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On almost hyperHermitian structures on Riemannian manifolds and tangent bundles

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Abstract: Some results concerning almost hyperHermitian structures are considered, using the notions of the canonical connection and the second fundamental tensor field h of a structure on a Riemannian manifold which were introduced by the second author.
With the help of any metric connection $\widetilde{\nabla}$ on an almost Hermitian manifold M an almost hyperHermitian structure can be constructed in the defined way on the tangent bundle *TM*. A similar construction was considered in [6], [7]. This structure includes two basic anticommutative almost Hermitian structures for which the second fundamental tensor fields h^1 and h^2 are computed. It allows us to consider various classes of almost hyperHermitian structures on *TM*. In particular, there exists an infinite-dimensional set of almost hyperHermitian structures on TTM where M is any Riemannian manifold.

Keywords: Riemannian manifolds, almost hyperHermitian structures, tangent bundle
MSC (2000): 53C15, 53C26

1 Some remarks on almost hyperHermitian structures

1^0. Let (M, J, g) be an almost Hermitian manifold i.e. $J^2 = -I$ and $g(JX, JY) = g(X, Y)$ for $X, Y \in \chi(M)$, where g is a fixed Riemannian metric on M. For any Riemannian metric $\tilde{g}$ on M such a metric g is defined by the formula

$$g(X, Y) = \tfrac{1}{2}(\tilde{g}(X,Y) + \tilde{g}(JX, JY)),\ X, Y \in \chi(M).$$

Let ∇ be the Riemannian connection of the metric g. Then one can define a connection $\bar{\nabla}$ on M by

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$$\bar{\nabla}_X Y = \frac{1}{2}(\nabla_X Y - J\nabla_X JY) = \nabla_X Y + \frac{1}{2}\nabla_X(J)JY, X, Y \in \chi(M). \tag{1}$$

The connection $\bar{\nabla}$ is called the canonical connection of the pair $(J,\ g)$, or more precisely of the corresponding G-structure, where $G = U(n)$, [2]. In particular, $\bar{\nabla}g = 0$, $\bar{\nabla}J = 0$.

The tensor field h is called the second fundamental tensor field of the pair $(J,\ g)$ [2], where

$$h_X Y = \nabla_X Y - \bar{\nabla}_X Y = -\frac{1}{2}\nabla_X(J)JY = \frac{1}{2}(\nabla_X Y + J\nabla_X JY), X, Y \in \chi(M); \tag{2}$$

$$h_{XYZ} = g(h_X Y, Z) = -h_{XZY}. \tag{3}$$

In particular, the classification given in [3] can be rewritten in terms of the tensor field h, [2]. Let dim $M \geq 6$ and $2\beta(X) = \delta\Phi(JX)$, where $\Phi(X,\ Y) = g(JX,\ Y)$. Then we have

Class	Defining condition
K	$h = 0$
$\mathbf{U}_1$=**NK**	$h_X X = 0$
$\mathbf{U}_2$=**AK**	$\sigma h_{XYZ} = 0$
$\mathbf{U}_3$=**SK**∩**H**	$h_{XYZ} - h_{JXJYZ} = \beta(Z) = 0$
$\mathbf{U}_4$	$h_{XYZ} = \frac{1}{2(n-1)}[<X, Y>\beta(Z) - <X, Z>\beta(Y) - <X, JY>\beta(JZ) + $ $+ <X, JZ>\beta(JY)]$
$\mathbf{U}_1 \oplus \mathbf{U}_2$=**QK**	$h_{XYJZ} = h_{JXYZ}$
$\mathbf{U}_3 \oplus \mathbf{U}_4$=**H**	$N(J) = 0$ or $h_{XYJZ} = -h_{JXYZ}$
$\mathbf{U}_1 \oplus \mathbf{U}_3$	$h_{XXY} - h_{JXJXY} = \beta(Z) = 0$
$\mathbf{U}_2 \oplus \mathbf{U}_4$	$\sigma[h_{XYJZ} - \frac{1}{(n-1)} <JX, Y>\beta(Z)] = 0$
$\mathbf{U}_1 \oplus \mathbf{U}_4$	$h_{XXY} = -\frac{1}{2(n-1)}[<X, Y>\beta(X) - \Vert X\Vert^2\beta(Y) - <X, JY>\beta(JX)]$
$\mathbf{U}_2 \oplus \mathbf{U}_3$	$\sigma[h_{XYJZ} + h_{JXYZ}] = \beta(Z) = 0$
$\mathbf{U}_1 \oplus \mathbf{U}_2 \oplus \mathbf{U}_3$=**SK**	$\beta = 0$
$\mathbf{U}_1 \oplus \mathbf{U}_2 \oplus \mathbf{U}_4$	$h_{XYJZ} - h_{JXYZ} = \frac{1}{(n-1)}[<X, Y>\beta(JZ) - <X, Z>\beta(JY) + $ $+ <X, JY>\beta(Z) - <X, JZ>\beta(Y)]$
$\mathbf{U}_1 \oplus \mathbf{U}_3 \oplus \mathbf{U}_4$	$h_{XJXY} + h_{JXXY} = 0$
$\mathbf{U}_2 \oplus \mathbf{U}_3 \oplus \mathbf{U}_4$	$\sigma[h_{XYJZ} + h_{JXYZ}] = 0$
U	No condition

Table 1

2^0. We consider an almost hyperHermitian structure (ahHs) on a manifold M consisting of $(J_1,\ J_2,\ J_3,\ g)$, where $J_i^2 = -I$, $J_1 J_2 = -J_2 J_1 = J_3$, $g(J_i X,\ J_i Y) = g(X,\ Y)$, $i = 1, 2, 3$.

For any Riemannian metric $\tilde{g}$ such a metric g can be defined by the formula
$g(X, Y) = \frac{1}{4}(\tilde{g}(X,Y) + \tilde{g}(J_1X, J_1Y) + \tilde{g}(J_2X, J_2Y) + \tilde{g}(J_3X, J_3Y)),\ X, Y \in \chi(M)$.

If ∇ is the Riemannian connection of the metric g, then the canonical connection $\bar{\nabla}$ in the sense of [**2**] of the ahHs has the following form

$$\bar{\nabla}_X Y = \frac{1}{4}(\nabla_X Y - J_1\nabla_X J_1 Y - J_2\nabla_X J_2 Y - J_3\nabla_X J_3 Y), X, Y \in \chi(M). \quad (4)$$

In particular, $\bar{\nabla}g = 0$, $\bar{\nabla}J_i = 0$, $i = 1, 2, 3$.

Proposition 1.1. *Let (M, J_1, g) be a Kaehlerian manifold i.e. $\nabla J_1 = 0$ on M. Then the connection given by* (4) *coincides with those defined by* (1) *for (M, J_2, g) and (M, J_3, g). In particular, the second fundamental tensor fields of (M, J_2, g) and (M, J_3, g) are the same.*

Proof.

$$\begin{aligned}\bar{\nabla}_X Y &= \frac{1}{4}(\nabla_X Y - J_1^2\nabla_X Y - J_2\nabla_X J_2 Y - J_1J_2\nabla_X J_1J_2Y) = \\ &= \frac{1}{4}(2\nabla_X Y - J_2\nabla_X J_2 Y - J_1J_2J_1\nabla_X J_2 Y) = \\ &= \frac{1}{2}(\nabla_X Y - J_2\nabla_X J_2 Y) = \\ &= \frac{1}{2}(\nabla_X Y - J_3J_1\nabla_X J_3J_1 Y) = \\ &= \frac{1}{2}(\nabla_X Y - J_3\nabla_X J_3 Y).\end{aligned}$$

□

To illustrate the situation described in *proposition* 1.1 we consider the following example.

Example 1.2. Let (M, J, g) be an almost Hermitian manifold and let (J, g) belong to one of the classes in T*able* 1, dim$M = 4n$. Further, define orthonormal vector fields $X_1, ..., X_{2n}, JX_1, ..., JX_{2n}$ on some open neighborhood $U$of a point $p \in M$. Assuming $J_1 = J$ on U, we get an almost Hermitian manifold (U, J_1, g) which also belongs to the corresponding class. Then, we can define J_2 by the following equalities

$$\begin{aligned}J_2X_1 &= X_{n+1}, J_2X_2 = X_{n+2}, \ldots, J_2X_n = X_{2n}; \\ J_2X_{n+1} &= -X_1, J_2X_{n+2} = -X_2, \ldots, J_2X_{2n} = -X_n; \\ J_2(J_1X_1) &= -J_1X_{n+1}, J_2(J_1X_2) = -J_1X_{n+2}, \ldots, J_2(J_1X_n) = -J_1X_{2n}; \\ J_2(J_1X_{n+1}) &= J_1X_1, J_2(J_1X_{n+2}) = J_1X_2, \ldots, J_2(J_1X_{2n}) = J_1X_n.\end{aligned}$$

It is clear that $J_1J_2 = -J_2J_1$, $J_1^2 = J_2^2 = -I$ and $g(J_iX, J_iY) = g(X, Y)$, $i = 1, 2, 3$ on U, where $J_3 = J_1J_2$, $X, Y \in \chi(U)$.

If (M, J, g) is a Kaehlerian manifold (class $\boldsymbol{K}$) and dim$M = 4n$ then we obtain the situation given in *proposition* 1.1 on the almost hyperHermitian manifold
(U, J_1, J_2, J_3, g).

One can find examples of the structures from Table 1 in [2], [3].

To get an almost Hermitian manifold of dimension $4n$ we can take a Riemannian product $M \times M$.

Problem 1.3. Let $(\boldsymbol{U}_\alpha,\ \boldsymbol{U}_\beta)$ be any pair of the classes from Table 1 ($\alpha, \beta = 0, ..., 15$). Can one construct such examples of ahHs $(J_1,\ J_2,\ J_3,\ g)$ on manifolds such that $(J_1,\ g)$ belongs to the class $\boldsymbol{U}_\alpha$ and $(J_2,\ g)$ belongs to the class $\boldsymbol{U}_\beta$? The cases $(\boldsymbol{K},\ \boldsymbol{K})$ and $(\boldsymbol{U},\ \boldsymbol{U})$ are easily illustrated.

3^0. **Definition 1.4.** [5] *A connected Riemannian manifold (M, g) with a family of local isometries $\{s_x\colon x \in M\}$ is called a locally k-symmetric Riemannian space ($k-s.$ $l.$ $R.$ $s.$) if the following axioms are fulfilled:*

(a) *$s_x(x) = x$ and x is the isolated fixed point of the local symmetry s_x;*

(b) *the tensor field S: $S_x = (s_x)_{*x}$ is smooth and invariant under any local isometry s_x;*

(c) *$S^k = I$ and k is the least of such positive integers.*

If M is a k-s. l. R. s. then the unique canonical connection $\widetilde{\nabla}$ can be defined by the following formula (see [2])

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{k}\sum_{j=1}^{k-1} \nabla_X(S^j)S^{k-j}Y = \frac{1}{k}\sum_{j=0}^{k-1} S^j \nabla_X S^{k-j}Y, X, Y \in \chi(M). \tag{5}$$

Further, we have $\widetilde{\nabla} g = \widetilde{\nabla}\tilde{R} = \widetilde{\nabla} h = \widetilde{\nabla} S = 0$, $S(\tilde{R}) = \tilde{R}$, $S(h) = h$, $S(g) = g$, where $h = \nabla - \widetilde{\nabla}$ and $\tilde{R}$ is the curvature tensor field of $\widetilde{\nabla}$.

Let M be such a k-s. l. R. s. that $S_x = (s_x)_{*x}$ has only complex eigenvalues $a_1 \pm b_1 i$, ..., $a_r \pm b_r i$. We define distributions D_i, $i = 1, ..., r$ by

$$D_i = ker(S^2 - 2a_i S + I).$$

Every $X \in \chi(M)$ has the unique decomposition $X = X_1 + ... + X_r$, where $X_i \in D_i$, $i = 1, ..., r$. An almost complex structure J on M is defined by

$$JX = \sum_{i=1}^{r} \frac{1}{b_i}(S - a_i I)X_i. \tag{6}$$

It is proved in [2] that (J, g) is an almost Hermitian structure and the connections $\bar{\nabla}$ and $\widetilde{\nabla}$ defined by (1) and (5) respectively coincide i.e. $\widetilde{\nabla} = \bar{\nabla}$ on M.

Proposition 1.5. *Let $(J_1,\ J_2,\ J_3,\ g)$ be such an ahHs on a k-s. l. R. s. (M, g) that $J_1 = J$, where J is defined by (6). In this case J_2 and J_3 are not invariant with respect to the family $\{s_x\colon x \in M\}$.*

Proof. Otherwise we have $(s_x)_{*x} \cdot J_2 X = J_2 \cdot (s_x)_{*x} X$ i.e. $S \cdot J_2 X = J_2 \cdot SX$. Using (6) we get

$$J_2 J_1 X = \sum_{i=1}^{r} \frac{1}{b_i}(J_2 S - a_i J_2 X_i) = \sum_{i=1}^{r} \frac{1}{b_i}(S - a_i I)J_2 X_i,$$

$$J_1 J_2 X = \sum_{i=1}^{r} \frac{1}{b_i}(S - a_i I)(J_2 X)_i,$$
$$(S^2 - 2a_i S + I) J_2 X_i = J_2 (S^2 - 2a_i S + I) X_i = 0,$$

therefore $J_2X_i = (J_2X)_i$ and $J_1J_2X = J_2J_1X$ for any $X \in \chi(M)$. But we have $J_1J_2X = - \quad J_2J_1X$, hence $J_3X = J_1J_2X = 0$ i.e. $J_3 = 0$.

We obtained the contradiction because J_3 is nonsingular. By similar arguments J_3 can not be invariant with respect to the family $\{s_x\colon x \in M\}$. □

2 HyperHermitian structures on tangent bundles

0^0. Let (M, g) be a Riemannian manifold and TM be its tangent bundle. For a metric connection $\widetilde{\nabla}$ $(\widetilde{\nabla} g = 0)$ we consider the connection map $\tilde{K}$ of $\widetilde{\nabla}$ [1], [4], defined by the formula

$$\widetilde{\nabla}_X Z = \tilde{K} Z_* X, \tag{7}$$

where Z is considered as a map from M into TM and the right side means a vector field on M assigning to $p \in M$ the vector $\tilde{K} Z_* X_p \in M_p$.

If $U \in TM$, we denote by H_U the kernel of $\tilde{K}_{|TM_U}$ and this n-dimensional subspace of TM_U is called the horizontal subspace of TM_U.

Let π denote the natural projection of TM onto M, then π_* is a C^∞-map of TTM onto TM. If $U \in TM$, we denote by V_U the kernel of $\pi_{*|TM_U}$ and this n-dimensional subspace of TM_U is called the vertical subspace of TM_U $(\dim TM_U = 2\dim M = 2n)$. The following maps are isomorphisms of corresponding vector spaces $(p = \pi\ (U))$

$$\pi_{*|TM_U} : \ H_U \to M_p, \quad \tilde{K}_{|TM_U} : \ V_U \to M_p$$

and we have

$$TM_U = H_U \oplus V_U.$$

If $X \in \chi(M)$, then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X and denoted by $\bar{X}^h$ $(\bar{X}^v)$, such that for all $U \in TM$:

$$\pi_* \bar{X}^h_U = X_{\pi(U)}, \quad \tilde{K} \bar{X}^h_U = 0_{\pi(U)}, \tag{8}$$

$$\pi_* \bar{X}^v_U = 0_{\pi(U)}, \quad \tilde{K} \bar{X}^v_U = X_{\pi(U)}. \tag{9}$$

Let $\tilde{R}$ be the curvature tensor field of $\widetilde{\nabla}$, then following [1] we write

$$[\bar{X}^v, \bar{Y}^v] = 0, \tag{10}$$

$$[\bar{X}^h, \bar{Y}^v] = \left(\bar{\tilde{\nabla}}_X Y\right)^v, \tag{11}$$

$$\pi_*([\bar{X}^h, \bar{Y}^h]_U) = [X, Y], \tag{12}$$

$$\tilde{K}([\bar{X}^h, \bar{Y}^h]_U) = \tilde{R}(X, Y) U. \tag{13}$$

For vector fields $\bar{X} = \bar{X}^h \oplus \bar{X}^v$ and $\bar{Y} = \bar{Y}^h \oplus \bar{Y}^v$ on TM the natural Riemannian metric $< , >$ is defined on TM by the formula

$$< \bar{X},\ \bar{Y} >= g(\pi_*\bar{X},\ \pi_*\bar{Y}) + g(\tilde{K}\bar{X},\ \tilde{K}\bar{Y}). \tag{14}$$

It is clear that the subspaces H_U and V_U are orthogonal with respect to $< , >$.

It is easy to verify that $\bar{X}_1^h, \bar{X}_2^h, \ldots, \bar{X}_n^h, \bar{X}_1^v, \bar{X}_2^v, \ldots, \bar{X}_n^v$ are orthonormal vector fields on TM if $X_1, X_2, \ldots, X_n$ are those on M i.e. $g(X_i, X_j) = \delta_j^i$.

$\mathbf{1}^0$. We define a tensor field J_1 on TM by the equalities

$$J_1\bar{X}^h = \bar{X}^v, J_1\bar{X}^v = -\bar{X}^h, X \in \chi(M). \tag{15}$$

For $X \in \chi(M)$ we get

$$J_1^2\bar{X} = J_1(J_1(\bar{X}^h \oplus \bar{X}^v)) = J_1(-\bar{X}^h \oplus \bar{X}^v) = -(\bar{X}^h \oplus \bar{X}^v) = -I\bar{X}$$

and

$$J_1^2 = -I.$$

For $X, Y \in \chi(M)$ we obtain

$$< J_1\bar{X}, J_1\bar{Y} >=< -\bar{X}^h \oplus \bar{X}^v, -\bar{Y}^h \oplus \bar{Y}^v >=< -\bar{X}^h, -\bar{Y}^h > + < \bar{X}^v, \bar{Y}^v >,$$

$$< \bar{X}, \bar{Y} >=< \bar{X}^h \oplus \bar{X}^v, \bar{Y}^h \oplus \bar{Y}^v >=< \bar{X}^h, \bar{Y}^h > + < \bar{X}^v, \bar{Y}^v >$$

and it follows that $<J_1\bar{X},\ J_1\bar{Y}> = <\bar{X},\ \bar{Y}>$, $(TM,\ J_1,\ < , >)$ is an almost Hermitian manifold.

Further, we want to analyze the second fundamental tensor field h^1 of the pair $(J_1, < , >)$.

The Riemannian connection $\hat{\nabla}$ of the metric $< , >$ on TM is defined by the formula (see [4])

$$\begin{aligned} < \hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} > = \frac{1}{2}(&\bar{X} < \bar{Y}, \bar{Z} > + \bar{Y} < \bar{Z}, \bar{X} > - \bar{Z} < \bar{X}, \bar{Y} > + < \bar{Z}, [\bar{X}, \bar{Y}] > + \\ &+ < \bar{Y}, [\bar{Z}, \bar{X}] > + < \bar{X}, [\bar{Z}, \bar{Y}] >),\ X, Y, Z \in \chi(M). \end{aligned} \tag{16}$$

Using (2), (3) for orthonormal vector fields $\bar{X},\ \bar{Y},\ \bar{Z}$ on TM we obtain

$$\begin{aligned} h^1_{\bar{X}\bar{Y}\bar{Z}} &= < h^1_{\bar{X}}\bar{Y}, \bar{Z} >= \frac{1}{2} < \hat{\nabla}_{\bar{X}}\bar{Y} + J_1\hat{\nabla}_{\bar{X}}J_1\bar{Y}, \bar{Z} >= \\ &= \frac{1}{2}(< \hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} > - < \hat{\nabla}_{\bar{X}}J_1\bar{Y}, J_1\bar{Z} >) = \\ &= \frac{1}{4}(< [\bar{X}, \bar{Y}], \bar{Z} > + < [\bar{Z}, \bar{X}], \bar{Y} > + < [\bar{Z}, \bar{Y}], \bar{X} > - \\ &\quad - < [\bar{X},\ J_1\bar{Y}],\ J_1\bar{Z} > - < [J_1\bar{Z}, \bar{X}], J_1\bar{Y} > - < [J_1\bar{Z}, J_1\bar{Y}], \bar{X} >). \end{aligned} \tag{17}$$

Using (10)–(13) and (17) we consider the following cases for the tensor field h^1 assuming all the vector fields to be orthonormal.

For $X,\, Y \in \chi(M)$ we get

$$
\begin{aligned}
< J_2\bar{X}, J_2\bar{Y} > &= < \left(\overline{JX}\right)^h \oplus - \left(\overline{JX}\right)^v, \left(\overline{JY}\right)^h \oplus - \left(\overline{JY}\right)^v > = < \left(\overline{JX}\right)^h, \left(\overline{JY}\right)^h > + \\
&+ < \left(\overline{JX}\right)^v, \left(\overline{JY}\right)^v > = g(JX, JY) + g(JX, JY) = \\
&= g(X,Y) + g(X,Y) = < \bar{X}^h, \bar{Y}^h > + < \bar{X}^v, \bar{Y}^v > \\
&= < \bar{X}^h \oplus \bar{X}^v, \bar{Y}^h \oplus \bar{Y}^v > = < \bar{X}, \bar{Y} > .
\end{aligned}
$$

Further, we obtain

$$J_1(J_2\bar{X}) = J_1(\left(\overline{JX}\right)^h \oplus - \left(\overline{JX}\right)^v) = \left(\overline{JX}\right)^h \oplus \left(\overline{JX}\right)^v,$$

$$J_2(J_1\bar{X}) = J_2(-\bar{X}^h \oplus \bar{X}^v) = -\left(\overline{JX}\right)^h \oplus - \left(\overline{JX}\right)^v.$$

Thus, we get $J_1J_2 = -J_2J_1 = J_3$ and ahHs (J_1, J_2, J_3, $<\ ,\ >$) on TM has been constructed.

Using (2), (3) for orthonormal vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ on TM we obtain

$$
\begin{aligned}
h^2_{\bar{X}\bar{Y}\bar{Z}} &= < h^2_{\bar{X}}\bar{Y}, \bar{Z} > = \frac{1}{2} < \hat{\nabla}_{\bar{X}}\bar{Y} + J_2\hat{\nabla}_{\bar{X}}J_2\bar{Y}, \bar{Z} > = \\
&= \frac{1}{2}(< \hat{\nabla}_{\bar{X}}\bar{Y}, \bar{Z} > - < \hat{\nabla}_{\bar{X}}J_2\bar{Y}, J_2\bar{Z} >) = \\
&= \frac{1}{4}(< [\bar{X}, \bar{Y}], \bar{Z} > + < [\bar{Z},\ \bar{X}], \bar{Y} > + < [\bar{Z}, \bar{Y}], \bar{X} > - \\
&- < [\bar{X},\ J_2\bar{Y}],\ J_2\bar{Z} > - < [J_2\bar{Z}, \bar{X}], J_2\bar{Y} > - < [J_2\bar{Z}, J_2\bar{Y}], \bar{X} >). \qquad (19)
\end{aligned}
$$

Using (10)–(13) and (19) we consider the following cases for the tensor field h^2 assuming all the vector fields to be orthonormal.

$$
\begin{aligned}
h^2_{\bar{X}^h\bar{Y}^h\bar{Z}^h} &= \frac{1}{4}(< [\bar{X}^h, \bar{Y}^h], \bar{Z}^h > + < [\bar{Z}^h, \bar{X}^h], \bar{Y}^h > + < [\bar{Z}^h, \bar{Y}^h], \bar{X}^h > - \\
&- < [\bar{X}^h, J_2\bar{Y}^h], J_2\bar{Z}^h > - < [J_2\bar{Z}^h, \bar{X}^h], J_2\bar{Y}^h > - \\
&- < [J_2\bar{Z}^h, J_2\bar{Y}^h], \bar{X}^h >) = \frac{1}{4}(g([X,Y],Z) + g([Z,X],Y) + \\
&+ g([Z,Y],X) - g([X,JY],JZ) - g([JZ,X],JY) - \\
&- g([JZ,JY],X)) = \frac{1}{2}(g(\nabla_X Y, Z) - g(\nabla_X JY, JZ)) = h_{XYZ}. \qquad (1.2^0)
\end{aligned}
$$

$$
\begin{aligned}
h^2_{\bar{X}^h\bar{Y}^h\bar{Z}^v} &= \frac{1}{4}(< [\bar{X}^h, \bar{Y}^h], \bar{Z}^v > + < [\bar{Z}^v, \bar{X}^h], \bar{Y}^h > + < [\bar{Z}^v, \bar{Y}^h], \bar{X}^h > - \\
&- < [\bar{X}^h, J_2\bar{Y}^h], J_2\bar{Z}^v > - < [J_2\bar{Z}^v, \bar{X}^h], J_2\bar{Y}^h > - \\
&- < [J_2\bar{Z}^v, J_2\bar{Y}^h], \bar{X}^h >) = \frac{1}{4}(g(\tilde{R}(X,Y)U, Z) + g(\tilde{R}(X,JY)U, JZ)) = \\
&= -\frac{1}{4}(g(\tilde{R}(X,Y)Z, U) + g(\tilde{R}(X,JY)JZ, U)). \qquad (2.2^0)
\end{aligned}
$$

By similar arguments we obtain

$$h^2_{\bar{X}^h\bar{Y}^v\bar{Z}^h} = \frac{1}{4}(g(\tilde{R}(X,Z)Y,U) + g(\tilde{R}(X,JZ)JY,U)). \qquad (3.2^0)$$

$$h^2_{\bar{X}^v\bar{Y}^h\bar{Z}^h} = -\frac{1}{4}(g(\tilde{R}(Z,Y)X,U) - g(\tilde{R}(JZ,JY)X,U)). \qquad (4.2^0)$$

$$h^2_{\bar{X}^v\bar{Y}^v\bar{Z}^v} = 0. \qquad (5.2^0)$$

$$h^2_{\bar{X}^v\bar{Y}^v\bar{Z}^h} = 0. \qquad (6.2^0)$$

$$h^2_{\bar{X}^v\bar{Y}^h\bar{Z}^v} = 0. \qquad (7.2^0)$$

$$h^2_{\bar{X}^h\bar{Y}^v\bar{Z}^v} = \frac{1}{2}(g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_X JY, JZ)). \qquad (8.2^0)$$

It is clear that the construction of the ahHs on TM strongly depends on the connection $\widetilde{\nabla}$ and we can obtain in this way an infinite dimensional set of ahHs.

Theorem 2.1. *Let* $(M,\, g,\, J)$ *be an almost Hermitian manifold. Then there exists an infinite family of ahHs on* TM (*in particular, such structures can be constructed by the method above*).

Corollary 2.2. *Let* $(M,\, g)$ *be a Riemannian manifold. Then there exists an infinite set of ahHs on* TTM.

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