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Normal Connections on Three-Dimensional Manifolds with Solvable Transformation Group

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Abstract—The purpose of the work is the classification of three-dimensional homogeneous spaces, allowing a normal connection, description of invariant affine connections on those spaces together with their curvature and torsion tensors, holonomy algebras. We consider only the case, when Lie group is solvable. The local classification of homogeneous spaces is equivalent to the description of the effective pairs of Lie algebras. We study the holonomy algebras of homogeneous spaces and find when the invariant connection is normal. Studies are based on the use of properties of the Lie algebras, Lie groups and homogeneous spaces and they mainly have local character.

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1. INTRODUCTION

The normal connection for Riemannian manifold was introduced by E. Cartan. Manifolds with zero torsion (i.e. flat normal connection) were studied almost simultaneously by D. Perepelkin [1], F. Fabricius-Bierre [2] and also E. Cartan. The results of their research are brought in the monography of B. Chena [3]. Some researches are devoted to the general questions of normal connections. Their interesting characteristic among metric linear connections was given by K. Nomizu [4]. Nguyen van Hai studied living conditions of invariant affine connection on (not necessarily reductive) homogeneous space. Its result in [5] generalizes some results of K. Nomizu [6, 7] and is connected with a problem of studying of affine connection which supposes transitive group of affine transformations. This problem was studied by W. Ambrose, J. Singer, K. Nomizu and others. Any kind of connection on a manifold gives rise, through its parallel displacement, to some notion of holonomy. Important examples include: holonomy of the Levi-Civita connection in Riemannian geometry (called Riemannian holonomy), holonomy of connections in vector bundles and holonomy of E. Cartan connections. In each of these cases the holonomy of the connection can be identified with a Lie group—the holonomy group. The holonomy of connection is closely related to the curvature of connection, via the Ambrose–Singer theorem. The holonomy was introduced by E. Cartan in order to study and classify symmetric spaces. Only a little later holonomy groups were used to study Riemannian geometry in a more general setting. Affine holonomy groups are the groups arising as holonomies of torsion-free affine connections; those groups which are not Riemannian or pseudo-Riemannian holonomy groups are also known as non-metric holonomy groups. The purpose of the work is the classification of three-dimensional homogeneous spaces, allowing a normal connection, description of invariant affine connections on those spaces together with their curvature and torsion tensors, holonomy algebras. We only consider case, when Lie group is solvable, other cases see in [8, 9].

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2. BASIC DEFINITIONS

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, and let $\overline{G}$ be a solvable Lie group. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. A large class of homogeneous spaces is spaces with solvable transformation group. Since we are interested in only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. In the study of homogeneous spaces it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only the effective action of the group $\overline{G}$ on the manifold M. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$, a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective. An *isotropic* $\mathfrak{g}$*-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y + \mathfrak{g}) = [x, y] + \mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. We divide the solution of our problem of classification all three-dimensional isotropically–faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ into the following parts. We classify (up to isomorphism) all faithful three–dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic. All of there pairs are described in [10].

Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [11] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [12]. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas:

$$R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}), \quad (x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)] - \Lambda([x_1, x_2]);$$

$$T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m}, \quad (x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto \Lambda(x_1)(x_2 + \mathfrak{g}) - \Lambda(x_2)(x_1 + \mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}.$$

We restate the theorem of Wang on the holonomy algebra of an invariant connection: the Lie algebra of the holonomy group of the invariant connection defined by $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(3, \mathbb{R})$ on $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given by $V + [\Lambda(\bar{\mathfrak{g}}), V] + [\Lambda(\bar{\mathfrak{g}}), [\Lambda(\bar{\mathfrak{g}}), V]] + \ldots$, where V is the subspace spanned by $\{[\Lambda(x), \Lambda(y)] - \Lambda([x, y]) | x, y \in \bar{\mathfrak{g}}\}$. Let $a_{\bar{\mathfrak{g}}}$ be the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ generated by $\{\Lambda(x); x \in \bar{\mathfrak{g}}\}$. Originally, $a_{\bar{\mathfrak{g}}}$ was introduced as such in the Riemannian case by B. Kostant [13], and has been used by A. Lichnerowicz [14] and H. Wang [15] under more general circumstances. The basic properties of $a_{\bar{\mathfrak{g}}}$ are given by: let $\mathfrak{h}^*$ be the Lie algebra of the holonomy group, then $\mathfrak{h}^* \subset a_{\bar{\mathfrak{g}}} \subset \mathbb{N}(\mathfrak{h}^*)$, where $\mathbb{N}(\mathfrak{h}^*)$ is the normalizer of $\mathfrak{h}^*$ in $\mathfrak{gl}(3, \mathbb{R})$. We will say that a invariant connection is *normal* if $\mathfrak{h}^* = a_{\bar{\mathfrak{g}}}$.

A geometric interpretation of the notion of normal connection is the following: let P be an invariant structure on M. Fixing an invariant connection in P, let $P(u_0)$ be the holonomy bundle through a frame $u_0 \in P$. Then the connection is normal if and only if every element of $\overline{G}$ maps $P(u_0)$ into itself. By virtue of the reduction theorem [12] for certain types of problems concerning a connection in a principal bundle we can assume that P is the holonomy bundle. Such simplification is not in general available unless $\overline{G}$ maps the holonomy bundle into itself. The result means that if an invariant connection on a homogeneous space is normal, then the reduction theorem can be still used advantageously. If an invariant connection is normal, then every parallel tensor field on M is invariant by $\overline{G}$. This result has been proved by A. Lichnerowicz [14].

3. THE LOCAL CLASSIFICATION OF HOMOGENEOUS SPACES

We define $(\bar{\mathfrak{g}}, \mathfrak{g})$ by the commutation table of $\bar{\mathfrak{g}}$. Here by $\{e_1, ..., e_n\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n = \dim \bar{\mathfrak{g}}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1, ..., e_{n-3}$. Let $\{u_1 = e_{n-2}, u_2 = e_{n-1}, u_3 = e_n\}$ be a basis of $\mathfrak{m}$. We describe affine connection by $\Lambda(e_{n-2})$, $\Lambda(e_{n-1})$, $\Lambda(e_n)$, curvature tensor

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R by $R(e_{n-2}, e_{n-1})$, $R(e_{n-2}, e_n)$, $R(e_{n-1}, e_n)$ and torsion tensor T by $T(e_{n-2}, e_{n-1})$, $T(e_{n-2}, e_n)$, $T(e_{n-1}, e_n)$. To refer to the pair we use the notation $d.n.m$, where d is the dimension of the subalgebra, n is the number of the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$, m is the number of $(\bar{\mathfrak{g}}, \mathfrak{g})$ in [10].

The information about the affine connections, the curvature and torsion tensors, the holonomy algebras is contained in the proof of the theorem.

Theorem 1. *If the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *allows a normal connection,* $\bar{\mathfrak{g}}$ *is solvable and* $\dim \mathfrak{g} > 1$ *then* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *is equivalent to one of the pairs*:

3.20.12, 3.20.14.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$\pm e_2$	$e_3 + u_1$
u_1	$-u_1$	0	0	0	0	$-u_1$
u_2	0	$-u_1$	$\mp e_2$	0	0	$-u_2$
u_3	0	0	$-e_3 - u_1$	u_1	u_2	0

,

3.20.15.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	$e_3 + u_1$
u_1	$-u_1$	0	0	0	$\frac{3}{2}e_2 - e_3 - \frac{3}{2}u_1$	$-\frac{1}{2}e_2 + \frac{1}{2}u_1$
u_2	0	$-u_1$	e_2	$-\frac{3}{2}e_2 + e_3 + \frac{3}{2}u_1$	0	$\frac{1}{2}e_1 + \frac{1}{2}u_2 + \frac{3}{2}u_3$
u_3	0	0	$-e_3 - u_1$	$\frac{1}{2}e_2 - \frac{1}{2}u_1$	$-\frac{1}{2}e_1 - \frac{1}{2}u_2 - \frac{3}{2}u_3$	0

,

3.20.20.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	αu_1
u_2	0	$-u_1$	0	0	0	$(\alpha - 1)u_2$
u_3	0	$-e_2$	$-u_1$	$-\alpha u_1$	$(1 - \alpha)u_2$	0

,

3.20.24.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_3
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	0	$-e_3$	$-u_1$	$-u_1$	$-u_2$	0

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2.9.1.	e_1	e_2	u_1	u_2	u_3
e_1	0	$2e_2$	u_1	0	$-u_3$
e_2	$-2e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	0
u_3	u_3	$-u_1$	0	0	0

,

2.9.2.	e_1	e_2	u_1	u_2	u_3
e_1	0	$2e_2$	u_1	0	$-u_3$
e_2	$-2e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_2
u_2	0	0	0	0	0
u_3	u_3	$-u_1$	$-u_2$	0	0

,

$2.9.4, \mu = 0, -1.$	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	u_1	0	μu_3
e_2	$(\mu-1)e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	u_1	0
u_2	0	0	$-u_1$	0	$-u_3$
u_3	$-\mu u_3$	$-u_1$	0	u_3	0

, $\mu \neq 1$,

2.9.5, 2.9.6.	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$\pm e_2$
u_2	0	0	0	0	αu_2
u_3	0	$-u_1$	$\mp e_2$	$-\alpha u_2$	0

, $\alpha \geq 0$,

2.9.7.	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	0	$-u_1$	0	$-u_2$	0

,

2.17.2(3).	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\pm e_1$
u_2	0	0	0	0	αe_2
u_3	$-u_1$	$-u_2$	$\mp e_1$	$-\alpha e_2$	0

,

2.17.6(7).	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\pm e_1$
u_2	0	0	0	0	$e_1 \pm e_2$
u_3	$-u_1$	$-u_2$	$\mp e_1$	$-e_1 \mp e_2$	0

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2.17.4.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 - e_2$
u_2	0	0	0	0	$e_1 + \alpha e_2$
u_3	$-u_1$	$-u_2$	$e_2 - \alpha e_1$	$-e_1 - \alpha e_2$	0

, $\alpha \geq 0$,

2.17.8.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 + \alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1 - u_1$	$-\beta e_2 - \alpha u_2$	0

, $-1 < \alpha < 1$,

2.17.9.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1 + e_2 + u_1$
u_2	0	0	0	0	$\gamma e_1 + \beta e_2 + \alpha u_1$
u_3	$-u_1$	$-u_2$	$-\delta e_1 - e_2 - u_1$	$-\gamma e_1 - \beta e_2 - \alpha u_1$	0

, $-1 < \alpha < 1$,

2.17.10.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1 + u_1$
u_2	0	0	0	0	$e_1 + \beta e_2 + \alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1 - u_1$	$-e_1 - \beta e_2 - \alpha u_2$	0

, $-1 < \alpha < 1$,

2.17.15.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	αe_1
u_2	0	0	0	0	$e_1 + \alpha e_2 + u_1$
u_3	$-u_1$	$-u_2$	$-\alpha e_1$	$-e_1 - \alpha e_2 - u_1$	0

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2.17.17.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	e_2
u_2	0	0	0	0	$\alpha e_1 + \beta e_2 + u_1$
u_3	$-u_1$	$-u_2$	$-e_2$	$-\alpha e_1 - \beta e_2 - u_1$	0

,

2.17.18.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\gamma e_2 + u_1$
u_2	0	0	0	0	$\alpha e_1 + \beta e_2 + u_1 + u_2$
u_3	$-u_1$	$-u_2$	$-\gamma e_2 - u_1$	$-\alpha e_1 - \beta e_2 - u_1 - u_2$	0

,

2.17.19.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 + u_1 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_2 - u_1 - u_2$	0

,

2.17.20.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_1 + \alpha e_2 + u_1 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_1 - \alpha e_2 - u_1 - u_2$	0

,

2.17.21.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$e_1 + \alpha e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-e_1 - \alpha e_2 - u_2$	0

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2.17.22.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 - \beta e_2 + u_1$
u_2	0	0	0	0	$\beta e_1 + \alpha e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 + \beta e_2 - u_1$	$-\beta e_1 - \alpha e_2 - u_2$	0

, $\beta > 0$,

2.17.23.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + u_1$
u_2	0	0	0	0	$\beta e_2 + u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - u_1$	$-\beta e_2 - u_2$	0

, $|\alpha| \le |\beta|$,

2.17.24.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\delta e_1 + \gamma e_2 + \alpha u_1 - u_2$
u_2	0	0	0	0	$\beta e_1 + \delta e_2 + u_1 + \alpha u_2$
u_3	$-u_1$	$-u_2$	$-\delta e_1 - \gamma e_2 - \alpha u_1 + u_2$	$-\beta e_1 - \delta e_2 - u_1 - \alpha u_2$	0

,

$|\beta| \le |\gamma|$,

2.17.26.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\alpha e_1 + \beta e_2 + u_1$
u_2	0	0	0	0	$e_1 + \gamma e_2 - u_2$
u_3	$-u_1$	$-u_2$	$-\alpha e_1 - \beta e_2 - u_1$	$-e_1 - \gamma e_2 + u_2$	0

, $-1 \le \beta \le 1$,

2.17.13, 2.17.14.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	0	u_1
e_2	0	0	0	0	u_2
u_1	0	0	0	0	$\pm e_1$
u_2	0	0	0	0	$\alpha e_2 + u_1$
u_3	$-u_1$	$-u_2$	$\mp e_1$	$-\alpha e_2 - u_1$	0

,

2.20.15, 2.20.16.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1 \pm e_1$	e_2
e_2	0	0	0	$\pm e_2$	u_1
u_1	0	0	0	$\pm u_1$	0
u_2	$-u_1 \mp e_1$	$\mp e_2$	$\mp u_1$	0	0
u_3	$-e_2$	$-u_1$	0	0	0

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2.20.3, 2.20.4.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$e_1 + u_1$	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	$2u_1$	0
u_2	$-e_1 - u_1$	$-e_2$	$-2u_1$	0	$e_i - u_3$
u_3	0	$-u_1$	0	$-e_i + u_3$	0

, $i = 2, 1$,

2.20.9.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1 + (\alpha + 1)e_2$
u_1	0	0	0	0	$2\alpha u_1$
u_2	$-u_1$	0	0	0	$e_1 + \alpha u_2$
u_3	$-\alpha e_1$	$-u_1 - (\alpha + 1)e_2$	$-2\alpha u_1$	$-e_1 - \alpha u_2$	0

,

2.20.10.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1 + (\alpha + 1)e_2$
u_1	0	0	0	0	$(2\alpha + 1)u_1$
u_2	$-u_1$	0	0	0	$e_2 + (\alpha + 1)u_2$
u_3	$-\alpha e_1$	$-u_1 - (\alpha + 1)e_2$	$-(2\alpha + 1)u_1$	$-e_2 - (\alpha + 1)u_2$	0

,

2.20.11, 2.20.13.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-e_1$
e_2	0	0	0	$\pm e_1$	u_1
u_1	0	0	0	$\mp e_1$	$-u_1$
u_2	$-u_1$	$\mp e_1$	$\pm e_1$	0	$\mp e_2$
u_3	e_1	$-u_1$	u_1	$\pm e_2$	0

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2.20.12, 2.20.14.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-2e_1$
e_2	0	0	0	$\pm e_1$	$-e_2+u_1$
u_1	0	0	0	0	$-3u_1$
u_2	$-u_1$	$\mp e_1$	0	0	e_2-u_2
u_3	$2e_1$	e_2-u_1	$3u_1$	u_2-e_2	0

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2.20.17.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1+\alpha e_1$	e_1+e_2
e_2	0	0	0	αe_2	u_1+e_2
u_1	0	0	0	αu_1	u_1
u_2	$-u_1-\alpha e_1$	$-\alpha e_2$	$-\alpha u_1$	0	0
u_3	$-e_1-e_2$	$-u_1-e_2$	$-u_1$	0	0

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2.20.18.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	0	$u_1+\alpha e_2$
u_1	0	0	0	0	$(\alpha+1)u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-\alpha e_1$	$-u_1-\alpha e_2$	$-(\alpha+1)u_1$	$-u_2$	0

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2.20.19.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$(\beta+1)e_1$
e_2	0	0	0	0	$u_1+\beta e_2$
u_1	0	0	0	0	$(\alpha+\beta)u_1$
u_2	$-u_1$	0	0	0	$(\alpha-1)u_2$
u_3	$-(\beta+1)e_1$	$-u_1-\beta e_2$	$-(\alpha+\beta)u_1$	$(1-\alpha)u_2$	0

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2.20.20.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1+e_1	$(\beta+1)e_1$
e_2	0	0	0	e_2	$u_1+\beta e_2$
u_1	0	0	0	u_1	$(\beta+1)u_1$
u_2	$-u_1-e_1$	$-e_2$	$-u_1$	0	0
u_3	$-(\beta+1)e_1$	$-u_1-\beta e_2$	$-(\beta+1)u_1$	0	0

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2.20.21, 2.20.23.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1+\beta e_1$	αe_1
e_2	0	0	0	$\pm e_1+\beta e_2$	$u_1+(\alpha+1)e_2$
u_1	0	0	0	$\beta u_1\mp e_1$	αu_1
u_2	$-u_1-\beta e_1$	$\mp e_1-\beta e_2$	$-\beta u_1\pm e_1$	0	0
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$-\alpha u_1$	0	0

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2.20.22, 2.20.24.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	αe_1
e_2	0	0	0	$\pm e_1$	$u_1+(\alpha+1)e_2$
u_1	0	0	0	0	$(\alpha-1)u_1$
u_2	$-u_1$	$\mp e_1$	0	0	$-u_2$
u_3	$-\alpha e_1$	$-u_1-(\alpha+1)e_2$	$(1-\alpha)u_1$	u_2	0

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2.20.25.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$u_1-\alpha e_1$	$\frac{\alpha-1}{3}e_1$
e_2	0	0	0	$-e_1-\alpha e_2$	$u_1+\frac{\alpha+2}{3}e_2$
u_1	0	0	0	$\frac{3}{2}e_1-e_2-\frac{3+2\alpha}{2}u_1$	$\frac{2\alpha+1}{6}u_1-\frac{1}{2}e_1$
u_2	$-u_1+\alpha e_1$	$e_1+\alpha e_2$	$-\frac{3}{2}e_1+e_2+\frac{3+2\alpha}{2}u_1$	0	$\frac{1}{2}(u_2+3u_3)$
u_3	$\frac{1-\alpha}{3}e_1$	$-u_1-\frac{\alpha+2}{3}e_2$	$\frac{1}{2}e_1-\frac{2\alpha+1}{6}u_1$	$-\frac{1}{2}(u_2+3u_3)$	0

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2.20.26.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$e_2+\alpha e_1$
e_2	0	0	0	0	$u_1+\alpha e_2$
u_1	0	0	0	0	$(\alpha+1)u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-e_2-\alpha e_1$	$-u_1-\alpha e_2$	$-(\alpha+1)u_1$	$-u_2$	0

,

2.20.8.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_1+e_2	u_1
u_1	0	0	0	$2u_1$	0
u_2	$-e_1-u_1$	$-e_1-e_2$	$-2u_1$	0	e_2-u_3
u_3	0	$-u_1$	0	$-e_2+u_3$	0

,

2.21.1.	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	$-u_3$
e_2	$-e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	0	$-u_1$	0	0	0
u_3	u_3	$-u_2$	0	0	0

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Proof. For the pairs in [10] we choose allowing a normal connection, find affine connections, holonomy algebras, also we find when the invariant connection is normal.

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For example, let isotropic representation has the form 2.21, $\{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 2\lambda - 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda \neq 1.$$

By $\mathfrak{h}$ we denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1, then $\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1, U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_2, U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_2, U^{(2\lambda-1)}(\mathfrak{h}) \supset \mathbb{R}u_3$.

1° Consider now the case, when $\lambda = 1/2$. Using the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1, 2$, $1 \leq j < k \leq 3$, we see that the pair has the form:

	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1/2)e_2$	u_1	$(1/2)u_2$	0
e_2	$-(1/2)e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	$\beta_1 u_1$
u_2	$-(1/2)u_2$	$-u_1$	0	0	$c_2e_2 + \beta_1 u_2$
u_3	0	$-u_2$	$-\beta_1 u_1$	$-c_2e_2 - \beta_1 u_2$	0

.

1.1° $4c_2 + \beta_1^2 \neq 0$. The mapping $\pi : \bar{\mathfrak{g}}_{2,3} \to \bar{\mathfrak{g}}$, where $\pi(e_1) = e_1$, $\pi(e_2) = e_2$, $\pi(u_1) = (1/t)u_1$, $\pi(u_2) = (1/t)u_2 + (\beta_1/2)e_2$, $\pi(u_3) = (1/t)u_3 - \beta_1 e_1$, $t = 2|4c_2 + \beta_1^2|^{-1/2}$, establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_{2,3}, \mathfrak{g}_{2,3})$.

1.2° $4c_2 + \beta_1^2 = 0$. The mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, $\pi(e_1) = e_1$, $\pi(e_2) = e_2$, $\pi(u_1) = u_1$, $\pi(u_2) = u_2 + (\beta_1/2)e_2$, $\pi(u_3) = u_3 - \beta_1 e_1$, establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2° $\lambda = 2/3$. Similarly, we obtain:

	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1/3)e_2$	u_1	$(2/3)u_2$	$(1/3)u_3$
e_2	$-(1/3)e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	$-(2/3)u_2$	$-u_1$	0	0	$\gamma_1 u_1$
u_3	$-(1/3)u_3$	$-u_2$	0	$-\gamma_1 u_1$	0

.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, $\pi(e_1) = e_1$, $\pi(e_2) = e_2$, $\pi(u_1) = u_1$, $\pi(u_2) = u_2$, $\pi(u_3) = u_3 - \gamma_1 e_2$.

3° $\lambda \notin \{1/2, 2/3\}$. Continuing in the same way as in the cases 1° and 2°, we obtain:

	e_1	e_2	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$(2\lambda-1)u_3$
e_2	$(\lambda-1)e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	$\alpha_1 u_1$	$\alpha_1 u_2$
u_2	$-\lambda u_2$	$-u_1$	$-\alpha_1 u_1$	0	$\alpha_1 u_3$
u_3	$(1-2\lambda)u_3$	$-u_2$	$-\alpha_1 u_2$	$-\alpha_1 u_3$	0

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where the coefficients α_1 and λ satisfy the equation $\alpha_1\lambda = 0$.

4.1° $\alpha_1 \neq 0$. In this case $\bar{\mathfrak{g}}$ in nonsolvable.

4.2° $\alpha_1 = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Now it remains to show that the pairs are not equivalent to each other. Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$, $i = 2, 3$, where $f_i(x)$ is the matrix of the mapping $adx|_{D\bar{\mathfrak{g}}_i}$ in the basis $\{e_2, u_1, u_2\}$ $D\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

Let

$$\Lambda(u_1) = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix}, \quad \Lambda(u_2) = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix}, \quad \Lambda(u_3) = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

for $p_{i,j}$, $q_{i,j}$, $r_{i,j} \in \mathbb{R}$ $(i, j = \overline{1, 3})$. If $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the three-dimensional homogeneous space 2.21.1 ($\lambda = 0$) then $\Lambda|_{\mathfrak{g}}$ is the isotropic representation of $\mathfrak{g}$. Λ is $\mathfrak{g}$-invariant $\Rightarrow [\Lambda(e_2), \Lambda(u_1)] = \Lambda([e_2, u_1]) \Rightarrow [\Lambda(e_2), \Lambda(u_1)] = 0$, we have $p_{2,1} = 0$, $p_{2,2} = p_{1,1}$, $p_{2,3} = p_{1,2}$, $p_{3,1} = 0$, $p_{3,2} = p_{2,1}$, $p_{3,3} = p_{1,1}$, $p_{3,2} = 0$. $[\Lambda(e_1), \Lambda(u_1)] = \Lambda([e_1, u_1]) \Rightarrow [\Lambda(e_1), \Lambda(u_1)] = \Lambda(u_1)$, $p_{1,1} = p_{1,3} = 0$. $[\Lambda(e_2), \Lambda(u_2)] = \Lambda([e_2, u_2]) \Rightarrow [\Lambda(e_2), \Lambda(u_2)] = \Lambda(u_1)$, $q_{2,2} = q_{1,1} + p_{1,2}$, $q_{2,3} = q_{1,2} + p_{1,3}$, $q_{3,3} = q_{2,2} + p_{1,2}$, $q_{2,1} = q_{3,1} = q_{3,2} = 0$. If $[\Lambda(e_1), \Lambda(u_2)] = \Lambda([e_1, u_2]) \Rightarrow [\Lambda(e_1), \Lambda(u_2)] = 0$ then $q_{1,2} = q_{1,3} = 0$. $[\Lambda(e_2), \Lambda(u_3)] = \Lambda([e_2, u_3]) \Rightarrow [\Lambda(e_2), \Lambda(u_3)] = \Lambda(u_2)$, $q_{1,1} = -p_{1,2} = r_{2,1} = r_{3,2}$, $r_{2,2} = r_{1,1}$, $r_{2,3} = r_{1,2}$, $r_{3,1} = 0$, $r_{3,3} = r_{2,2}$. If $[\Lambda(e_1), \Lambda(u_3)] = \Lambda([e_1, u_3]) \Rightarrow [\Lambda(e_1), \Lambda(u_3)] = -\Lambda(u_3)$ then $r_{1,1} = r_{1,2} = r_{1,3} = 0$, affine connection has the form

$$\begin{pmatrix} 0 & p_{1,2} & 0 \\ 0 & 0 & p_{1,2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -p_{1,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{1,2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -p_{1,2} & 0 & 0 \\ 0 & -p_{1,2} & 0 \end{pmatrix},$$

then curvature tensor has the form

$$\begin{pmatrix} 0 & {p_{1,2}}^2 & 0 \\ 0 & 0 & {p_{1,2}}^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -{p_{1,2}}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {p_{1,2}}^2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -{p_{1,2}}^2 & 0 & 0 \\ 0 & -{p_{1,2}}^2 & 0 \end{pmatrix},$$

torsion tensor $(2p_{1,2}, 0, 0), (0, 2p_{1,2}, 0), (0, 0, 2p_{1,2})$, if $p_{1,2} = 0$ then holonomy algebra is equal to zero, if $p_{1,2} \neq 0$ then holonomy algebra has the form $\begin{pmatrix} p_3 & p_1 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_2 & -p_3 \end{pmatrix}$. Connection is normal if $\mathfrak{h}^*$ is equal to $\mathfrak{a}_{\bar{\mathfrak{g}}}$, then $p_{1,2} \neq 0$. Similarly in the cases 2.21.1 ($\lambda = 1/2$), 2.21.2, 2.21.3 affine connection has the form

$$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & q_{1,2} & 0 \\ 0 & 0 & q_{1,2} + p_{1,3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} + q_{1,2} & 0 \\ 0 & 0 & r_{1,1} + 2q_{1,2} + p_{1,3} \end{pmatrix},$$

curvature tensor in the case 2.21.1 ($\lambda = 1/2$)—

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2p_{1,3}q_{1,2} + {p_{1,3}}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & {q_{1,2}}^2 & 0 \\ 0 & 0 & {q_{1,2}}^2 + 2p_{1,3}q_{1,2} + {p_{1,3}}^2 \\ 0 & 0 & 0 \end{pmatrix},$$

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in the case 2.21.2—

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2p_{1,3}q_{1,2}+{p_{1,3}}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & {q_{1,2}}^2-1 & 0 \\ 0 & 0 & {q_{1,2}}^2+2p_{1,3}q_{1,2}+{p_{1,3}}^2-1 \\ 0 & 0 & 0 \end{pmatrix},$$

in the case 2.21.3—

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2p_{1,3}q_{1,2}+{p_{1,3}}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & {q_{1,2}}^2+1 & 0 \\ 0 & 0 & {q_{1,2}}^2+2p_{1,3}q_{1,2}+{p_{1,3}}^2+1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the case 2.21.1 ($\lambda \neq 0, 1/2$) curvature tensor is equal to zero.

In the cases 2.21.1 ($\lambda = 0$) $p_{1,2} = 0$, 2.21.1 ($\lambda = 1/2$) $p_{1,3} = q_{1,2} = 0$, 2.21.2 $p_{1,3} = 0$, $q_{1,2}^2 = 1$ holonomy algebra is equal to zero. In the cases 2.21.1 ($\lambda = 1/2$) $p_{1,3} \neq 0$, $q_{1,2} = 0$, 2.21.2 $q_{1,2}^2 = 1$, $p_{1,3} \neq -2q_{1,2}, 0$ holonomy algebra has the form $\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix}$, in the cases 2.21.1 ($\lambda = 1/2$) $p_{1,3} \neq 0$, $q_{1,2} = -p_{1,3}$, 2.21.2 $p_{1,3} \neq 0$, $q_{1,2} = -p_{1,3} \pm 1$— $\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, in the cases 2.21.1 ($\lambda = 1/2$) $p_{1,3} \neq 0$, $q_{1,2} \neq 0, -p_{1,3}$, 2.21.2 $p_{1,3} \neq 0$, $q_{1,2}^2 \neq 1$, $q_{1,2} \neq -p_{1,3} \pm 1$, 2.21.3 $p_{1,3} \neq 0$— $\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & p_3 \\ 0 & 0 & 0 \end{pmatrix}$, in the cases 2.21.1 ($\lambda = 1/2$) $p_{1,3} = 0$, $q_{1,2} \neq 0$, 2.21.2 $p_{1,3} = 0$, $q_{1,2}^2 \neq 1$, 2.21.3 $p_{1,3} = 0$— $\begin{pmatrix} 0 & p_1 & 0 \\ 0 & 0 & p_1 \\ 0 & 0 & 0 \end{pmatrix}$. Connection is normal if $\mathfrak{h}^*$ is equal to $\mathfrak{a}_{\bar{\mathfrak{g}}}$, where $a_{\bar{\mathfrak{g}}}$ = the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$ generated by $\{\Lambda(x); x \in \bar{\mathfrak{g}}\}$. In this cases $\mathfrak{h}^* \subset \begin{pmatrix} 0 & p_1 & p_2 \\ 0 & 0 & p_3 \\ 0 & 0 & 0 \end{pmatrix}$, $\Lambda(e_1) \notin \mathfrak{h}^* \Rightarrow \mathfrak{h}^* \neq \mathfrak{a}_{\bar{\mathfrak{g}}}$.

Similarly, we obtain in the cases 3.20.12 ($\delta = 1$), 3.20.14, 3.20.15 ($\delta = -1$) affine connection has the form

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & 0 & 0 \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & \delta & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+1 \end{pmatrix},$$

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in cases 3.20.20, $\lambda = 0\,(\delta = 1)$, 3.20.24 $(\delta = 0)$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & 0 & 0 \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1}+\delta & 1-\delta \\ 0 & p_{1,2} & r_{1,1}+p_{1,3} \end{pmatrix},$$

2.9.4, $\mu = 0$, 2.9.5, $\lambda = 0$, 2.9.6, $\lambda = 0$, 2.9.7, $\lambda = 0$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & 0 & 0 \\ 0 & q_{2,2} & q_{2,3} \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{2,2} & r_{2,3} \\ 0 & p_{1,2} & r_{1,1}+p_{1,3} \end{pmatrix},$$

2.9.1, $\lambda = 0, \mu = -1$, 2.9.2, $\mu = -1$, 2.9.4, $\mu = -1$

$$\begin{pmatrix} 0 & p_{1,2} & 0 \\ 0 & 0 & p_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & 0 & 0 \\ 0 & q_{2,2} & 0 \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ -p_{2,3} & 0 & 0 \\ 0 & p_{1,2} & 0 \end{pmatrix},$$

2.17.2–2.17.4, 2.17.6–2.17.10, 2.17.13–2.17.15, 2.17.17–2.17.24, 2.17.26

$$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & p_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & q_{1,3} \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & -q_{1,3} & r_{1,3} \\ -p_{2,3} & r_{1,1}+p_{1,3}-q_{2,3} & r_{2,3} \\ 0 & 0 & r_{1,1}+p_{1,3} \end{pmatrix},$$

2.20.3, 2.20.4 $(\delta = 0)$, 2.20.8 $(\delta = 1)$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}+1 & p_{1,3} \\ 0 & \delta & q_{1,1}+1 \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1} & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3} \end{pmatrix},$$

2.20.9, 2.20.10

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+a & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a+1 \end{pmatrix},$$

2.20.11 $(\delta = 0)$, 2.20.12 $(\delta = 1)$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & 1 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}-1-\delta & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}-\delta \end{pmatrix},$$

2.20.13 $(\delta = 0)$, 2.20.14 $(\delta = 1)$

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & -1 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}-1-\delta & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}-\delta \end{pmatrix},$$

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2.20.15, 2.20.16

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}\pm 1 & p_{1,3} \\ 0 & 0 & q_{1,1}\pm 1 \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1} & 1 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3} \end{pmatrix},$$

2.20.17

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}+a & p_{1,3} \\ 0 & 0 & q_{1,1}+a \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+1 & 1 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+1 \end{pmatrix},$$

2.20.18

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+a & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a \end{pmatrix},$$

2.20.19 ($\delta = 0$), 2.20.20 ($\delta = 1$)

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}+\delta & p_{1,3} \\ 0 & 0 & q_{1,1}+\delta \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+b+1 & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+b \end{pmatrix},$$

2.20.21, 2.20.23

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}+b & p_{1,3} \\ 0 & \pm 1 & q_{1,1}+b \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+a & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a+1 \end{pmatrix},$$

2.20.22, 2.20.24

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & \pm 1 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+a & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a+1 \end{pmatrix},$$

2.20.25

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2}-a & p_{1,3} \\ 0 & -1 & q_{1,1}-a \end{pmatrix},$$

$$\begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}-1/3+a/3 & 0 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a/3+2/3 \end{pmatrix},$$

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2.20.26

$$\begin{pmatrix} 0 & p_{1,2} & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ 0 & q_{1,1}+p_{1,2} & p_{1,3} \\ 0 & 0 & q_{1,1} \end{pmatrix}, \quad \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{1,1}+a & 1 \\ 0 & p_{1,2} & r_{1,1}+p_{1,3}+a \end{pmatrix},$$

in other cases just as above.

In the case 3.20.12 connection is normal if $q_{1,1} = -p_{1,2}/2 \neq 0$, $r_{1,1} = -(p_{1,3}+1)/2$, $p_{1,3} \neq 0$; 3.20.14—$p_{1,3} \neq 0$, $q_{1,1} = -p_{1,2}/2 \neq 0$, $r_{1,1} = -(p_{1,3}+1)/2$, $p_{1,2}^2 \neq (p_{1,3}+1)^2$; 3.20.15—$p_{1,3} \neq 0$, $q_{1,1} = -p_{1,2}/2$, $q_{1,1} + 1 + 3r_{1,1} \neq 0$, $r_{1,1} = -(p_{1,3}+1)/2$, $p_{1,2}^2 \neq (p_{1,3}+1)^2$; 3.20.20 ($\lambda = 0$)—$a \neq 1$, $p_{1,3} \neq 0$, $q_{1,1} = -p_{1,2}/2 \neq 0$, $r_{1,1} = -(p_{1,3}+1)/2$; 3.20.24—$q_{1,1} = -p_{1,2}/2 \neq 0$, $r_{1,1} = -p_{1,3}/2$, then holonomy algebra

$$\begin{pmatrix} p_6 & p_1 & p_2 \\ 0 & p_3 & p_5 \\ 0 & p_4 & -p_3 \end{pmatrix}.$$

Consider the pair 2.9.1 ($\lambda = 0, \mu = -1$), connection is normal if $p_{1,2} \neq 0$, $p_{2,3} \neq 0$, $q_{2,2} = -2q_{1,1}$, holonomy algebra—$\mathfrak{sl}(3,\mathbb{R})$. In the case 2.9.2 ($\mu = -1$) connection is normal if $p_{1,2} \neq 0$, $p_{2,3} \neq 0$, $2q_{1,1} + q_{2,2} \neq 0$, holonomy algebra—$\mathfrak{gl}(3,\mathbb{R})$, or if $p_{1,2} \neq 0$, $p_{2,3} \neq 0$, $2q_{1,1} + q_{2,2} = 0$, holonomy algebra—$\mathfrak{sl}(3,\mathbb{R})$. In the case 2.9.4 ($\mu = -1$) connection is normal if $p_{1,2} \neq 0$, $p_{2,3} \neq 0$, $2q_{1,1} + q_{2,2} = 0$, holonomy algebra—$\mathfrak{sl}(3,\mathbb{R})$.

For the pair 2.17.2 connection is normal if $a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.3—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.4—$p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.6 connection is normal if $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.7—$p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.8—$b \neq 0$, $\delta \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.9—$b\delta \neq \gamma$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.10—$b\delta \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.13—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.14—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.15—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.17—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.18—$a\delta \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.19—$ab \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.20—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.21—$a \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.22—$a^2 + b^2 \neq 0$ (always, $b > 0$), $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.23—$ab \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.24—$\delta^2 \neq bc$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.25—$ab \neq 0$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$; 2.17.26—$b \neq ac$, $p_{1,3} = r_{1,1} = p_{2,3} = q_{2,3} = q_{1,3} = 0$, then holonomy algebra

$$\begin{pmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

For 2.20.11 connection is normal if $q_{1,1} = p_{1,2} = r_{1,1} = 0$, $p_{1,3} = 1$; 2.20.13—$q_{1,1} = p_{1,2} = r_{1,1} = 0$, $p_{1,3} = 1$; 2.20.15—$q_{1,1} = p_{1,3} = r_{1,1} = 0$, $p_{1,2} = -2$; 2.20.16—$q_{1,1} = p_{1,3} = r_{1,1} = 0$, $p_{1,2} = 2$; 2.20.17—$a \neq 0$, $q_{1,1} = r_{1,1} = 0$, $p_{1,3} = -2$, $p_{1,2} = -2a$; 2.20.20—$b \neq -1/2$, $q_{1,1} = r_{1,1} = 0$, $p_{1,3} = -2b - 1$, $p_{1,2} = -2$; 2.20.21—$a \neq -1/2$, $q_{1,1} = r_{1,1} = 0$, $p_{1,3} = -2a - 1$, $p_{1,2} = -2b$; 2.20.23—$b \neq -1/2$, $a \neq \pm b$, $q_{1,1} = r_{1,1} = 0$, $p_{1,3} = -2b - 1$, $p_{1,2} = -2a$, then holonomy algebra

$$\begin{pmatrix} 0 & p_1 & p_2 \\ 0 & p_3 & p_4 \\ 0 & p_5 & -p_3 \end{pmatrix}.$$

Continuing in the same way for all solvable pairs, we have, that homogeneous spaces, allowing a normal connection, except presented in the theorem, do not exist. □

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