1,2} & 0 & 0 \\ 0 & -p_{1,2} & 0 \end{pmatrix}$
3.5.2 3.5.3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{2,3} \\ 0 & -p_{2,3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -p_{2,3} \\ 0 & 0 & 0 \\ p_{2,3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_{2,3} & 0 \\ -p_{2,3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Pair	**Curvature tensor**
3.4.2	$\begin{pmatrix} 0 & {p_{1,2}}^2-1 & 0 \\ 0 & 0 & {p_{1,2}}^2-1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -{p_{1,2}}^2+1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {p_{1,2}}^2-1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -{p_{1,2}}^2+1 & 0 & 0 \\ 0 & -{p_{1,2}}^2+1 & 0 \end{pmatrix}$
3.4.3	$\begin{pmatrix} 0 & {p_{1,2}}^2+1 & 0 \\ 0 & 0 & {p_{1,2}}^2+1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -{p_{1,2}}^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {p_{1,2}}^2+1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -{p_{1,2}}^2-1 & 0 & 0 \\ 0 & -{p_{1,2}}^2-1 & 0 \end{pmatrix}$
3.5.2	$\begin{pmatrix} 0 & -{p_{2,3}}^2-1 & 0 \\ {p_{2,3}}^2+1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -{p_{2,3}}^2-1 \\ 0 & 0 & 0 \\ {p_{2,3}}^2+1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -{p_{2,3}}^2-1 \\ 0 & {p_{2,3}}^2+1 & 0 \end{pmatrix}$
3.5.3	$\begin{pmatrix} 0 & -{p_{2,3}}^2+1 & 0 \\ {p_{2,3}}^2-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -{p_{2,3}}^2+1 \\ 0 & 0 & 0 \\ {p_{2,3}}^2-1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -{p_{2,3}}^2+1 \\ 0 & {p_{2,3}}^2-1 & 0 \end{pmatrix}$

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Pair	Torsion tensor
3.4.2, 3.4.3	$(2p_{1,2},0,0)\,,(0,2p_{1,2},0)\,,(0,0,2p_{1,2})$
3.5.2, 3.5.3	$(0,0,-2p_{2,3})\,,(0,2p_{2,3},0)\,,(-2p_{2,3},0,0)$

Pair		Holonomy algebra	Pair	Holonomy algebra
3.4.2	${p_{1,2}}^2 \neq 1$	$\begin{pmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & p_1 \\ 0 & p_3 & -p_2 \end{pmatrix}$	3.4.3	$\begin{pmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & p_1 \\ 0 & p_3 & -p_2 \end{pmatrix}$
	${p_{1,2}}^2 = 1$	is equal to zero		
3.5.3	${p_{2,3}}^2 \neq 1$	$\begin{pmatrix} 0 & -p_1 & -p_2 \\ p_1 & 0 & -p_3 \\ p_2 & p_3 & 0 \end{pmatrix}$	3.5.2	$\begin{pmatrix} 0 & -p_1 & -p_2 \\ p_1 & 0 & -p_3 \\ p_2 & p_3 & 0 \end{pmatrix}$
	${p_{2,3}}^2 = 1$	is equal to zero		

Theorem 16. If $(\bar{\mathfrak{g}},\mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections, $\bar{\mathfrak{g}}$ is semisimple and $\mathfrak{g}$ is not semisimple then $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to 5.3.2 (Levi decomposition $\{\{e_1,e_2\},\{-(1/2)e_1+e_5,e_2-2e_3,2e_4\}\}$), the connection $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & -r_{1,2} \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & r_{1,2} & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, the curvature tensor $\begin{pmatrix} 0 & 0 & -6r_{1,2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 6r_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -4r_{1,2} & 0 & 0 \\ 0 & 2r_{1,2} & 0 \\ 0 & 0 & 2r_{1,2} \end{pmatrix}$, the torsion tensor $(0,0,0),(0,0,0),(-2r_{1,2},0,0)$, if $r_{1,2}\neq 0$ then holonomy algebra $\mathfrak{sl}(3,\mathbb{R})$ else is equal to zero.

Theorem 17. *If the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections, $\bar{\mathfrak{g}}$ is not semisimple (or solvable), $\mathfrak{g}$ is semisimple and the radical of $\bar{\mathfrak{g}}$ is noncommutative (in this case commutant of the radical of $\bar{\mathfrak{g}}$ is commutative) then $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition
3.3.2	$\{\{u_3,-u_2,u_1\},\{2e_2+2u_1+u_3,-2e_3,e_1-u_2\}\}$
3.3.3	$\{\{-2u_2,-2u_1,u_3\},\{-4e_1+2u_2,-4e_2-2u_1,-4e_3\}\}$

Pair	Connection
3.3.2 3.3.3	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & p_{3,2} & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ -p_{3,2} & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{3,3} \end{pmatrix}$

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Pair	Curvature tensor
3.3.2	$\begin{pmatrix} -p_{1,3}p_{3,2}-r_{1,1} & 0 & 0 \\ 0 & -p_{1,3}p_{3,2}-r_{1,1} & 0 \\ 0 & 0 & 2p_{1,3}p_{3,2}-r_{3,3} \end{pmatrix}, \begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3} \\ 0 & 0 & 0 \\ 0 & p_{3,2}r_{1,1}-r_{3,3}p_{3,2} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3} \\ -p_{3,2}r_{1,1}+r_{3,3}p_{3,2} & 0 & 0 \end{pmatrix}$
3.3.3	$\begin{pmatrix} -p_{1,3}p_{3,2} & 0 & 0 \\ 0 & -p_{1,3}p_{3,2} & 0 \\ 0 & 0 & 2p_{1,3}p_{3,2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-p_{1,3} \\ 0 & 0 & 0 \\ 0 & p_{3,2}r_{1,1}-r_{3,3}p_{3,2}-p_{3,2} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-p_{1,3} \\ -p_{3,2}r_{1,1}+r_{3,3}p_{3,2}+p_{3,2} & 0 & 0 \end{pmatrix}$

Pair	Torsion tensor
3.3.2	$(0,0,2p_{3,2}-1), (p_{1,3}-r_{1,1},0,0), (0,p_{1,3}-r_{1,1},0)$
3.3.3	$(0,0,2p_{3,2}), (p_{1,3}-r_{1,1}-1,0,0), (0,p_{1,3}-r_{1,1}-1,0)$

Pair		Holonomy algebra
3.3.2	$p_{1,3}\neq 0, p_{3,2}\neq 0, r_{3,3}\neq -2r_{1,1}$	$\mathfrak{gl}(3,\mathbb{R})$
	$p_{1,3}\neq 0, p_{3,2}\neq 0, r_{3,3}= -2r_{1,1}$	$\mathfrak{sl}(3,\mathbb{R})$
	$p_{1,3}=0, p_{3,2}\neq 0, r_{3,3}\neq r_{1,1}, r_{1,1}\neq 0$	$\begin{pmatrix} p_3 & 0 & 0 \\ 0 & p_3 & 0 \\ p_1 & p_2 & r_{3,3}p_3/r_{1,1} \end{pmatrix}$
	$p_{1,3}=0, p_{3,2}\neq 0, r_{3,3}\neq r_{1,1}, r_{1,1}= 0$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_1 & p_2 & p_3 \end{pmatrix}$
	$p_{1,3}\neq 0, p_{3,2}= 0, r_{3,3}\neq r_{1,1}, r_{1,1}\neq 0$	$\begin{pmatrix} p_3 & 0 & p_1 \\ 0 & p_3 & p_2 \\ 0 & 0 & r_{3,3}p_3/r_{1,1} \end{pmatrix}$
	$p_{1,3}\neq 0, p_{3,2}=r_{1,1}=0, r_{3,3}\neq r_{1,1}$	$\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ 0 & 0 & p_3 \end{pmatrix}$
	$p_{1,3}=p_{3,2}=0, r_{3,3}\neq r_{1,1}, r_{1,1}\neq 0$	$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & r_{3,3}p_1/r_{1,1} \end{pmatrix}$
	$p_{1,3}=p_{3,2}=0, r_{3,3}\neq r_{1,1}=0$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$
	$p_{1,3}p_{3,2}=0, r_{3,3}=r_{1,1}, r_{1,1}\neq 0$	$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$
	else	is equal to zero
3.3.3	as in 3.3.2, but	$\operatorname{tr}\mathfrak{h}^* = 0$

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Theorem 18. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections, $\bar{\mathfrak{g}}$ is not semisimple (or solvable), $\mathfrak{g}$ is semisimple and the radical of $\bar{\mathfrak{g}}$ is commutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the following pairs:*

Pair	Levi decomposition
3.3.1	$\{\{-2u_2, -2u_1, u_3\}, \{-4e_1 + 2u_2, -4e_2 - 2u_1, -4e_3\}\}$
3.4.1	$\{\{u_2, -u_3, u_1\}, \{e_2, -e_3 - u_2, e_1 + u_1\}\}$
3.5.1	$\{\{-u_2, u_1, -u_3\}, \{e_3, -e_2 + u_2, e_1 - u_3\}\}$
Pair	**Connection**
3.3.1	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & p_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ -p_{3,2} & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{3,3} \end{pmatrix}$
3.4.1	$\begin{pmatrix} 0 & p_{1,2} & 0 \\ 0 & 0 & p_{1,2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -p_{1,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{1,2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -p_{1,2} & 0 & 0 \\ 0 & -p_{1,2} & 0 \end{pmatrix}$
3.5.1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{2,3} \\ 0 & -p_{2,3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -p_{2,3} \\ 0 & 0 & 0 \\ p_{2,3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_{2,3} & 0 \\ -p_{2,3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Pair	**Curvature tensor**
3.3.1	$\begin{pmatrix} -p_{1,3}p_{3,2} & 0 & 0 \\ 0 & -p_{1,3}p_{3,2} & 0 \\ 0 & 0 & 2p_{1,3}p_{3,2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3} - r_{1,1}p_{1,3} \\ 0 & 0 & 0 \\ 0 & p_{3,2}r_{1,1} - r_{3,3}p_{3,2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3}r_{3,3} - r_{1,1}p_{1,3} \\ -p_{3,2}r_{1,1} + r_{3,3}p_{3,2} & 0 & 0 \end{pmatrix}$
3.4.1	$\begin{pmatrix} 0 & {p_{1,2}}^2 & 0 \\ 0 & 0 & {p_{1,2}}^2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -{p_{1,2}}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {p_{1,2}}^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -{p_{1,2}}^2 & 0 & 0 \\ 0 & -{p_{1,2}}^2 & 0 \end{pmatrix}$
3.5.1	$\begin{pmatrix} 0 & -{p_{2,3}}^2 & 0 \\ {p_{2,3}}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -{p_{2,3}}^2 \\ 0 & 0 & 0 \\ {p_{2,3}}^2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -{p_{2,3}}^2 \\ 0 & {p_{2,3}}^2 & 0 \end{pmatrix}$
Pair	**Torsion tensor**
3.3.1	$(0, 0, 2p_{3,2}), (p_{1,3} - r_{1,1}, 0, 0), (0, p_{1,3} - r_{1,1}, 0)$
3.4.1	$(2p_{1,2}, 0, 0), (0, 2p_{1,2}, 0), (0, 0, 2p_{1,2})$
3.5.1	$(0, 0, -2p_{2,3}), (0, 2p_{2,3}, 0), (-2p_{2,3}, 0, 0)$

Pair		Holonomy algebra
3.4.1	$p_{1,2} \neq 0$	$\begin{pmatrix} p_2 & p_1 & 0 \\ p_3 & 0 & p_1 \\ 0 & p_3 & -p_2 \end{pmatrix}$
	$p_{1,2} = 0$	is equal to zero
3.5.1	$p_{2,3} \neq 0$	$\begin{pmatrix} 0 & -p_1 & -p_2 \\ p_1 & 0 & -p_3 \\ p_2 & p_3 & 0 \end{pmatrix}$
	$p_{2,3} = 0$	is equal to zero
3.3.1	as in 3.3.2,	but $\operatorname{tr} \mathfrak{h}^* = 0$

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Theorem 19. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections,* $\bar{\mathfrak{g}}$ *is not semisimple (or solvable),* $\mathfrak{g}$ *is not semisimple or solvable then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition $\bar{\mathfrak{g}}$
6.1.1	$\{\{-2u_1, -2e_5, u_3, e_4, -2e_6, -2u_2\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
6.1.2	$\{\{e_4, -2u_2, -2e_6, -2u_1, -2e_5, u_3\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
6.1.3	$\{\{-2u_2, -2u_1, -2e_6, -2e_5, u_3, e_4\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
5.2.1	$\{\{-2e_5, -2u_1, u_3, -2u_2, -2e_4\}, \{-4e_1 + 2e_5, -4e_2 - 2e_4, -4e_3\}\}$
5.2.2	$\{\{-2e_4, -2u_1, u_3, -2u_2, -2e_5\}, \{-4e_1 + 2e_5 - 4e_2 - 2e_4, -4e_3\}\}$
5.2.3	$\{\{u_3, -2u_2, -2u_1, -2e_5, -2e_4\}, \{-4e_1 + 2e_5, -4e_2 - 2e_4, -4e_3\}\}$
4.1.1	$\{\{-u_1, 2u_2, u_3, e_1\}, \{-4e_2 + 2u_1, -4e_3, -4e_4 + 2u_2\}\}$
Pair	**Levi decomposition $\mathfrak{g}$**
6.1.1	$\{\{-2e_6, -2e_5, e_4\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
6.1.2	$\{\{-2e_5, e_4, -2e_6\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
6.1.3	$\{\{-2e_6, -2e_5, e_4\}, \{-4e_1 + 2e_6, -4e_2 - 2e_5, -4e_3\}\}$
5.2.1	$\{\{-e_5, e_4\}, \{2e_2, -2e_3 - 2e_5, e_1 + e_4\}\}$
5.2.2	$\{\{-e_5, e_4\}, \{2e_2, -2e_3 - 2e_{5,1} + e_4\}\}$
5.2.3	$\{\{-e_5, e_4\}, \{2e_2, -2e_3 - 2e_5, e_1 + e_4\}\}$
4.1.1	$\{\{e_1\}, \{-4u_1, -4u_2, 4u_3\}\}$

Pair	Connection
6.1.1 6.1.2 6.1.3	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1} + p_{1,3} \end{pmatrix}$
5.2.1	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1} + q_{2,3} \end{pmatrix}$
5.2.2	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1} + q_{2,3} + 1 \end{pmatrix}$
5.2.3	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1} + q_{2,3} + \lambda \end{pmatrix}$
4.1.1	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{3,3} \end{pmatrix}$

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Pair	Curvature tensor
6.1.1	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{p_{1,3}}^2\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{p_{1,3}}^2\\0&0&0\end{pmatrix}$
6.1.2	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{p_{1,3}}^2-1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{p_{1,3}}^2-1\\0&0&0\end{pmatrix}$
6.1.3	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{p_{1,3}}^2+1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{p_{1,3}}^2+1\\0&0&0\end{pmatrix}$
5.2.1	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{q_{2,3}}^2\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{q_{2,3}}^2\\0&0&0\end{pmatrix}$
5.2.2	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{q_{2,3}}^2+q_{2,3}-\lambda q_{2,3}\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{q_{2,3}}^2+q_{2,3}-\lambda q_{2,3}\\0&0&0\end{pmatrix}$
5.2.3	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&{q_{2,3}}^2+1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&{q_{2,3}}^2+1\\0&0&0\end{pmatrix}$
4.1.1	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&p_{1,3}r_{3,3}-r_{1,1}p_{1,3}\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&p_{1,3}r_{3,3}-r_{1,1}p_{1,3}\\0&0&0\end{pmatrix}$

Pair	Torsion tensor
6.1.1	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(0,p_{1,3}-r_{1,1},0)$
6.1.2	$(0,0,0),(-p_{1,3}+r_{1,1},0,0),(0,-p_{1,3}+r_{1,1},0)$
6.1.3	$(0,0,0),(-p_{1,3}+r_{1,1},0,0),(0,-p_{1,3}+r_{1,1},0)$
5.2.1	$(0,0,0),(q_{2,3}-r_{1,1},0,0),(0,q_{2,3}-r_{1,1},0)$
5.2.2	$(0,0,0),(q_{2,3}-r_{1,1}-\lambda,0,0),(0,q_{2,3}-r_{1,1}-\lambda,0)$
5.2.3	$(0,0,0),(q_{2,3}-r_{1,1}-\lambda,0,0),(0,q_{2,3}-r_{1,1}-\lambda,0)$
4.1.1	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(0,p_{1,3}-r_{1,1},0)$

Pair	Holonomy	algebra	Pair	Holonomy	algebra
6.1.1	$p_{1,3} \neq 0$	$\mathfrak{p}$	6.1.2	${p_{1,3}}^2 \neq 1$	$\mathfrak{p}$
	$p_{1,3} = 0$	is equal to zero		${p_{1,3}}^2 = 1$	is equal to zero
6.1.3		$\mathfrak{p}$	5.2.1	$q_{2,3} \neq 0$	$\mathfrak{p}$
				$q_{2,3} = 0$	is equal to zero
5.2.2	$q_{2,3} \neq 0, \lambda - 1$	$\mathfrak{p}$	4.1.1	$p_{1,3} \neq 0, r_{3,3} \neq r_{1,1}$	$\mathfrak{p}$
	$q_{2,3} = 0, \lambda - 1$	is equal to zero		$p_{1,3}(r_{3,3} - r_{1,1}) = 0$	is equal to zero
5.2.3		$\mathfrak{p}$			

Here $\mathfrak{p} = \begin{pmatrix}0&0&p_1\\0&0&p_2\\0&0&0\end{pmatrix}$.

In this case if the $\mathfrak{g}$ is not semisimple then the radical of $\bar{\mathfrak{g}}$ is not commutative and commutant of the radical of $\bar{\mathfrak{g}}$ is commutative.

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Theorem 20. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections,* $\bar{\mathfrak{g}}$ *is not semisimple (or solvable),* $\mathfrak{g}$ *is solvable and the radical of* $\bar{\mathfrak{g}}$ *is commutative then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

Pair	Levi decomposition
3.19.14	$\{\{-e_2, u_1, e_3\}, \{-u_1+u_2, -u_3, e_1-e_2\}\}$
3.21.6	$\{\{-e_3, e_2, u_1\}, \{-u_3, -u_1+u_2, e_1-e_3\}\}$
3.21.7	$\{\{u_1, e_2, -e_3\}, \{-u_3, u_1+u_2, -e_1-e_3\}\}$
3.25.30	$\{\{2u_1, e_1, e_2\}, \{4e_3, e_2+4u_2, u_1+4u_3\}\}$
2.7.2	$\{\{u_1, u_2\}, \{e_1, e_2+u_1, u_3\}\}$
2.17.27	$\{\{u_2, e_2\}, \{2e_1, e_2+u_1, 2e_2+2u_3\}\}$
1.1.5	$\{\{u_3\}, \{-e_1, -u_1, -u_2\}\}$
1.1.7	$\{\{u_3\}, \{-e_1-u_3, -u_1, -u_2\}\}$
1.3.3	$\{\{u_3\}, \{e_1+u_3, u_1, u_2\}\}$
1.3.4	$\{\{u_3\}, \{-e_1+u_3, -u_1, -u_2\}\}$
1.3.5	$\{\{u_3\}, \{e_1, u_1, u_2\}\}$
1.3.6	$\{\{u_3\}, \{-e_1, -u_1, -u_2\}\}$
1.5.19	$\{\{u_2\}, \{e_1, u_1, u_3\}\}$
1.8.2	$\{\{-e_1+u_1\}, \{u_1, u_2, u_3\}\}$

Pair	Connection
3.19.14	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&q_{1,3}\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&r_{1,2}&0\\0&0&0\\0&0&0\end{pmatrix}$
3.21.6 3.21.7	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&q_{1,2}&q_{1,3}\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&-q_{1,3}&q_{1,2}\\0&0&0\\0&0&0\end{pmatrix}$
3.25.30	$\begin{pmatrix}0&-1&p_{1,3}\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&q_{1,3}\\0&-1&p_{1,3}\\0&0&1\end{pmatrix}, \begin{pmatrix}r_{1,1}&-q_{1,3}&r_{1,3}\\0&r_{1,1}&0\\0&-1&r_{1,1}+p_{1,3}\end{pmatrix}$
2.7.2	$\begin{pmatrix}-1/2&p_{1,2}&0\\0&0&0\\0&0&-1/2\end{pmatrix}, \begin{pmatrix}q_{1,1}&q_{1,2}&0\\0&q_{2,2}&0\\0&0&q_{1,1}\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\-1/2&p_{1,2}&0\end{pmatrix}$
2.17.27	$\begin{pmatrix}-1&0&p_{1,3}\\0&0&p_{2,3}\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&q_{1,3}\\-1&0&q_{2,3}\\0&0&0\end{pmatrix}, \begin{pmatrix}r_{1,1}&-q_{1,3}&r_{1,3}\\-p_{2,3}&r_{1,1}+p_{1,3}-q_{2,3}&r_{2,3}\\-1&0&r_{1,1}+p_{1,3}\end{pmatrix}$
1.1.5 1.1.7	$\begin{pmatrix}0&0&p_{1,3}\\0&0&0\\0&p_{3,2}&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&q_{2,3}\\q_{3,1}&0&0\end{pmatrix}, \begin{pmatrix}r_{1,1}&0&0\\0&r_{2,2}&0\\0&0&r_{3,3}\end{pmatrix}$
1.3.3 1.3.4	$\begin{pmatrix}0&0&p_{1,3}\\0&0&p_{2,3}\\p_{3,1}&p_{3,2}&0\end{pmatrix}, \begin{pmatrix}0&0&-p_{2,3}\\0&0&p_{1,3}\\-p_{3,2}&p_{3,1}&0\end{pmatrix}, \begin{pmatrix}r_{1,1}&r_{1,2}&0\\-r_{1,2}&r_{1,1}&0\\0&0&r_{3,3}\end{pmatrix}$
1.3.5 1.3.6	$\begin{pmatrix}0&0&p_{1,3}\\0&0&p_{2,3}\\p_{3,1}&p_{3,2}&0\end{pmatrix}, \begin{pmatrix}0&0&-p_{2,3}\\0&0&p_{1,3}\\-p_{3,2}&p_{3,1}&0\end{pmatrix}, \begin{pmatrix}r_{1,1}&r_{1,2}&0\\-r_{1,2}&r_{1,1}&0\\0&0&r_{3,3}\end{pmatrix}$
1.5.19	$\begin{pmatrix}-1/2&p_{1,2}&p_{1,3}\\0&0&p_{2,3}\\0&0&p_{1,1}+1\end{pmatrix}, \begin{pmatrix}q_{1,1}&q_{1,2}&q_{1,3}\\0&q_{2,2}&q_{2,3}\\0&0&q_{1,1}\end{pmatrix}, \begin{pmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\-p_{2,3}&r_{2,2}&r_{2,3}\\-1/2&p_{1,2}&r_{1,1}+p_{1,3}\end{pmatrix}$
1.8.2	$\begin{pmatrix}0&p_{1,2}&p_{1,3}\\0&0&p_{1,2}\\0&0&0\end{pmatrix}, \begin{pmatrix}-p_{1,2}&q_{1,2}&q_{1,3}\\0&0&q_{1,2}+p_{1,3}\\0&0&p_{1,2}\end{pmatrix}, \begin{pmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\-p_{1,2}&r_{1,1}+q_{1,2}&r_{1,2}+q_{1,3}\\0&-p_{1,2}&r_{1,1}+2q_{1,2}+p_{1,3}\end{pmatrix}$

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Pair	Curvature tensor
3.19.14	$\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix}$
3.21.6	$\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$
3.21.7	$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$
3.25.30	$\begin{pmatrix}0&0&p_{1,3}\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&-p_{1,3}&{p_{1,3}}^2\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}-2r_{1,1}&0&2q_{1,3}p_{1,3}-3r_{1,3}\\0&-p_{1,3}-2r_{1,1}&{p_{1,3}}^2\\0&0&-p_{1,3}-2r_{1,1}\end{pmatrix}$
2.7.2	$\begin{pmatrix}0&-q_{1,2}/2+p_{1,2}q_{2,2}-q_{1,1}p_{1,2}&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},$ $\begin{pmatrix}0&0&0\\0&0&0\\0&q_{1,1}p_{1,2}+q_{1,2}/2-p_{1,2}q_{2,2}&0\end{pmatrix}$
2.17.27	$\begin{pmatrix}0&0&-3q_{1,3}\\0&0&p_{1,3}-2q_{2,3}\\0&0&0\end{pmatrix},$ $\begin{pmatrix}-p_{1,3}-2r_{1,1}&3q_{1,3}&-4r_{1,3}+{p_{1,3}}^2+q_{1,3}p_{2,3}\\0&-2r_{1,1}-2p_{1,3}+2q_{2,3}&p_{2,3}p_{1,3}+p_{2,3}q_{2,3}-3r_{2,3}\\0&0&-p_{1,3}-2r_{1,1}\end{pmatrix},$ $\begin{pmatrix}-2q_{1,3}&0&q_{1,3}p_{1,3}+q_{1,3}q_{2,3}\\p_{1,3}-2q_{2,3}&q_{1,3}&-r_{1,3}+q_{1,3}p_{2,3}+{q_{2,3}}^2\\0&0&q_{1,3}\end{pmatrix}$
1.1.5	$\begin{pmatrix}p_{1,3}q_{3,1}-1&0&0\\0&-p_{3,2}q_{2,3}+1&0\\0&0&p_{3,2}q_{2,3}-p_{1,3}q_{3,1}\end{pmatrix},\begin{pmatrix}0&0&p_{1,3}r_{3,3}-r_{1,1}p_{1,3}\\0&0&0\\0&p_{3,2}r_{2,2}-r_{3,3}p_{3,2}&0\end{pmatrix},$ $\begin{pmatrix}0&0&0\\0&0&q_{2,3}r_{3,3}-r_{2,2}q_{2,3}\\q_{3,1}r_{1,1}-r_{3,3}q_{3,1}&0&0\end{pmatrix}$
1.1.7	$\begin{pmatrix}p_{1,3}q_{3,1}-1-r_{1,1}&0&0\\0&-p_{3,2}q_{2,3}+1-r_{2,2}&0\\0&0&p_{3,2}q_{2,3}-p_{1,3}q_{3,1}-r_{3,3}\end{pmatrix},$ $\begin{pmatrix}0&0&p_{1,3}r_{3,3}-r_{1,1}p_{1,3}\\0&0&0\\0&p_{3,2}r_{2,2}-r_{3,3}p_{3,2}&0\end{pmatrix},$ $\begin{pmatrix}0&0&0\\0&0&q_{2,3}r_{3,3}-r_{2,2}q_{2,3}\\q_{3,1}r_{1,1}-r_{3,3}q_{3,1}&0&0\end{pmatrix}$

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1.3.3	$\begin{pmatrix} -p_{1,3}p_{3,2}+p_{2,3}p_{3,1}-r_{1,1} & p_{1,3}p_{3,1}+p_{2,3}p_{3,2}-1-r_{1,2} & 0 \\ -p_{2,3}p_{3,2}-p_{1,3}p_{3,1}+1+r_{1,2} & -p_{1,3}p_{3,2}+p_{2,3}p_{3,1}-r_{1,1} & 0 \\ 0 & 0 & -2p_{2,3}p_{3,1}+2p_{1,3}p_{3,2}-r_{3,3} \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ 0 & 0 & p_{2,3}r_{3,3}+r_{1,2}p_{1,3}-r_{1,1}p_{2,3} \\ p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & p_{3,1}r_{1,2}+p_{3,2}r_{1,1}-r_{3,3}p_{3,2} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & -p_{2,3}r_{3,3}-r_{1,2}p_{1,3}+r_{1,1}p_{2,3} \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ -p_{3,2}r_{1,1}-p_{3,1}r_{1,2}+r_{3,3}p_{3,2} & p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & 0 \end{pmatrix}$
1.3.4	$\begin{pmatrix} -p_{1,3}p_{3,2}+p_{2,3}p_{3,1}-r_{1,1} & p_{1,3}p_{3,1}+p_{2,3}p_{3,2}+1-r_{1,2} & 0 \\ -p_{2,3}p_{3,2}-p_{1,3}p_{3,1}-1+r_{1,2} & -p_{1,3}p_{3,2}+p_{2,3}p_{3,1}-r_{1,1} & 0 \\ 0 & 0 & -2p_{2,3}p_{3,1}+2p_{1,3}p_{3,2}-r_{3,3} \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ 0 & 0 & p_{2,3}r_{3,3}+r_{1,2}p_{1,3}-r_{1,1}p_{2,3} \\ p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & p_{3,1}r_{1,2}+p_{3,2}r_{1,1}-r_{3,3}p_{3,2} & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & -p_{2,3}r_{3,3}-r_{1,2}p_{1,3}+r_{1,1}p_{2,3} \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ -p_{3,2}r_{1,1}-p_{3,1}r_{1,2}+r_{3,3}p_{3,2} & p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & 0 \end{pmatrix}$
1.3.5	$\begin{pmatrix} -p_{1,3}p_{3,2}+p_{2,3}p_{3,1} & p_{1,3}p_{3,1}+p_{2,3}p_{3,2}-1 & 0 \\ -p_{2,3}p_{3,2}-p_{1,3}p_{3,1}+1 & -p_{1,3}p_{3,2}+p_{2,3}p_{3,1} & 0 \\ 0 & 0 & -2p_{2,3}p_{3,1}+2p_{1,3}p_{3,2} \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ 0 & 0 & p_{2,3}r_{3,3}+r_{1,2}p_{1,3}-r_{1,1}p_{2,3} \\ p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & p_{3,1}r_{1,2}+p_{3,2}r_{1,1}-r_{3,3}p_{3,2} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & -p_{2,3}r_{3,3}-r_{1,2}p_{1,3}+r_{1,1}p_{2,3} \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ -p_{3,2}r_{1,1}-p_{3,1}r_{1,2}+r_{3,3}p_{3,2} & p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & 0 \end{pmatrix}$
1.3.6	$\begin{pmatrix} -p_{1,3}p_{3,2}+p_{2,3}p_{3,1} & p_{1,3}p_{3,1}+p_{2,3}p_{3,2}+1 & 0 \\ -p_{2,3}p_{3,2}-p_{1,3}p_{3,1}-1 & -p_{1,3}p_{3,2}+p_{2,3}p_{3,1} & 0 \\ 0 & 0 & -2p_{2,3}p_{3,1}+2p_{1,3}p_{3,2} \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ 0 & 0 & p_{2,3}r_{3,3}+r_{1,2}p_{1,3}-r_{1,1}p_{2,3} \\ p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & p_{3,1}r_{1,2}+p_{3,2}r_{1,1}-r_{3,3}p_{3,2} & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & 0 & -p_{2,3}r_{3,3}-r_{1,2}p_{1,3}+r_{1,1}p_{2,3} \\ 0 & 0 & p_{1,3}r_{3,3}-r_{1,1}p_{1,3}-r_{1,2}p_{2,3} \\ -p_{3,2}r_{1,1}-p_{3,1}r_{1,2}+r_{3,3}p_{3,2} & p_{3,1}r_{1,1}-p_{3,2}r_{1,2}-r_{3,3}p_{3,1} & 0 \end{pmatrix}$
1.5.19	$\begin{pmatrix} 0 & -q_{1,2}/2+p_{1,2}q_{2,2}-q_{1,1}p_{1,2} & -q_{1,3}+p_{1,2}q_{2,3}-q_{1,2}p_{2,3} \\ 0 & 0 & p_{2,3}q_{1,1}-q_{2,2}p_{2,3}-q_{2,3}/2 \\ 0 & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} -p_{1,2}p_{2,3}-p_{1,3}/2-r_{1,1} & -3r_{1,2}/2+p_{1,2}r_{2,2}+p_{1,3}p_{1,2}-r_{1,1}p_{1,2} & A \\ 0 & 2p_{1,2}p_{2,3}-r_{2,2} & B \\ 0 & 0 & C \end{pmatrix},$ $A=-2r_{1,3}+p_{1,2}r_{2,3}+{p_{1,3}}^2-r_{1,2}p_{2,3},$ $B=p_{2,3}r_{1,1}+2p_{2,3}p_{1,3}-r_{2,2}p_{2,3}-3r_{2,3}/2,$ $C=-p_{1,2}p_{2,3}-p_{1,3}/2-r_{1,1},$

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1.8.2	$\begin{pmatrix} -q_{1,2}p_{2,3}-q_{1,3}/2 & q_{1,1}r_{1,2}+q_{1,2}r_{2,2}+ \\ & +q_{1,3}p_{1,2}-r_{1,1}q_{1,2}-r_{1,2}q_{2,2} & D \\ p_{2,3}q_{1,1}-q_{2,2}p_{2,3}-q_{2,3}/2 & p_{1,2}q_{2,3}+q_{1,2}p_{2,3} & H \\ 0 & q_{1,1}p_{1,2}+q_{1,2}/2-p_{1,2}q_{2,2} & q_{1,3}/2-p_{1,2}q_{2,3} \end{pmatrix}$, $D=q_{1,2}r_{2,3}+q_{1,3}p_{1,3}-r_{1,2}q_{2,3}$, $H=q_{2,2}r_{2,3}+q_{2,3}r_{1,1}+q_{2,3}p_{1,3}+p_{2,3}q_{1,3}-r_{2,2}q_{2,3}-r_{2,3}q_{1,1}$ $\begin{pmatrix} 0 & {p_{1,2}}^2-p_{1,2} & 3p_{1,3}p_{1,2}-p_{1,3} \\ 0 & 0 & {p_{1,2}}^2-p_{1,2} \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -{p_{1,2}}^2+p_{1,2} & q_{1,2}p_{1,2}-p_{1,3}p_{1,2}-q_{1,2} & p_{1,2}q_{1,3}+2p_{1,3}q_{1,2}+{p_{1,3}}^2-q_{1,3} \\ 0 & 0 & q_{1,2}p_{1,2}+2p_{1,3}p_{1,2}-q_{1,2}-p_{1,3} \\ 0 & 0 & {p_{1,2}}^2-p_{1,2} \end{pmatrix}$, $\begin{pmatrix} -q_{1,2}p_{1,2}-r_{1,1} & -r_{1,2}p_{1,2}+{q_{1,2}}^2-p_{1,2}q_{1,3}-r_{1,2} & A \\ -{p_{1,2}}^2+p_{1,2} & -p_{1,3}p_{1,2}-r_{1,1}-q_{1,2} & B \\ 0 & -{p_{1,2}}^2+p_{1,2} & C \end{pmatrix}$, $A=-2p_{1,2}r_{1,3}+3q_{1,2}q_{1,3}+q_{1,3}p_{1,3}-r_{1,2}p_{1,3}-r_{1,3}$, $B={q_{1,2}}^2+2p_{1,3}q_{1,2}+{p_{1,3}}^2-r_{1,2}p_{1,2}-r_{1,2}-q_{1,3}$, $C=q_{1,2}p_{1,2}+p_{1,3}p_{1,2}-r_{1,1}-2q_{1,2}-p_{1,3}$

Pair	Torsion tensor
3.19.14	$(0,0,0),(0,0,0),(q_{1,3}-r_{1,2},0,0)$
3.21.6	$(0,0,0),(0,0,0),(2q_{1,3},0,0)$
3.21.7	$(0,0,0),(0,0,0),(2q_{1,3},0,0)$
3.25.30	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(2q_{1,3},p_{1,3}-r_{1,1},0)$
2.7.2	$(p_{1,2}-q_{1,1},0,0),(0,0,0),(0,0,q_{1,1}-p_{1,2})$
2.17.27	$(0,0,0),(p_{1,3}-r_{1,1},2p_{2,3},0),(2q_{1,3},2q_{2,3}-r_{1,1}-p_{1,3},0)$
1.1.5	$(0,0,p_{3,2}-q_{3,1}),(p_{1,3}-r_{1,1},0,0),(0,q_{2,3}-r_{2,2},0)$
1.1.7	$(0,0,p_{3,2}-q_{3,1}-1),(p_{1,3}-r_{1,1},0,0),(0,q_{2,3}-r_{2,2},0)$
1.3.3	$(0,0,2p_{3,2}-1),(p_{1,3}-r_{1,1},p_{2,3}+r_{1,2},0),(-p_{2,3}-r_{1,2},p_{1,3}-r_{1,1},0)$
1.3.4	$(0,0,2p_{3,2}-1),(p_{1,3}-r_{1,1},p_{2,3}+r_{1,2},0),(-p_{2,3}-r_{1,2},p_{1,3}-r_{1,1},0)$
1.3.5	$(0,0,2p_{3,2}),(p_{1,3}-r_{1,1},p_{2,3}+r_{1,2},0),(-p_{2,3}-r_{1,2},p_{1,3}-r_{1,1},0)$
1.3.6	$(0,0,2p_{3,2}),(p_{1,3}-r_{1,1},p_{2,3}+r_{1,2},0),(-p_{2,3}-r_{1,2},p_{1,3}-r_{1,1},0)$
1.5.19	$(p_{1,2}-q_{1,1},0,0),(p_{1,3}-r_{1,1},2p_{2,3},0),(q_{1,3}-r_{1,2},q_{2,3}-r_{2,2},q_{1,1}-p_{1,2})$
1.8.2	$(2p_{1,2}-1,0,0),(p_{1,3}-r_{1,1},2p_{1,2}-1,0),(q_{1,3}-r_{1,2},p_{1,3}-r_{1,1},2p_{1,2}-1)$

Pair	Holonomy	algebra
3.19.14		$\begin{pmatrix} 0 & p_2 & p_1 \\ 0 & p_3 & 0 \\ 0 & 0 & -p_3 \end{pmatrix}$
3.21.6 3.21.7		$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & -p_3 \\ 0 & p_3 & 0 \end{pmatrix}$
3.25.30	$p_{1,3}=r_{1,3}=0, r_{1,1}\neq 0$	$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$

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	$p_{1,3} = r_{1,1} = 0, r_{1,3} \neq 0$	$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
	$p_{1,3} = 0, r_{1,3} \neq 0, r_{1,1} \neq 0$	$\begin{pmatrix} p_1 & p_2 & p_3 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$
	$p_{1,3} \neq 0$	$\begin{pmatrix} (2r_{1,1}/p_{1,3})p_6 & p_2 & p_1 \\ 0 & p_4 + (1 + 2r_{1,1}/p_{1,3})p_6 & p_3 \\ 0 & p_5 & -p_4 + (1 + 2r_{1,1}/p_{1,3})p_6 \end{pmatrix}$
	else	is equal to zero
2.7.2	$2p_{1,2}(q_{2,2} - q_{1,1}) \neq q_{1,2}$	$\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & p_2 & 0 \end{pmatrix}$
	$2p_{1,2}(q_{2,2} - q_{1,1}) = q_{1,2}$	is equal to zero

Theorem 21. *If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ allows nontrivial affine connections, the curvature and torsion tensors are not zero for some connections, $\bar{\mathfrak{g}}$ is not semisimple (or solvable), $\mathfrak{g}$ is solvable and the radical of $\bar{\mathfrak{g}}$ is not commutative then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the following pairs:*

Pair	Levi decomposition
4.21.11, $\mu = 0$	$\{\{8e_4, -8e_2 - 8u_1, e_1, u_1\}, \{-32e_3, 8e_4 + 16u_2, 16e_2 + 16u_1 - 32u_3\}\}$
3.13.6, $\mu = 0$	$\{\{2u_1, e_3, e_1\}, \{4e_2, e3 + 4u_2, u_1 + 4u_3\}\}$
2.8.7, $\lambda \neq 0$	$\{\{u_2, -(1/\lambda)e_2 + u_1\}, \{\lambda^3 e_1, \lambda^2 u_1, \lambda^3 u_3\}\}$
$\lambda = 0$	$\{\{u_2, e_2\}, \{e_1, u_1, u_3\}\}$
2.18.3	$\{\{u_2, u_1\}, \{e_1, e_2 + u_1, u_3\}\}$
2.21.4	$\{\{e_1 + u_2, -e_2 + u_1\}, \{u_1, -u_3, u_2\}\}$

Pair	Connection
4.21.11, $\mu = 0$	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & p_{1,3} \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 1 & r_{1,1} + p_{1,3} \end{pmatrix}$
3.13.6, $\mu = 0$	$\begin{pmatrix} 0 & -1 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & q_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} + p_{1,3} - q_{2,3} & r_{2,3} \\ 0 & -1 & r_{1,1} + p_{1,3} \end{pmatrix}$
3.13.6, $\mu = -1$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & p_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -p_{2,3} & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$
2.8.7, $\lambda = 0$	$\begin{pmatrix} -1/2 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & r_{1,3} \\ 0 & r_{2,2} & 0 \\ -1/2 & 0 & r_{1,1} + p_{1,3} \end{pmatrix}$
2.8.7, $\lambda = -1$	$\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & q_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_{1,2} & 0 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}$

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2.18.3	$\begin{pmatrix}0 & p_{1,2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}q_{1,1} & q_{1,2} & 0\\ 0 & q_{1,1}+p_{1,2} & 0\\ 0 & 0 & q_{1,1}\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & p_{1,2}+1 & 0\end{pmatrix}$
2.21.4	$\begin{pmatrix}0 & p_{1,2} & 0\\ 0 & 0 & p_{1,2}\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}-p_{1,2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & p_{1,2}\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ -p_{1,2} & 0 & 0\\ 0 & -p_{1,2} & 0\end{pmatrix}$

Pair	Curvature tensor
4.21.11, $\mu=0$	$\begin{pmatrix}0 & 0 & -p_{1,3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & p_{1,3} & {p_{1,3}}^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}2r_{1,1} & 0 & 0\\ 0 & p_{1,3}+2r_{1,1} & {p_{1,3}}^2\\ 0 & 0 & p_{1,3}+2r_{1,1}\end{pmatrix}$
3.13.6, $\mu=0$	$\begin{pmatrix}0 & 0 & -q_{2,3}+2p_{1,3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -2p_{1,3}+q_{2,3} & -r_{2,3}+{p_{1,3}}^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$ $\begin{pmatrix}-2r_{1,1} & 0 & 0\\ 0 & q_{2,3}-2r_{1,1}-2p_{1,3} & -4r_{2,3}+{q_{2,3}}^2\\ 0 & 0 & q_{2,3}-2r_{1,1}-2p_{1,3}\end{pmatrix}$
3.13.6, $\mu=-1$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 3p_{2,3}\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}p_{2,3} & 0 & 0\\ 0 & -2p_{2,3} & 0\\ 0 & 0 & p_{2,3}\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 3p_{2,3} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$
2.8.7, $\lambda=0$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -q_{2,3}/2\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}-p_{1,3}/2-r_{1,1} & 0 & -2r_{1,3}+{p_{1,3}}^2\\ 0 & -r_{2,2} & 0\\ 0 & 0 & -p_{1,3}/2-r_{1,1}\end{pmatrix},$ $\begin{pmatrix}0 & 0 & 0\\ -q_{2,3}/2 & 0 & q_{2,3}r_{1,1}+q_{2,3}p_{1,3}-r_{2,2}q_{2,3}\\ 0 & 0 & 0\end{pmatrix}$
2.8.7, $\lambda=-1$	$\begin{pmatrix}0 & 0 & -q_{1,3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -3r_{1,2}/2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}-q_{1,3}/2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & q_{1,3}/2\end{pmatrix}$
2.18.3	$\begin{pmatrix}0 & {p_{1,2}}^2+p_{1,2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -{p_{1,2}}^2-p_{1,2} & 0\end{pmatrix}$
2.21.4	$\begin{pmatrix}0 & {p_{1,2}}^2-p_{1,2} & 0\\ 0 & 0 & {p_{1,2}}^2-p_{1,2}\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}-{p_{1,2}}^2+p_{1,2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & {p_{1,2}}^2-p_{1,2}\end{pmatrix},$ $\begin{pmatrix}0 & 0 & 0\\ -{p_{1,2}}^2+p_{1,2} & 0 & 0\\ 0 & -{p_{1,2}}^2+p_{1,2} & 0\end{pmatrix}$

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Pair	Torsion tensor
4.21.11, $\mu=0$	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(0,p_{1,3}-r_{1,1},0)$
3.13.6, $\mu=0$	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(0,2q_{2,3}-r_{1,1}-p_{1,3},0)$
3.13.6, $\mu=-1$	$(0,0,0),(0,2p_{2,3},0),(0,0,0)$
2.8.7, $\lambda=0$	$(0,0,0),(p_{1,3}-r_{1,1},0,0),(0,q_{2,3}-r_{2,2},0)$
2.8.7, $\lambda=-1$	$(0,0,0),(0,0,0),(q_{1,3}-r_{1,2},0,0)$
2.18.3	$(p_{1,2}-q_{1,1}+1,0,0),(0,0,0),(0,0,q_{1,1}-p_{1,2}-1)$
2.21.4	$(2p_{1,2}-1,0,0),(0,2p_{1,2}-1,0),(0,0,2p_{1,2}-1)$

Pair	Holonomy algebra
4.21.11, $\mu=0$	$r_{1,1}\neq 0, p_{1,3}=0$ $\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$
	$p_{1,3}\neq 0, \gamma=2r_{1,1}/p_{1,3}$ $\begin{pmatrix} \gamma p_6 & p_2 & p_1 \\ 0 & p_5+(1+\gamma)p_6 & p_3 \\ 0 & p_4 & -p_5+(1+\gamma)p_6 \end{pmatrix}$
	else is equal to zero
3.13.6, $\mu=0$	$q_{2,3}=2p_{1,3}, r_{2,3}=p_{1,3}^2, r_{1,1}\neq 0$ $\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}$
	$q_{2,3}=2p_{1,3}, r_{2,3}\neq p_{1,3}^2, r_{1,1}\neq 0$ $\begin{pmatrix} p_6 & p_2 & p_1 \\ 0 & p_5+p_6 & p_3 \\ 0 & p_4 & -p_5+p_6 \end{pmatrix}$
	$q_{2,3}=2p_{1,3}, r_{2,3}\neq p_{1,3}^2, r_{1,1}=0$ $\begin{pmatrix} 0 & p_2 & p_1 \\ 0 & p_5 & p_3 \\ 0 & p_4 & -p_5 \end{pmatrix}$
	$q_{2,3}\neq 2p_{1,3}, r_{2,3}=(q_{2,3}^2+q_{1,3}^2)/4, \gamma=2r_{1,1}/(2p_{1,3}-q_{2,3})$ $\begin{pmatrix} \gamma p_3 & p_2 & p_1 \\ 0 & (\gamma+1)p_3 & 0 \\ 0 & 0 & (\gamma+1)p_3 \end{pmatrix}$
	$q_{2,3}\neq 2p_{1,3}, r_{2,3}\neq (q_{2,3}^2+q_{1,3}^2)/4, \gamma=2r_{1,1}/(2p_{1,3}-q_{2,3})$ $\begin{pmatrix} \gamma p_6 & p_2 & p_1 \\ 0 & (\gamma+1)p_6+p_5 & p_3 \\ 0 & p_4 & (\gamma+1)p_6-p_5 \end{pmatrix}$, else is equal to zero
3.13.6, $\mu=-1$	$p_{2,3}\neq 0$ $\mathfrak{sl}(3,\mathbb{R})$
	$p_{2,3}=0$ is equal to zero
2.8.7, $\lambda=-1$	$q_{1,3}\neq 0, r_{1,2}\neq 0$ $\begin{pmatrix} p_2 & p_3 & p_1 \\ 0 & 0 & 0 \\ p_4 & p_5 & -p_2 \end{pmatrix}$
	$q_{1,3}\neq 0, r_{1,2}=0$ $\begin{pmatrix} p_2 & 0 & p_1 \\ 0 & 0 & 0 \\ p_3 & 0 & -p_2 \end{pmatrix}$
	$q_{1,3}=0, r_{1,2}\neq 0$ $\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & p_2 & 0 \end{pmatrix}$
	else is equal to zero

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2.18.3	$p_{1,2} \neq 0, -1$	$\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & 0 \\ 0 & p_2 & 0 \end{pmatrix}$
	$p_{1,2} = 0, -1$	is equal to zero
2.21.4	$p_{1,2} \neq 0, 1$	$\begin{pmatrix} p_3 & p_1 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_2 & -p_3 \end{pmatrix}$
	$p_{1,2} = 0, 1$	is equal to zero

If the radical of $\bar{\mathfrak{g}}$ is not commutative then commutant of the radical of $\bar{\mathfrak{g}}$ is commutative.

We describe all invariant affine connections on three-dimensional homogeneous spaces with non-solvable transformation group together with their curvature, torsion tensors and holonomy algebras. In this work we use the algebraic approach for description of connections, methods of the theory of Lie groups, Lie algebras and homogeneous spaces.

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